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Capacitated transit assignment with loading priorities

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Abstract. In a transit network involving vehicles with rigid capacities, we advocate the use of strategies for describing consumer behavior. At each boarding node, a user sorts the transit lines in decreasing order of preference, and boards the first vehicle in this list whose residual capacity is nonzero. Since a user's position in the queue varies from day to day, the delay experienced is stochastic. This leads to an equilibrium problem where, at a solution, users are assigned to strategies that minimize their expected delay. This situation is formulated as a variational inequality, whose cost mapping is discontinuous and strongly asymmetric, due to the priority of current passengers over incoming users. We prove that the solution set is nonempty and provide numerical results obtained by an efficient solution algorithm.

Key words. transit networks – equilibrium assignment – strategy – hyperpath – capacities – priorities – variational inequalities

1. Introduction

A transit network consists of a set of lines and stops where passengers board and alight. At a 'multistop', the choice of a vehicle is contingent on its attractiveness and available capacity. In such situations, which involve randomness due to the arrival processes of both customers and vehicles, user behavior is well captured by the concept of a 'travel strategy'. Travel strategies were initially introduced to address the problem of assigning passengers to common transit lines. Roughly speaking, a strategy minimizes the expected transit delay (queueing and travel time) by specifying a set of attractive lines. While a strategy is a deterministic object, the actual line travelled, which depends on the first incoming vehicle in the attractive set, is stochastic. In early studies (Dial [8], LeClercq [15]), the concept of a strategy is restricted to common transit lines with identical features. It was then extended to transit lines with different travel delays by Chriqui and Robillard [4], and to the general case of diverging lines by Spiess [24] and Gendreau [10]. The correspondence between strategies and *hyperpaths* in some suitably defined hypergraph is due to Nguyen and Pallottino [19]. In a similar vein, and with the aim of solving large scale problems, De Cea and Fernández [6] proposed a solution algorithm based on a transit route representation of the network.

The abovementioned studies do not take into account vehicle capacities, which could prevent users from boarding a vehicle belonging to an attractive line. Although

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congestion has been considered as an externality (Gendreau [10], Wu et al. [27], De Cea and Fernández [7]), or as having the effect of reducing frequency (Cominetti and Correa [5], Cepeda [3]), rigid capacities have not yet been considered explicitly in transit equilibrium models. While it can be argued that the integration of upper bound constraints within static, steady-state user-equilibrium models is sensible (see Hearn [12] or Patriks-son [22]), this approach does not apply to fixed-schedule transit networks, where system performance is measured by *passenger volumes* (numbers), not *flow rates*. Alternatively, in networks with fixed schedules, it has been proposed by Nguyen et al. [21] to penalize the capacity constraints for no-priority flows, and to solve the resulting asymmetric variational inequality. However, this approach has several pitfalls, among which the nature or even existence of a limiting sequence of equilibrium paths.

We argue that the very definition of a Wardrop equilibrium in networks with rigid capacities and/or fixed schedules calls for a different notion of strategy, where the ordering of attractive lines matters. More precisely, we adopt the point of view of Marcotte and Nguyen [17], and extend it to priority networks. At each node, users travel along the first unsaturated arc within an ordered set of outgoing arcs; the probability of accessing an arc is proportional to its (residual) capacity and inversely proportional to the number of users that want to access that arc. An equilibrium is then characterized as the solution of a variational inequality, whose nonlinear cost mapping is induced by the travel delays and capacities of the network's links.

The aim of this paper is to extend the work of Marcotte and Nguyen [17] to capacitated networks with priority loading and to provide a rigorous treatment of equilibrium in that context. The paper is organized as follows. We start by giving a simple example that illustrates our strategic approach. Then, we introduce the nontrivial process that maps strategic flows to priority-complient path flows. Next, we prove the nonvacuity of the solution set. Finally we describe a solution algorithm, together with numerical results obtained through its implementation.

