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# Existence and computation of optimal tolls in multiclass network equilibrium problems

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# ABSTRACT

In this work we provide a simple proof of the existence of optimal tolls for multiclass network equilibrium problems where the value-of-time parameter varies continuously throughout the population. The main argument, based on a finite-dimensional reformulation of the problem, also allows us to determine in a simple fashion revenue minimizing link toll vectors.

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# 1. Introduction

For a transportation network subject to congestion it is a well known fact (see Beckmann et al. [1]) that, if tolls are set to the marginal costs corresponding to link usage, then the resulting equilibrium will minimize total travel delay. Although this approach is no longer valid when the valuation of travel delay varies across the population, it is still true that the set of tolls that induces a system-optimal use of the network is nonempty. For the case of finitely many classes of customers, each characterized by its own value-of-time (VOT) parameter, Yang and Huang [2] have shown that such tolls could be set to the optimal dual vector of a network-structured linear program (see also Marcotte and Patriksson [3]). Slightly later, although the publishing year pre-dates that of [2], Cole et al. [4] extended the result to the infinite-dimensional case (a single origin-destination pair however), using a two-step approach. First, they provided an alternative and existential proof of Yang and Huang's result, followed by a limiting argument, where they let the number of user classes go to infinity. Independently, Fleischer et al. [5] and Karakostas and Kolliopoulos [6] obtained similar results and made the connection with the more general class of problems known as congestion games.

The aim of this work is to provide a simple and unifying proof of these results in the slightly more general framework of

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variational inequalities. Initially, we believed that this could be achieved by transposing in a trivial fashion the technique of Yang and Huang to the infinite-dimensional case. Unfortunately, in the absence of constraint qualifications, the duality argument could not be invoked any longer. However, it turns out that an implicit reformulation of the problem in finite dimension is amenable to a proof almost as direct as that of Yang and Huang, and considerably shorter and simpler than that of Cole et al. As a corollary, duality results for a class of semi-infinite linear programs are derived.

The work is structured as follows. In Section 2, we introduce the finite-dimensional multiclass problem, together with its infinite-dimensional counterpart. We describe Yang and Huang's linear programming approach and discuss the difficulty of extending it to the infinite case. In Section 3, we propose a new formulation of the problem, which lends itself to treatment with Yang and Huang's technique. In Section 4, we address the computation of minimal toll schedules, an issue that had previously been considered in the finite-dimensional case.

# 2. The multiclass network equilibrium problem

Let us consider a multicommodity network characterized by its node set N, arc set A and set K of origin–destination pairs, each pair  $k \in K$  being endowed with a fixed demand  $b_k$ . Let  $v_p$  denote the flow along path p. Let  $P_k$  denote the set of paths linking an OD pair  $k \in K$ . Let  $\Omega_v$  be the set of demand-feasible path flow vectors v, defined as

$$\Omega_{v} = \left\{ v \geq 0 : \sum_{p \in P_{k}} v_{p} = b_{k}, \forall k \in K \right\}$$

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or, in vector-matrix form:

$$\Omega_v = \{v \ge 0 : Av = b\}$$

In parallel, we introduce arc flows  $x_a$ , together with the compatibility relationship

$$x_a = \sum_{k \in K} \sum_{p \in P_k} \delta_{ap} v_p,$$

where  $\delta_{ap}$ , an entry of the arc–path incidence matrix  $\Delta$ , takes value 1 if arc *a* belongs to path *p*, and value 0 otherwise. The set of feasible arc flows is defined, implicitly, as

$$\Omega_{x} = \{ x : \exists v \in \Omega_{v} : x = \Delta v \}.$$

With each vector x we associate a generalized cost function  $\alpha F(x)$  + T, where the delay function F assumes positive values, but is not required to be separable, T is a toll vector, and the scalar  $\alpha$  represents the valuation of one unit of delay by the users. On the basis of this notation, a feasible arc flow vector x satisfies Wardrop's equilibrium conditions with respect to T if and only if it solves the variational inequality

$$\langle \alpha F(x) + T, x - y \rangle \le 0 \quad \forall y \in \Omega_x.$$
 (1)

Equivalently, a feasible path flow vector v is Wardropian if and only if it solves the variational inequality

$$\langle \Delta^{t}(F(\Delta v) + T), v - w \rangle \leq 0 \quad \forall w \in \Omega_{v}.$$
 (2)

Alternatively, a vector *x*<sup>\*</sup> is said to be *system optimal* if it minimizes total travel delay, i.e.,

$$x^* \in \arg\min_{x\in\Omega_x}\langle F(x),x\rangle$$

and a corresponding system-optimal path flow vector is denoted by  $v^*$ . The reader is referred to Marcotte and Patriksson [3] for further details concerning these standard formulations.

