

A NOTE ON THE UNIQUENESS OF SOLUTIONS TO THE TRANSPORTATION PROBLEM¹

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ABSTRACT

Let μ and ν be two probability measures on the real line and c be a lower semicontinuous function on the plane. The mass transfer problem consists in determining a measure ξ with respective marginals μ and ν that minimizes the functional $\int c d\xi$. In this paper we show that, whenever the function c is *strictly* superadditive, the solution corresponding to the lower Fréchet bound is the unique optimal solution. This result also holds for the discrete version of the problem.

RÉSUMÉ

Soient μ et ν deux mesures de probabilité sur la droite réelle, et soit c une fonction semicontinue inférieurement définie sur le plan. Le problème du transfert des masses consiste à déterminer une mesure ξ de marges μ et ν qui minimise la fonctionnelle $\int c d\xi$. Dans cet article nous montrons que, si la fonction c est *strictement* superadditive, alors la solution correspondant à la borne inférieure de Fréchet est l'unique solution optimale de notre problème. En fait, ce résultat est aussi valable pour le problème de Hitchcock, c'est-à-dire la version discrète du problème considéré.

1. INTRODUCTION

Together with the shortest path problem, Hitchcock's transportation problem is one of the most studied problems in the field of network flows. It constitutes a special case of the transshipment problem, which pervades most transportation studies.

Given a bipartite network whose node set is partitioned into origin and destination nodes, Hitchcock's problem consists in satisfying, at minimum transportation cost, the demand for a commodity that is supplied from the origin nodes. Mathematically, it takes the form of the linear program:

$$\begin{aligned} \min \rho &= \sum_{i=1}^m \sum_{j=1}^n c_{ij} x_{ij} \\ \text{s.t.} \quad \sum_{j=1}^n x_{ij} &= a_i \quad 1 \leq i \leq m \\ \sum_{i=1}^m x_{ij} &= b_j \quad 1 \leq j \leq n \\ x_{ij} &\geq 0 \quad 1 \leq i \leq m, \quad 1 \leq j \leq n, \end{aligned} \tag{1}$$

where a_i denotes the supply at origin node i , b_j the demand at destination node j , c_{ij} the unit transportation cost on arc (i, j) and x_{ij} the amount of the commodity being

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shipped on arc (i, j) . In order that the problem be feasible, we require that supply and demand be equal, i.e.:

$$\sum_{i=1}^m a_i = \sum_{j=1}^n b_j.$$

Its linear programming dual, to be used in the sequel, takes the form:

$$\begin{aligned} \max \rho^* &= \sum_{i=1}^m a_i u_i + \sum_{j=1}^n b_j v_j \\ \text{s.t.} \quad u_i + v_j &\leq c_{ij} \quad 1 \leq i \leq m, \quad 1 \leq j \leq n. \end{aligned} \quad (2)$$

The mass transfer problem, also known as the Monge-Kantorovitch problem, and first been studied by Monge [7] in 1781, constitutes a continuous extension of the transportation problem. The work of Monge has been pursued by several mathematicians, among which Appell [1, 2] and Kantorovitch [6], who formulated it as a mathematical program in a functional space. This latter approach has been adopted in the modern literature.

To be specific, let us consider two topological spaces X and Y endowed with respective regular measures μ and ν , with $\mu(X) = \nu(Y)$. These measures correspond, respectively, to the supply vector a and the demand vector b of the discrete version. Let c denote a lower semicontinuous function over the product set $X \times Y$. The mass transfer problem consists in determining a measure ξ over $X \times Y$ with marginals μ and ν and such that the transportation cost (primal objective)

$$\gamma = \iint_{X \times Y} c(x, y) d\xi(x, y) \quad (3)$$

be minimal.

The dual problem associated with (3) consists in determining two real functions u and v , defined over X and Y respectively, that maximize the dual objective

$$\gamma^* = \int_X u(x) d\mu(x) + \int_Y v(y) d\nu(y) \quad (4)$$

subject to the constraints

$$u(x) + v(y) \leq c(x, y) \quad \forall (x, y) \in X \times Y. \quad (5)$$

The above problems are equivalent in the sense that their optimal values coincide. If γ is finite, the optima are achieved.

A similar problem has been considered in probability theory (consult Cambanis, Simons and Stout [3], Rachev and Rüschendorf [8] and Tchen [9]), where one is required to minimize a functional $\int c d\xi$, where ξ is a probability distribution function over R^2 with given marginals μ and ν . In particular, Tchen proved that if c is continuous and superadditive, then the joint distribution defined as

$$\xi_*(x, y) = \max\{\mu(x) + \nu(y) - 1, 0\}$$

is optimal.