2. Strategic equilibrium in priority networks

To gain some insight into the distinction between the priority and no-priority models in networks involving arcs with rigid capacities, let us consider the graph illustrated in Figure 1, where we associate with each arc a delay (cost) and a capacity (parenthesized). The subpaths 1-3 and 2-3-5 correspond to two bus lines A and B with rigid capacities equal to 10, and the remaining arcs can be interpreted as walking arcs, i.e., arcs with infinite capacities. Demand for the sole origin-destination pair (1,5) is set to 15.

In the strategic approach of Marcotte and Nguyen [17], user behavior is dictated by a strategy that specifies, at each node of the network, an ordered set of adjacent nodes. In our example, the only two strategies of interest¹ are shown in Table 1.

First, we search for an equilibrium assignment of commuters to strategies when priorities are not taken into account. At equilibrium, the (expected) cost of active strategies must be identical, and less than or equal to the cost of inactive strategies. If u_a denotes

¹ We discard strategies that do not contain an 'escape route', as they might not be able to carry all flow. We also discard irrational strategies that will clearly carry no flow at equilibrium, such as ([2, 3], [3, 5], [5, 4], [5], []). See [18] for a detailed discussion.



Fig. 1. A network example

Table 1. The two strategies of interest for the network of Figure 1

nodes :	1	2	3	4	5
s_1 :	[2]	[3,5]	[5,4]	[5]	[]
s_2 :	[3,2]	[3,5]	[5,4]	[5]	[]

the capacity of arc *a*, the probability π of accessing arc *a* for each of *x* competing users is given by $\pi = \min\{1, u_a/x\}$. Let x_i denote the number of passengers associated with strategy s_i . In our example, the respective probabilities π_{13} , π_{23} and π_{35} of accessing arcs (1, 3), (2, 3) and (3, 5) are:

$$\begin{aligned} \pi_{13} &= \min\left\{1, \frac{\text{capacity of arc }(1, 2)}{\text{incoming flow}}\right\} = \min\left\{1, \frac{10}{x_2}\right\},\\ \pi_{23} &= \min\left\{1, \frac{\text{capacity of arc}(2, 3)}{\text{incoming flow}}\right\} = \min\left\{1, \frac{10}{x_1 + (1 - \pi_{13})x_2}\right\},\\ \pi_{35} &= \min\left\{1, \frac{\text{capacity of arc}(3, 5)}{\text{incoming flow}}\right\} = \min\left\{1, \frac{10}{\pi_{23}x_1 + (\pi_{13} + (1 - \pi_{13})\pi_{23})x_2}\right\}\end{aligned}$$

These access probabilities allow us to compute the expected costs C_1 and C_2 of strategies s_1 and s_2 :

$$C_1 = 150 + \pi_{23}(110 + \pi_{35}(120) + (1 - \pi_{35})(200 + 400)) + (1 - \pi_{23})(800),$$

$$C_2 = \pi_{13}(100 + \pi_{35}(120) + (1 - \pi_{35})(200 + 400)) + (1 - \pi_{13})C_1.$$

It is not difficult to check on this small example that the unique equilibrium vector corresponds to $x^* = (x_1^*, x_2^*) = (0, 15)$. The expected cost of strategy s_2 is equal to $433\frac{1}{3}$, which is less than 540, the expected cost of the inactive strategy s_1 , hence the equilibrium conditions are satisfied.

The situation differs when priorities, which frequently arise in practice, are taken into account. It is clear from the network topology that users boarding a bus of line B at node 2 and travelling on arc (2, 3) can always access arc (3, 5) by staying on board, and have therefore a natural priority over competing users. We are interested in finding a strategic equilibrium that explicitly accounts for this natural priority in loading passengers to segments of transit lines.