In this framework, our problem consists in finding a toll vector T such that the equilibrium associated with the modified cost  $\alpha F(x) + T$  coincides with the efficient assignment  $x^*$ . If the VOT parameter  $\alpha$  is uniform throughout the population, this can be achieved by setting T to the vector of scaled marginal arc costs  $\alpha \langle F'(x^*)x^* \rangle$ .

Next, consider an extension of the basic framework where  $\alpha$  varies across users. To be precise, the population is partitioned into segments, where segment  $i \in I$  is endowed with its own VOT parameter  $\alpha_i$ . We associate with class i a probability  $h(\alpha_i)$ , a demand vector  $h(\alpha_i)b$  and the feasible sets  $\Omega_x(\alpha_i) = h(\alpha_i)\Omega_x$  and  $\Omega_v(\alpha_i) = h(\alpha_i)\Omega_v$ . Arc and path flow vectors associated with a class indexed by  $i \in I$  are denoted by  $x(\alpha_i)$  and  $v(\alpha_i)$ , respectively. The vector of total arc flows is defined as  $X = \sum_{i \in I} x(\alpha_i)$  and the vector of total path flows as  $V = \sum_{i \in I} v(\alpha_i)$ . In this framework, we let  $x = (x(\alpha_i))_i$  denote the concatenation of class link flow vectors, which must belong to the feasible set  $\Omega_x = \prod_{i \in I} \Omega_x(\alpha_i)$ . The path flow vector and its feasible set are defined in a similar manner. We say that a multiclass equilibrium is reached when *each* class is in equilibrium, i.e.,

$$\langle \alpha_i F(X) + T, x(\alpha_i) - y(\alpha_i) \rangle \le 0 \quad \forall y(\alpha_i) \in \Omega_x(\alpha_i), \ \forall i \in I.$$
 (3)

The latter variational inequality can be aggregated to yield the equivalent

$$\sum_{i \in I} \langle \alpha_i F(X) + T, x - y \rangle \le 0 \quad \forall y \in \Omega_x.$$
(4)

In this context, and for the sake of notational consistency, we denote by  $X^*$  the vector of system-optimal total link flows. Yang

and Huang [2] have shown that the dual vector associated with the capacity constraints of the linear program

$$\min_{\mathbf{x}\in\Omega_{\mathbf{x}}} \qquad \sum_{i\in I} \alpha_i \langle F(\mathbf{X}^*), \mathbf{x}(\alpha_i) \rangle \tag{5}$$

subject to 
$$\sum_{i \in I} x(\alpha_i) \le X^*$$
 (6)

induces X\*. Actually, this technique can be used to induce any flow pattern that satisfies some mild conditions, not only system-optimal ones.

In this work, we are concerned with the situation where the mass vector  $(h(\alpha_i))_i$  is replaced by a measurable, almost everywhere positive and square-integrable function having compact support  $[0, \alpha_{\max}]$ , that we still denote as *h*. The variables of the problem are then flow densities  $x(\alpha) \in \Omega_x(\alpha)$  and  $v(\alpha) \in \Omega_v(\alpha)$ , with

$$\Omega_{v}(\alpha) = \left\{ v(\alpha) \ge 0 : \sum_{p \in \mathcal{K}} v_{p}(\alpha) = h(\alpha)b, \text{ a.e. } \alpha \in [0, \alpha_{\max}] \right\} (7)$$
$$\Omega_{x}(\alpha) = \Delta \Omega_{v}(\alpha)$$

$$= \{ x(\alpha) : \exists v(\alpha) \in \Omega(\alpha) : x(\alpha) = \Delta v(\alpha) \}.$$
 (8)

The parallel with the finite-dimensional case is straightforward, with class flow vectors being replaced by square-integrable flow densities, and the feasible sets being members of the class of square-integrable vector functions having compact support  $[0, \alpha_{max}]$ . Total flow vectors are then obtained by integrating flow densities over  $[0, \alpha_{max}]$  and an equilibrium *x* is reached when, almost everywhere on  $[0, \alpha_{max}]$ , there holds

$$\langle \alpha F(X) + T, x(\alpha) - y(\alpha) \rangle \leq 0 \quad \forall y(\alpha) \in \Omega_x(\alpha),$$
  
a.e.  $\alpha \in [0, \alpha_{\max}]$  (9)

which, through aggregation (integration), is equivalent to the (multiclass) variational inequality