Following some preliminary results (Section 2) we show in Section 3 that if c is lower semicontinuous and *strictly* superadditive, and if γ is finite, then the optimal solution is unique. In Section 4 we show that this result also holds for Hitchcock's transportation problem.

2. PRELIMINARY RESULTS

Throughout the paper we denote by $\Gamma(\mu, \nu)$ the space of measures over $X \times Y$ with marginals μ and ν . We begin by recalling the main duality theorems for the Monge-Kantorovitch problem as given in a thorough analysis of Kellerer [4].

Theorem 1 [4]

If c is lower semicontinuous, the two optimal values γ and γ^* in (3) and (4) are equal.

Theorem 2 [4]

If c is lower semicontinuous, then there is a measure $\xi \in \Gamma(\mu, \nu)$ which are optimal for the primal problem 3.

Theorems 1 and 2 are given in a slightly more general form as Theorem 2.6 and 2.19 in [4].

A pair of functions u and v with respective domains X and Y defines a *cost partition* or *dual-feasible solution* if there holds:

$$u(x) + v(y) \leq c(x, y)$$

for all (x, y) in $X \times Y$. Kellerer [4] proved the existence of a cost partition which is optimal for the dual problem (4-5) if γ^* is finite. Before stating this result, we need a definition.

A function c fulfills the condition (6) if there exist two functions $p(x)$ and $q(y)$ such that

$$|c(x, y)| \leq p(x) + q(y) \quad (6)$$

for all (x, y) in $X \times Y$, while $p \in L^1(\mu)$, $q \in L^1(\nu)$, for all $x \in X$, $p(x) < \infty$, for all $y \in Y$, $q(y) < \infty$.

Theorem 3 [4]

If the condition 6 is satisfied by c , then there exist $u \in L^1(\mu)$, $v \in L^1(\nu)$ which are optimal for the dual problem (4-5).

Theorem 3 is given in a slightly different form as Theorem 2.21 in [4].

Remark

If condition (6) holds for the function c , then the primal problem has an optimal solution giving rise to a finite number γ . Indeed we have:

$$\begin{aligned} \left| \int_{X \times Y} c(x, y) d\xi(x, y) \right| &\leq \int_{X \times Y} |c(x, y)| d\xi(x, y) \\ &\leq \int_{X \times Y} p(x) d\xi(x, y) + \int_{X \times Y} q(y) d\xi(x, y) \\ &\leq \int_X p(x) d\mu(x) + \int_Y q(y) d\nu(y) \end{aligned}$$

Now let S_ξ denote the support of the measure ξ , i.e., the set of points (x, y) for which ξ is positive on any neighborhood of (x, y) . If (u_0, v_0) is an optimal cost partition and

ξ_0 an associated optimal measure, we have:

$$\begin{aligned} \int_{X \times Y} c(x, y) d\xi_0(x, y) &= \int_X u_0(x) d\mu_0(x) + \int_Y v_0(y) d\nu_0(y) \\ &= \int_{X \times Y} u_0(x) d\xi_0(x, y) + \int_{X \times Y} v_0(y) d\xi_0(x, y) \\ &= \int_{X \times Y} (u_0(x) + v_0(y)) d\xi_0(x, y). \end{aligned}$$

Hence $u_0(x) + v_0(y) = c(x, y)$ almost everywhere with respect to ξ_0 on S_{ξ_0} .

We say that a two-variable function c is *superadditive* if

$$c(x, y) + c(x', y') - c(x', y) - c(x, y') \geq 0 \quad (7)$$

whenever $x \leq x'$ and $y \leq y'$. The functions xy , $(x + y)^2$, $\min\{x, y\}$, $-\max\{x, y\}$, $-|x - y|^p$ ($p \geq 1$), $f(x - y)$ (f concave) are examples of superadditive functions.

We now introduce the Fréchet bounds. The lower Fréchet bound is the measure whose distribution function is

$$\xi_*(x, y) = \max\{\mu(x) + \nu(y) - 1, 0\}. \quad (8)$$

The upper Fréchet bound is the measure whose distribution function is

$$\xi^*(x, y) = \min\{\mu(x), \nu(y)\}. \quad (9)$$

Both ξ_* and ξ^* have marginals μ and ν .

Based on them, Rachev and Rüschendorf [8] report the following characterization of $\Gamma(\mu, \nu)$.