At node 1, the passengers using strategy s_1 access arc (1,2) with probability 1 since this arc has infinite capacity. On the other hand, passengers using strategy s_2 access their preferred arc (1,3) with probability $\pi_{13} = \min\{1, \frac{10}{x_2}\}$. On the average, $\pi_{13}x_2$ passengers using s_2 access arc (1,3), and the remaining $(1 - \pi_{13})x_2$ users resort to arc (1,2). The total flow that has reached node 2 at this stage is $x_1 + (1 - \pi_{13})x_2$. The access probability to arc (2, 3) under strategies s_1 and s_2 is $\pi_{23} = \min\{1, \frac{10}{x_1 + (1 - \pi_{13})x_2}\}$. At node 3, $\pi_{23}[x_1 + (1 - \pi_{13})x_2]$ flow units coming from arc (2, 3) and $\pi_{13}x_2$ from arc (1, 3) compete for the residual capacity (10) of arc (3, 5). We enforce the priority rule at node 3 by first loading users of the first class, whose access probability $\pi_{35}^{(1)}$ is clearly equal to one². Next, we update the residual capacity \overline{u}_{35} of arc (3,5)

$$\overline{u}_{35} = 10 - \pi_{23}[x_1 + (1 - \pi_{13})x_2]$$

and obtain the access probability $\pi_{35}^{(2)}$ for the second class users:

$$\pi_{35}^{(2)} = \begin{cases} 0 & \text{if } \overline{u}_{35} = 0\\ \min\{1, \frac{\overline{u}_{35}}{\pi_{13}x_2}\} & \text{if } \overline{u}_{35} > 0. \end{cases}$$

Based on these access probabilities, we derive the cost function

$$C_1 = 150 + \pi_{23}(110 + 120) + (1 - \pi_{23}) \cdot 800,$$

$$C_2 = \pi_{13}[100 + \pi_{35}^{(2)} \cdot 120 + (1 - \pi_{35}^{(2)}) \cdot 600] + (1 - \pi_{13})C_1.$$

Solving for x in the above system, we find $x^* = (x_1^*, x_2^*) = (15, 0)$ which is at odds with the solution obtained in the no-priority case. Note that the equilibrium travel time has, surprisingly, increased from $433\frac{1}{3}$ to 570.

This small example illustrates the difference between the two strategic models. The remaining sections of this paper address the model with loading priorities. The challenge is threefold:

- Does an equilibrium exist?
- How to load strategic flows compliant with priorities?
- How to compute a best response with respect to a current strategic flow assignment? This computation of a 'best' strategy plays the role of a shortest path procedure in static traffic assignment models.

 $^{^2}$ There is no priority at the initial arc of a transit line, while the access for the first class users is equal to one at all intermediate arcs.

3. The loading process

The loading process maps strategic flows into arc flows. In parallel, it generates the access probabilities that will be instrumental in computing the 'best' strategic response to given strategic flow conditions. We first give a list of the most frequently used notations; additional notations will be introduced as required.

G = (N, A) network with node set N and arc set A

 $L = \{L_1, L_2, \cdots, L_l\}$ set of transit lines

- d_{qr} demand from origin node q to destination node r
- \overline{u}_{jk} residual capacity of arc (j, k)
- E_i^s preference order associated with strategy s and node j
- z_{i}^{s} strategic flow at node j
- π_{jk}^{s} probability of accessing node k from node j using strategy s
- $C^{s}(x)$ (expected) cost of strategy s
- C(x) vector of strategic costs
 - W set of active strategies (working set)
 - j^{-} backward star at node j

By definition, a vector x^* of strategic flows is a **strategic equilibrium** if and only if it lies in the set X of demand-feasible vectors and satisfies the variational inequality VI(C, X):

$$\langle C(x^*), x^* - x \rangle \le 0 \quad \forall x \in X.$$
⁽¹⁾

Marcotte, Nguyen and Schoeb [18], showed that the computation of the strategic costs $C^s(x)$ requires a **loading mechanism** which simulates the assignment of strategic flow onto the arcs and paths of the network. In order to capture the priority effect of strategy s users that stay on board at some node j, we look for a predecessor node \overline{k} of j such that $(\overline{k}, j, E_j^s(1))$ belongs to some bus line L_i $(1 \le i \le l)$. If no such triplet exists, then z_j^s is a non-priority flow at node j. Indeed, the existence of a node \overline{k} (corresponding to the priority flow) induces the flows $z_j^{s1} = \pi_{\overline{k}j}^s z_{\overline{k}}^s$ and $z_j^{s2} = \sum_{k \in j^-, k \neq \overline{k}} \pi_{kj}^s z_k^s$.