MCVI: 
$$\langle \alpha F(X) + T, x - y \rangle_2 \le 0 \quad \forall y \in \Omega_x,$$
 (10)

where  $\langle \cdot, \, \cdot \rangle_2$  denotes the inner product of vector functions:

$$\langle f,g\rangle_2 = \int_0^{\alpha_{\max}} \langle f(\alpha),g(\alpha)\rangle \mathrm{d}\alpha.$$

The corresponding path flow formulation of equilibrium is then

$$\langle \Delta^{\mathfrak{l}}(\alpha F(\Delta V) + T), v - w \rangle \le 0 \quad \forall w \in \Omega_{v}.$$
<sup>(11)</sup>

The above problem has been investigated from a theoretical and computational point of view, and actually lends itself to a variety of finite-dimensional formulations, for which the reader is referred to Leurent [7], Marcotte [8], and Marcotte and Zhu [9].

To derive link tolls that induce the system-optimal flow pattern  $X^*$ , it is tempting to mimic the technique of Yang and Huang by considering the conic linear program

LPCON: 
$$\min_{x \in \Omega_x} \langle \alpha F(X^*), x \rangle_2$$
 (12)

subject to 
$$\int_0^{\alpha_{\max}} x(\alpha) d\alpha \le X^*.$$
 (13)

Unfortunately, no known constraint qualification applies to the above problem, that would allow us to derive optimal tolls from the dual vector associated with the capacity constraint (see Bonnans and Shapiro [10] or Shapiro [11]).

Alternatively, one could consider one of the finite-dimensional formulations proposed by Marcotte [8], such as the one based on the threshold values ("breakpoints") of the VOT parameter  $\alpha$  that define intervals where path flows are invariant. More precisely, consider the set of vertices  $x^e$ , indexed by *E* of the polyhedron  $\Omega_x$ 

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and a vector  $\hat{\alpha} = (\alpha_e)_{e \in E}$  of VOT values. Under mild conditions, it can be shown that the equilibrium is related to a vector  $\hat{\alpha}$  in the following manner:  $x(\alpha) = h(\alpha)x^e$  if and only if  $\alpha \in [\alpha_{e-1}, \alpha_e]$ , with  $\alpha_0 = 0$  and  $\alpha_{|E|} = \alpha_{\max}$ . Otherwise stated, the segment  $[0, \alpha_{\max}]$  can be partitioned into intervals such that, within each interval, origin–destination flows are assigned to a unique path.

This relationship between breakpoint vectors and flow densities allows for a finite-dimensional reformulation of the equilibrium problem. On the basis of this, we can write down an equivalent of LPCON, where the capacity constraints are expressed as

$$\Gamma\left(\int_{\alpha_{i-1}}^{\alpha_i}h(\alpha)\mathrm{d}\alpha\right)_{i=1,\ldots,|E|}\leq X^*,$$

and  $\Gamma$  denotes the arc–vertex incidence matrix of the multicommodity network. Unfortunately, and quite surprisingly, unless the density function *h* is uniform (constant over the interval [0,  $\alpha_{max}$ ]), no constraint qualification could be derived for these nonconvex constraints which would have ensured the existence of dual variables and hence system-optimal tolls.

## 3. An implicit formulation of the toll problem

In this section, we introduce a finite-dimensional and implicit formulation of the toll problem upon which a simple proof of the existence of system-optimal tolls can be based, and which paves the way to efficient algorithms for their computation. It relies on the simple fact that, in order to verify whether a flow density vector *x* satisfies the variational inequality MCVI, only finite quantities are required. Indeed, the left-hand side of (10) can be written as

$$\langle \alpha F(X) + T, x - y \rangle_2 = \left\langle F(X), \int_0^{\alpha_{\max}} \alpha(x(\alpha) - y(\alpha)) d\alpha \right\rangle$$
  
 
$$+ \left\langle T, \int_0^{\alpha_{\max}} (x(\alpha) - y(\alpha)) d\alpha \right\rangle$$
  
 
$$= \langle T, X - Y \rangle + \langle F(X), \bar{X} - \bar{Y} \rangle,$$

where X and Y represent, respectively, the *average* flow vectors associated with the flow density vectors x and y. It follows that the sole attributes of a density vector x required to check that it satisfies the conditions of a multiclass equilibrium are its total and average flows (moments of order 0 and 1). This prompts us to define the set of joint total and average flows

$$\bar{\Omega} = \left\{ (X, \bar{X}) : \exists x \in \Omega_x : X = \int_0^{\alpha_{\max}} x(\alpha) d\alpha, \\ \bar{X} = \int_0^{\alpha_{\max}} \alpha x(\alpha) d\alpha \right\}$$

for which there exist feasible flow densities. The situation parallels that of arc formulation of the standard traffic assignment problems, where the set of feasible arc flows must be compatible with flows disaggregated by origins and destinations. In contrast with the latter case, one must be careful that this set is actually a polyhedron. This is taken care of in the following lemma.