Theorem 4 [8]

The measure ξ is in $\Gamma(\mu, \nu)$ if and only if $\xi_(x, y) \leq \xi(x, y) \leq \xi^*(x, y)$ for all (x, y) in $X \times Y$.*

Tchen [9] proved the following key result.

Theorem 5 [9]

If $c : R^2 \rightarrow R$ is continuous and superadditive then every probability measure ξ in $\Gamma(\mu, \nu)$ satisfies:

$$\int_{X \times Y} c d\xi_* \leq \int_{X \times Y} c d\xi \leq \int_{X \times Y} c d\xi^*. \quad (10)$$

3. UNIQUENESS OF THE SOLUTION OF THE MASS TRANSFER PROBLEM

We say that the function c is *strictly* superadditive over $X \times Y$ if there holds

$$c(x, y) - c(x', y) + c(x', y') - c(x, y') > 0, \quad (11)$$

whenever $x' > x$ and $y' > y$. When this condition is fulfilled and when γ is finite, we will show that ξ_* is the *only* optimal solution to the mass transfer problem. We need the following lemma.

Lemma 1

Let $\xi \in \Gamma(\mu, \nu)$ be distinct from ξ_ . Then there exist two points (x', y') and (x'', y'') in the support S_ξ of the measure ξ such that $x' < x''$ and $y' < y''$.*

Proof

Let (x_0, y_0) be a point such that $\xi(x_0, y_0) \neq \xi_*(x_0, y_0)$. From Theorem 4 we have that $\xi(x_0, y_0) > \xi_*(x_0, y_0)$. Therefore:

$$\xi(x_0, y_0) > \xi_*(x_0, y_0) = \max\{\mu(x_0) + \nu(y_0) - 1, 0\} \geq 0, \quad (12)$$

from which it follows that $\xi(x_0, y_0)$ is positive.

Let us define the sets

$$\begin{aligned} A &= \{(x, y) \in S_\xi : y > y_0\} \\ B &= \{(x, y) \in S_\xi : x \leq x_0, y \leq y_0\} \\ C &= \{(x, y) \in S_\xi : x > x_0\}. \end{aligned}$$

If the result does not hold, there cannot exist a point (x, y) in the support of ξ such that $x > x_0$ and $y > y_0$. It follows that

$$A \subset \{(x, y) \in S_\xi : x \leq x_0, y > y_0\},$$

whence $\mu(x_0) = \xi(A) + \xi(B)$. Similarly we obtain that

$$C \subset \{(x, y) \in S_\xi : x > x_0, y \leq y_0\}$$

and $\nu(y_0) = \xi(C) + \xi(B)$, i.e.,

$$\mu(x_0) + \nu(y_0) = \xi(A) + 2\xi(B) + \xi(C) = 1 + \xi(B).$$

Therefore:

$$\xi(x_0, y_0) = \xi(B) = \mu(x_0) + \nu(y_0) - 1 = \xi_*(x_0, y_0),$$

which is a contradiction with (12). •

Theorem 6

Let μ and ν be two probability measures on the real line, and let $c : R^2 \rightarrow R$ be a lower semicontinuous strictly superadditive function: for any choice of numbers $x' > x, y' > y$

$$c(x, y) + c(x', y') > c(x', y) + c(x, y'),$$

we assume that the condition (6) is true for c . If ξ is a probability measure in the plane with μ and ν as marginals, if $\xi \neq \xi_*$, then

$$\int_{X \times Y} c d\xi_* < \int_{X \times Y} c d\xi.$$

Proof

According to the remark of Section 2, γ is a finite number. According to Theorem 2, there is a least one optimal measure for the primal problem 3. Moreover, $\gamma^* = \gamma$ (Theorem 1). Assume that ξ is optimal and that $\xi \neq \xi_*$ and let (u, v) be an optimal cost partition associated with ξ , whose existence is ensured by Theorem 3. We have:

$$u(x) + v(y) = c(x, y) \quad \text{almost everywhere on } S_\xi \quad (13)$$

with respect to the measure ξ . From Lemma 1, one can find points (x_0, y_0) and (x_1, y_1) in the support of ξ such that $x_0 < x_1$ and $y_0 < y_1$.