We now focus on the loading process at some node j, where users are divided into two priority classes \overline{W}^1 and \overline{W}^2 :

$$\forall j \quad \overline{W}^i = \{s \in W : E_j^s \neq \emptyset, \ z_j^{si} > 0\} \subset \overline{W} = \{s \in W : E_j^s \neq \emptyset\}, \quad i = 1, 2$$

where all users of the first class \overline{W}^1 have priority over those of the second class \overline{W}^2 . Furthermore we assume that, at boarding nodes, users are uniformly distributed in two distinct queues, one for each class.

The loading mechanism is illustrated on the example of Figure 2. At node j, 40 units of flow compete for access to the successor nodes k_1 , k_2 and k_3 ; 5 units from strategy s_1 and 8 from strategy s_3 have priority over 5 units from s_1 , 10 from s_2 and 12 from s_3 . At



the start of the process, we load the priority class $\overline{W}^1 = \{s_1, s_3 : z_j^{s_11} = 5, z_j^{s_31} = 8\}$. The entire strategic flows $z_j^{s_11}$ and $z_j^{s_31}$ are assigned to their respective preferred node because the respective capacities are not violated. The residual capacities are computed, and loading halts for users of class \overline{W}^1 .

Next, we perform the loading of the no-priority class

$$\overline{W}^2 = \{s_1, s_2, s_3 : z_j^{s_1 2} = 5, z_j^{s_2 2} = 10, z_j^{s_3 2} = 12\}$$

Since the ratio 2/12 is less than the ratio 10/15, the first saturated arc will be (j, k_2) . At that 'instant', 5/6 (probability (2/12) times flow (5)) units of flow from s_1 and 10/6 (probability (2/12) times flow (10)) from s_2 have reached node k_1 , while 2 units from s_3 have reached k_2 . Node k_2 is removed from the preference sets and we update the residual capacities of outgoing arcs. Now, 135/6 units of flow (25/6 from s_1 , 50/6 from s_2 and 10 from s_3) compete for 45/6 (10-15/6) units of residual capacity on arc (j, k_1) . According to the assumption of uniform distribution in the queue, flows are assigned to their preferred node k_1 according to a 1 for 3 ratio, i.e., 25/18 units of flow from s_1 , 25/9 from s_2 and 10/3 from s_3 reach node k_1 . Finally, the flow left unassigned is moved to the uncapacitated arc (j, k_3) . The arc access probabilities are then computed by dividing the flow having accessed that arc by the total flow having reached node j. The entire process is summarized in Table 2.

In acyclic networks, the loading process can be performed in accordance with any topological ordering of the nodes.³ Node processing goes through two successive stages. First, determining the set of active strategies \overline{W} and classes \overline{W}^i (i = 1, 2). Next, flow assignment, once for each priority class.

Note that the class \overline{W}^i might include strategic flows with zero values. Since the access probabilities of a zero-flow strategy cannot be computed using its values, we

 $^{^3}$ If this assumption is not met, network loading can yet be achieved through a fixed point process. See [18] for details.

	Table	2. Loading results at	node j	
	arc:	(j, k_1)	(j, k_2)	(j, k_3)
iteration 1	residual capacity:	15	10	∞
	flow:	$5(s_1)$	$8(s_3)$	0
iteration 2	residual capacity:	10	2	∞
	flow:	$\frac{35}{6}(s_1) + \frac{10}{6}(s_2)$	$10(s_3)$	0
iteration 3	residual capacity:	45/6 $\frac{65}{5}(s_1) \pm \frac{40}{5}(s_2)$	0	∞
	now.	$\frac{1}{9}(s_1) + \frac{1}{9}(s_2) + \frac{10}{3}(s_3)$	$10(s_3)$	0
iteration 4	residual capacity:	$\begin{pmatrix} 0 \\ 65 \\ (a) \end{pmatrix} + \begin{pmatrix} 40 \\ (a) \end{pmatrix}$	0	∞ $25(\alpha) + 50(\alpha)$
	now:	$\frac{\frac{1}{9}(s_1) + \frac{1}{9}(s_2)}{+\frac{10}{3}(s_3)}$	$10(s_3)$	$\frac{1}{9}(s_1) + \frac{1}{9}(s_2) + \frac{20}{3}(s_3)$
access	probabilities: $\pi_{jk_1}^{s_1}$	$=\frac{13}{18}$ $\pi_{jk_1}^{s_2}=\frac{4}{9}$	$\pi_{jk_1}^{s_3} = \frac{1}{6}$	$\pi_{jk_2}^{s_3} = \frac{3}{6}$
	$\pi^{s_1}_{jk_3}$	$= \frac{5}{18}$ $\pi_{jk_3}^{s_2} = \frac{5}{9}$	$\pi_{jk_3}^{s_3} = \frac{2}{6}$	