**Lemma 3.1.** Assume that the function h is measurable over the interval  $[0, \alpha_{max}]$ . Then the set  $\overline{\Omega}$  is a polyhedron.

**Proof.** The set  $\overline{\Omega}$  is trivially convex and bounded, so it remains to show that the number of its extreme points and extreme rays is finite. The argument is based on the relationship between extreme points of  $\overline{\Omega}$  and the set of vertices of  $\Omega_x$ . Note that a vertex of  $\Omega_x$  is characterized by an assignment of flows to unique paths, one for each origin–destination pair. Now, let  $(X, \overline{X}) \in \overline{\Omega}$  and assume that, for every OD pair, the total flow *X* corresponding to a vertex of  $\Omega$  is *not* assigned to a single path for each OD pair. It follows that there must exist a subset *S* of  $[0, \alpha_{max}]$  with positive measure over which the density function *h* also has positive measure, and for which the

flow density is not concentrated on *X*. Consequently, there exist total flows  $X^1$  and  $X^2$  such that

$$x(\alpha) = h(\alpha) \frac{X^1 + X^2}{2}$$
  
for  $\alpha \in S$ . Next we introduce, for  $i = 1, 2$ , the functions

$$x^{i}(\alpha) = \begin{cases} h(\alpha)X^{i} & \text{if } \alpha \in S \\ x(\alpha) & \text{else,} \end{cases}$$

the corresponding total flow vectors  $X^i = \int_0^{\alpha_{max}} x^i(\alpha) d\alpha$ , and finally the average flow vectors  $\bar{X}^i = \int_0^{\alpha_{max}} \alpha x^i(\alpha) d\alpha$ , for i = 1, 2. From this construction, we have that  $X = 1/2(X^1 + X^2)$  and  $\bar{X} = 1/2(\bar{X}^1 + \bar{X}^2)$ , with  $X^1 \neq X^2$ . Since  $(X^1, \bar{X}^1)$  and  $(X^2, \bar{X}^2)$  belong to  $\bar{\Omega}$  by construction,  $(X, \bar{X}) = 1/2((X^1, \bar{X}^1) + (X^2, \bar{X}^2))$ , and  $(X, \bar{X})$  is not extremal, as claimed. This is the desired contradiction that concludes the proof. The proof concerning extreme rays involves cycles in the original graph and is similar. Actually, cyclic flows are of little interest, as one may restrict one's attention to the convex envelope of extremal flows.  $\Box$ 

We are now in position to provide a simple proof of the existence of "optimal" tolls.

**Theorem 3.1.** Assume that the function h is measurable over the interval  $[0, \alpha_{max}]$ . Then there exists a system-optimal toll vector for the multiclass equilibrium problem.

Proof. Consider the mathematical program

$$\overline{\text{LP}}: \min_{\substack{(X,\bar{X})\in\bar{\Omega}}} \langle F(X^*), \bar{X} \rangle$$
  
subject to  $X \leq X^*.$ 

Since  $\Omega$  is a polyhedron by Lemma 3.1, the above program is linear. Since  $X^*$  is the solution that achieves the minimum system cost, the fact that the cost function F is positive implies that the total flow vector *cannot* be less, componentwise, than  $X^*$ . It follows that  $X^*$  is the only feasible, and therefore optimal, partial solution of  $\overline{LP}$ . From strong duality, there exists an optimal multiplier  $\lambda^*$  such that  $X^*$  is solution of the Lagrangean dual:

$$\min_{(X,\bar{X})\in\bar{\Omega}} \langle F(X^*),\bar{X}\rangle + \langle \lambda^*, X - X^* \rangle$$

or, equivalently, of the variational inequality

$$\langle F(X^*), \bar{X}^* - \bar{Y} \rangle + \langle \lambda^*, X^* - Y \rangle \leq 0 \quad \forall (Y, \bar{Y}) \in \bar{\Omega}.$$

This last inequality precisely states that the multiclass equilibrium  $x^*$  implicit in the definition of  $(X^*, \bar{X}^*)$  is compatible with the system-optimal total flow vector  $X^*$ , provided that the toll vector is set to  $\lambda^*$ , the optimal dual vector of  $\overline{LP}$ .  $\Box$ 

**Note:** As occurs in the discrete case, the linear program LP possesses a structure that can be exploited for its numerical resolution. This will actually be subsumed by the analysis of the next section.