We set $h = \min\{|y_1 - y_0|, |x_1 - x_0|\}$. For $i = 0$ or 1 , D_i is the disk of center (x_i, y_i) and of radius $h/2$. From (13) it follows that we can find two points $(x'_0, y'_0) \in D_0$ and $(x'_1, y'_1) \in D_1$ such that

$$\begin{aligned} u(x'_0) + v(y'_0) &= c(x'_0, y'_0) \\ u(x'_1) + v(y'_1) &= c(x'_1, y'_1) \end{aligned}$$

We have: $x'_0 < x'_1$ and $y'_0 < y'_1$. We may write:

$$\begin{aligned} &c(x'_0, y'_0) + c(x'_1, y'_1) - c(x'_0, y'_1) - c(x'_1, y'_0) \\ &= u(x'_0) + v(y'_0) + u(x'_1) + v(y'_1) - c(x'_0, y'_1) - c(x'_1, y'_0) \\ &= u(x'_0) + v(y'_1) - c(x'_0, y'_1) + u(x'_1) + v(y'_0) - c(x'_1, y'_0) \\ &\leq 0, \end{aligned}$$

in contradiction with the strict superadditivity of c . •

4. SOLUTION UNIQUENESS FOR HITCHCOCK'S PROBLEM

The Northwest corner rule is a greedy procedure that produces a feasible solution to Hitchcock's transportation problem. At the first iteration, one performs the following operations: If $a_1 < b_1$, set $x_{11} = a_1$, subtract a_1 from b_1 and delete row 1; if $a_1 \geq b_1$, set $x_{11} = b_1$, subtract b_1 from a_1 and delete column 1. The procedure is then repeated on the reduced matrix.

The *Monge condition* is the equivalent of the superadditivity property to the discrete case, modulo a sign reversal convention. We say that a cost matrix $c = (c_{ij})$ satisfies a Monge condition if

$$c_{ij} + c_{i'j'} - c_{ij'} - c_{i'j} \leq 0 \quad (14)$$

for all indices i, i' in $\{1, 2, \dots, m\}$, j, j' in $\{1, 2, \dots, n\}$ such that $i < i'$ and $j < j'$. Hoffman [5] has proved that if c satisfies the Monge condition, then the Northwest corner rule provides an optimal solution.

A *strict* Monge condition is obtained if (14) is strict. The next lemma is the equivalent of Lemma 1 in the discrete case.

Lemma 2

Let x^ be the solution provided by the Northwest corner rule and x any other feasible solution, distinct from x^* . Then there exists indices (i, j) and (i', j') , $i' < i$, $j < j'$ such that $x_{ij} > 0$ and $x_{i'j'} > 0$.*

Proof

Assume that there do not exist indices (i, j) and (i', j') , $i' < i$, $j < j'$ such that $x_{ij} > 0$ and $x_{i'j'} > 0$. Two cases can occur:

- $x_{1j} = 0$ for $j > 1$. Then $x_{11} = a_1 \leq b_1$ and $x_{1j} = x_{1j}^*$ for all j in $\{1, \dots, n\}$ and the first row of x coincides with the first row of x^* ;
- $x_{i1} = 0$ for $i > 1$. Then $x_{11} = b_1 \leq a_1$ and $x_{i1} = x_{i1}^*$ for all i in $\{1, 2, \dots, m\}$ and the first column of x coincides with the first column of x^* ;

One then repeats the process on a smaller matrix resulting from the deletion of a row (first case) or a column (second case). After a finite number of steps, we come to the conclusion that the matrices x and x^* coincide, which contradicts our hypothesis. •

Theorem 7

If the cost matrix c satisfies the strict Monge condition, then the solution x^* provided by the Northwest corner rule is the only solution of the Hitchcock transportation problem.

Proof

Hoffman's result ensures that x^* is optimal. Let x be an optimal solution distinct from x^* . From the previous lemma, there must exist pairs of indices (i, j) and (i', j') , $i' < i$, $j < j'$ such that $x_{ij} > 0$ and $x_{i'j'} > 0$.

Let (u_i, v_j) be an optimal cost partition associated with x . Since $x_{ij} > 0$ there follows, by complementary slackness, that

$$u_i + v_j = c_{ij}.$$

Similarly, since $x_{i'j'} > 0$,

$$u_{i'} + v_{j'} = c_{i'j'}.$$

Consequently:

$$\begin{aligned} c_{i'j} + c_{ij'} - c_{i'j'} - c_{ij} &= c_{i'j} + c_{ij'} - u_{i'} - v_{j'} - u_i - v_j \\ &= (c_{i'j} - u_{i'} - v_j) + (c_{ij'} - u_i - v_{j'}) \\ &\geq 0, \end{aligned}$$

in contradiction with the strict Monge condition for the matrix c . •

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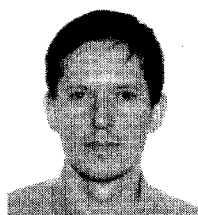
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