base their derivation on the residual proportion ρ_j^s of strategic flow z_j^{si} at node *j*. At the origin node of strategy *s*, this proportion is set to 1. The residual strategic flow \overline{z}_j^{si} and the probabilities π_{jk}^s are then recursively computed according to the formula

$$\begin{split} \bar{z}_{j}^{si} &:= (1 - \beta^{i}) \bar{z}_{j}^{si} \\ \pi_{jk}^{s} &:= \begin{cases} \pi_{jk}^{s} + \beta^{i} \frac{\bar{z}_{j}^{si}}{z_{j}^{s}} & \text{if } z_{j}^{s} > 0 \\ \pi_{jk}^{s} + \beta^{i} \rho_{j}^{si} & \text{if } z_{j}^{s} = 0. \end{cases} \end{split}$$

where β^i is the residual proportion of z_j^{si} assigned to node k. Next, the residual proportion is updated according to the formula

$$\rho_j^{si} := (1 - \beta^i) \rho_j^{si}.$$

The loading process for class \overline{W}^i terminates whenever ρ_j^{si} or \overline{z}_j^{si} vanish. A formal description of the loading algorithm **PCAPLOAD**(*x*) associated with a strategic vector *x* can be found in Appendix A.

The running time of the algorithm is determined by the **while** loop (see Appendix A), which is performed at most twice, once for each priority class. At each step within the loop (with the possible exception of the last step), an arc becomes saturated. Therefore, if j^+ denotes the forward star at node j, these loops are executed at most $|j^+| + 1$ times for each node j. It follows that the loops are executed at most $\sum_{j \in N} (|j^+|+1) = |A|+|N|$

times and that the total running time of the loading algorithm is bounded from above by a polynomial function of the number of nodes and origin-destination pairs.

At this point, one might question the relevance of the degenerate situation where the demand d_k for node $k \in K$ and the residual capacity \overline{u}_{jk} are both zero. Indeed, this particular situation only occurs when the loading is halted for the first class \overline{W}^1 and, at that very moment, the strategic flow of the second class \overline{W}^2 which prefers to access the saturated arc (j, k) is null. In this case, the node k must to be removed from all residual preference sets before executing the loading for the second class \overline{W}^2 . Therefore, we have $\pi_{jk}^s = 0$ for each strategy s within the set W_k . The importance of this unlikely situation will become clear in the next section.

4. Existence of an equilibrium

Marcotte et al. [18] have constructed an example where the set of equilibria is disconnected, thus ruling out 'good' properties (monotonicity) of the cost mapping underlying the variational inequality (1). In this section, we prove an existence result in the absence of any nondegeneracy assumption. This is achieved in two steps, first by proving existence for the model with no loading priorities, and next by extending this result to the priority case.

4.1. No-priority model

In the model without priorities, Brouwer's fixed point theorem [2] and the continuity of the cost mapping C ensure the existence of at least one equilibrium solution. Continuity of C follows directly from the continuity of the access probabilities as functions of incoming flow. We give an outline of the proof that

$$\lim_{z\to\overline{z}}\pi^s_{jk}(z_j)=\pi^s_{jk}(\overline{z}_j)\quad\forall k\in E^s_j,\,\forall s\in\overline{W},\,\overline{z}_j=(\overline{z}^s_j)_{s\in\overline{W}},\,z_j=(z^s_j)_{s\in\overline{W}}.$$

Let $j^+ = \{k_1, k_2, \dots, k_l\}$ denote the successor nodes at node j. At each iteration of the loading process, at least one arc (j, k_i) $(1 \le i \le l)$ becomes saturated. Hence the number of iterations is at most l, and exactly l in the nondegenerate situation.