#### 4. Optimization over the set of optimal tolls

The main result of the previous section is a constructive proof that the set of optimal tolls is nonempty. This set is actually a polyhedron upon which one may optimize a secondary objective. Motivated by the analysis of Hearn et al. [12], we address the problem that consists in finding a positive optimal toll vector whose sum is minimal, and propose an efficient algorithm for its determination. Throughout this section, we assume that the implicitly defined polyhedron  $\overline{\Omega}$  assumes the form

$$\bar{\Omega} = \{ (X, \bar{X}) \ge 0 : AX + \bar{A}\bar{X} = \bar{b} \}.$$

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Then, a toll vector *T* is system optimal if and only if  $(X^*, \overline{X}^*)$  is an optimal solution of the linear program

$$\min_{X,\bar{X}} \quad \langle T,X\rangle + \langle F(X^*),\bar{X}\rangle$$

subject to  $AX + \overline{A}\overline{X} = \overline{b}$ 

 $X, \overline{X} > 0,$ 

whose optimality conditions (dual feasibility and complementary slackness) are expressed as

 $\mu A \leq T$  $\langle T - \mu A, X \rangle = 0$  $\mu \bar{A} \leq F(X^*) \quad \langle F(X^*) - \mu \bar{A}, \bar{X} \rangle = 0.$ 

Replacing *X* and  $\overline{X}$  by  $X^*$  and  $\overline{X}^*$  in the above, we have that *T* is optimal if and only if there exists a dual vector  $\mu$  such that

$$\begin{split} \mu A &\leq T \qquad \langle T - \mu A, X^* \rangle = 0 \\ \mu \bar{A} &\leq F(X^*) \qquad \langle F(X^*) - \mu \bar{A}, \bar{X}^* \rangle = 0 \end{split}$$

A minimum-optimal toll can then be obtained by solving the linear program

MINTOLL: 
$$\min_{\substack{T,\mu\\ \text{subject to}}} \langle T, X^* \rangle$$
$$\max_{\substack{T,\mu\\ \mu\bar{A} \leq T\\ \mu\bar{A} \leq F(X^*)\\ \langle T - \mu A, X^* \rangle + \langle F(X^*) - \mu \bar{A}, \bar{X}^* \rangle = 0.$$

Using an exact penalty scheme for the complementarity constraint (this is always valid for finite-dimensional linear programs) and dividing the objective by the penalty constant M yields

$$\min_{T,\mu} \frac{1}{M} \langle T, X^* \rangle + \langle T - \mu A, X^* \rangle + \langle F(X^*) - \mu \bar{A}, \bar{X}^* \rangle$$
  
subject to  $\mu A - T \le 0$   
 $\mu \bar{A} \le F(X^*),$ 

whose linear programming dual is

$$\min_{\substack{(X,\bar{X})\geq 0}} \langle F(X^*), \bar{X} \rangle$$
subject to  $AX + \bar{A}\bar{X} = AX^* + \bar{A}\bar{X}^*$ 
 $X \leq (1 + 1/M)X^*$ 
or, since  $AX^* + \bar{A}\bar{X}^* = \bar{b}$ ,

 $\langle F(X^*), \bar{X} \rangle$ 

min  $(X,\bar{X}){\in}\bar{\Omega}$ 

subject to  $X \leq (1 + 1/M)X^*$ .

Using a convex and differentiable penalty function for the bound constraint, the latter mathematical program reduces to a convex multiclass problem that can be easily and efficiently solved by the techniques described in Marcotte and Zhu [9].

## 5. Conclusion

While we did not address some important modeling issues, such as equity (users of efficient paths might be penalized by the optimal toll scheme; see Lawphongpanich [13]) or cost non-separability (in some practical situations, tolls are set on paths rather than arcs of the network; see Agdeppa et al. [14], Maruyamaa and Sumalee [15]), we believe that our approach could be useful in addressing these situations as well.

From the theoretical point of view, a by-product of our analysis is a duality result for a class of conic linear programs. Note also that the Lebesgue measure that we adopted could be replaced by the Lebesgue-Stieltjes measure, thus providing a framework that unifies both the discrete and continuous cases.

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