Let \overline{t} $(1 \le t \le l)$ be the number of iterations required for the loading of \overline{z} at node j. At the start of iteration n $(n = 1, 2, ..., \overline{t})$, we construct the set $\overline{K}^{(n)}$ of 'active strategies' first choices at node j. Next, we compute the proportion of assigned strategic flow, which is common to all strategies:

$$\overline{\beta}^{(n)} = \min_{k_i \in \overline{K}^{(n)}} \left\{ \frac{\overline{u}_i^{(n)}}{\overline{d}_i^{(n)}} \right\},\,$$

where $\overline{u}_i^{(n)}$ is the residual capacity of arc (j, k_i) and $\overline{d}_i^{(n)}$ is the residual demand at node k_i . Let $\overline{I}^{(n)}$ denote the set of 'minimum' indices:

$$\overline{I}^{(n)} = \left\{ i : i \in \arg\min_{k_i \in \overline{K}^{(n)}} \{ \frac{\overline{u}_i^{(n)}}{\overline{d}_i^{(n)}} \} \right\}.$$

The number of saturated arcs at iteration n for \overline{z} is equal to $|\overline{I}^{(n)}|$, and $\overline{d}_i^{(n)}$ is nonzero for all $i \in \overline{I}^{(n)}$, by construction. Consider now a sequence z of strategic vectors that converges to \overline{z}^4 and whose loading at node j requires t iterations. We associate with z the residual variables $d_i^{(n)}, u_i^{(n)}$ and $\beta^{(n)}$, and the index set $I^{(n)}$ ($1 \le n \le t$).

In the nondegenerate case there is, from continuity arguments, a bijection between the arcs being saturated at z and \overline{z} . Consider now the general degenerate case. Let $m_1(1 \le m_1 \le |\overline{I}^{(1)}|)$ denote the number of iterations required for loading the proportion of z-flow corresponding to the proportion of \overline{z} -flow loaded during the first iteration. At the end of these m_1 iterations, the number of saturated arcs for z and \overline{z} is the same. In general, let $m_1 + \ldots + m_n$ denote the number of iterations required for loading of the flow z at node j given that n iterations have been performed for the flow \overline{z} .

An outline of the proof follows. First, we show by induction on $n \in [1..\bar{t}]$ that the proportion of assigned flow $\beta^{(m_1+...+m_{n-1}+1)}$ converges towards $\overline{\beta}^{(n)}$, and the proportion $\beta^{(m_1+...+m_{n-1}+m)}$ converges towards 0, for all $m \in [2..m_n]$. This key result states that the *m* additional iterations executed for the vector *z* in order to saturate the same number of arcs as \overline{z} induce almost no flow. Next, by computing the access probabilities after each iteration $n \in [1..\bar{t}]$, we prove continuity of probabilities $\pi = \{\pi_{jk_i}^s(z)\}_{k_i \in j^+, s \in W}$. As an obvious consequence, we obtain that the solution set, in the no-priority case, is nonempty. A complete proof can be found in [11].

4.2. Priority model

In the priority model, discontinuities in the cost function may arise when the entry flow and the residual capacity are both zero. Indeed, the access probability to a transit line for no-priority users may be discontinuous in the degenerate case. To see this, let $(x_1, x_2) = (u, 0)$ and $(x_1^{\epsilon}, x_2^{\epsilon}) = (u - \alpha \epsilon, \beta \epsilon)$, where x_1 and x_1^{ϵ} (x_2 and x_2^{ϵ}) represent the priority (no-priority) flows and u the capacity of the preferred outgoing arc. The access probability for the no-priority users x_2 is $\pi = 0$, while the access probability for x_2^{ϵ} is $\pi^{\epsilon} = \min\{1, \alpha/\beta\}$, which can take any value between 0 and 1. As a direct consequence, the cost function C is discontinuous at (x_1, x_2) .

To prove existence of a solution, we will show that the function C is lower semicontinuous⁵ over the set of feasible strategic flows.

Theorem 1 (Ky Fan Inequality). Suppose that X a convex compact subset of \mathbb{R}^n and that Φ is a function from $X \times X$ to R satisfying

$$\forall y \in X, \ x \to \Phi(x, y) \ is \ lower \ semi-continuous$$
$$\forall x \in X, \ y \to \Phi(x, y) \ is \ concave.$$
(2)

Then there exists $x^* \in X$ such that

$$\sup_{y \in X} \Phi(x^*, y) \le \sup_{y \in X} \Phi(y, y).$$
(3)

⁴ For ease of notation, we do not use a sequence index.

⁵ A function $f : \mathbb{R}^n \to \mathbb{R}$ is lower semi-continuous at point \overline{x} , if $\forall \epsilon > 0 \exists \eta > 0$ such that $||x - \overline{x}|| < \eta \Longrightarrow f(\overline{x}) - f(x) \le \epsilon$, which is equivalent to the inequality $f(\overline{x}) \le \liminf f(x)$.

Consider the vector of probabilities $\pi = {\pi_{jk}^{s}(z_{j}^{s})}_{j \in N, k \in E_{j}^{s}, s \in \overline{W}}$ induced by the loading procedure **PCAPLOAD**. The probabilities π_{jk}^{s} are computed in a manner very similar to the no-priority case except in the degenerate situation where the entry flow and the residual capacity are both zero. This special case corresponds to the triplet $(j, k, s) \in N \times E_{j}^{s} \times \overline{W}$ satisfying the conditions :

$$\overline{W}^1 \neq \emptyset, \ s \in \overline{W}^2, \ z_j^s = 0 \text{ and } \overline{u}_{jk} = 0.$$
 (4)

In this particular case, we have $\pi_{jk}^s(\tilde{z}_j) = 0$ (for a vector \tilde{z}_j defined by conditions (4)) and we obtain $\liminf_{z_j \to \tilde{z}_j} \tilde{\pi}_{jk}^s(z_j) = \tilde{\pi} \in [0, 1]$, implying that

$$\pi_{jk}^s(\tilde{z}_j) \le \liminf_{z_j \to \tilde{z}_j} \pi_{jk}^s(z_j).$$

Consequently, the function $z_j \to \pi_{ik}^s(z_j)$ is lower semi-continuous at point \tilde{z}_j .

On the other hand, we have, by results of the no-priority case, that for all $\overline{z}_j \neq \widetilde{z}_j$:

$$\lim_{z \to \overline{z}} \pi_{jk}^s(z_j^s) = \pi_{jk}^s(\overline{z}_j^s).$$

This shows that the function C^s , which depends continuously on the access probabilities $\pi_{ik}^s(z_j)$, is lower semi-continuous on the set X.

Theorem 2. There exists at least one solution satisfying the variational inequality VI(C, X).

Proof. Set $\Phi(x, y) = \langle C(x), x - y \rangle$. The function Φ is concave in y and clearly satisfies $\Phi(y, y) = 0$. We now show that $x \to \Phi(x, y)$ is lower semi-continuous for every value of y. Since C^s is lower semi-continuous we have:

$$\forall \epsilon' = \frac{\epsilon}{2||\overline{x} - y||} > 0 \ \exists \eta > 0 \ \text{such that} \ ||x - \overline{x}|| < \eta \Longrightarrow C^s(x) \ge C^s(\overline{x}) - \epsilon'$$
$$\iff \forall \epsilon' > 0 \ \exists \eta > 0 \ \text{such that} \ ||x - \overline{x}|| < \eta \Longrightarrow -C^s(x) \le -C^s(\overline{x}) + \epsilon'.$$

It follows that:

$$\begin{aligned} \langle -C(x), x - y \rangle &\leq \langle -C(\overline{x}), x - y \rangle + \epsilon' ||x - y|| \\ &\leq \langle -C(\overline{x}), \overline{x} - y \rangle + \langle -C(\overline{x}), x - \overline{x} \rangle + \epsilon' ||x - y|| \\ &\leq \langle -C(\overline{x}), \overline{x} - y \rangle + ||C(\overline{x})||||x - \overline{x}|| + \epsilon' ||x - \overline{x}|| \\ &+ \epsilon' ||\overline{x} - y|| \\ &\leq \langle -C(\overline{x}), \overline{x} - y \rangle + (\epsilon' + ||C(\overline{x})||)\eta + \frac{\epsilon}{2} \\ &\leq \langle -C(\overline{x}), \overline{x} - y \rangle + \epsilon \quad \text{if } \eta \leq \frac{\epsilon}{2(\epsilon' + ||C(\overline{x})||)}, \end{aligned}$$

where the last term is uniformly bounded, since $C^{s}(\overline{x})$ is bounded by $M = \sum_{(j,k)\in A^{s}} c_{jk}$

for every *s*. This shows that the function $x \to \Phi(x, y)$ is lower semi-continuous on the compact set *X*. Ky Fan's Inequality may then applied (Theorem 2). Thus, there exists $x^* \in X$ such that :

$$\forall y \in X, \quad \langle C(x^*), x^* - y \rangle \le 0.$$

5. Computing an optimal strategy

In order to check whether a given strategic vector is an equilibrium, one has to find the best response to a given strategic flow x, i.e., to solve the linear program

$$\min_{y \in X} \langle C(x), y \rangle. \tag{5}$$

It is also of tantamount importance to perform this task efficiently, as it plays a role similar to the shortest path procedure in static traffic assignment.

The construction of an optimal strategy s^* is based on the recursive process developed by Nguyen and Pallottino [20]. At node j, let $E_j^{s^*}$ be the optimal preference order at node j, which is obtained by sorting in increasing order the labels $c_{jk} + \omega_k^{s^*}$ (see Marcotte et al. [18]). The expected travel costs $\omega_i^{s^*}$ from node j to the destination r are then computed by scanning the nodes in reverse topological order, and applying the recursion

$$\omega_{j}^{s^{*}} = \begin{cases} \infty & \text{if } j > r \\ 0 & \text{if } j = r \\ \sum_{k \in E_{j}^{s^{*}}} \pi_{jk}^{s^{*}}(c_{jk} + \omega_{k}^{s^{*}}) \text{ if } j < r, \end{cases}$$
(6)

The resulting procedure resembles the loading process described in Section 3. However, in order that a strategy that carries zero flow be generated by the procedure, one must be able assign access probabilities to null flows. While this **micro-loading** phase (loading of virtual flow) has been implemented in the no-priority case, new challenges occur when priorities are taken into account. Indeed, since loading is performed in reverse topological order, one might be unaware of the priority status of a strategic flow at loading time. To make up for this, we analyze both situations:

- 1. The virtual (zero) flow $z_k^{s^*}$ has priority and the micro-loading is performed over the set $\overline{W}^1 \cup \{s^*\}$, yielding $\omega_{1i}^{s^*}$;
- 2. $z_k^{s^*}$ has no priority and the micro-loading is performed successively over the sets \overline{W}^1 and $\overline{W}^{(2)} \cup \{s^*\}$, yielding $\omega_{2j}^{s^*}$.

Next, we compute the costs $\omega_i^{s^*p}$ (p = 1, 2) in accordance with the dynamic programming equation:

$$\omega_{j}^{s^{*}p} = \sum_{k \in E_{i}^{s^{*}}} \pi_{jk}^{s^{*}}(c_{jk} + \omega_{k}^{s^{*}}), \text{ with } \omega_{k}^{s^{*}} = \begin{cases} \omega_{k}^{s^{*}1} \text{ if } (j,k, E_{k}^{s^{*}}(1)) \in L_{i} \ (1 \le i \le l) \\ \omega_{k}^{s^{*}2} \text{ otherwise.} \end{cases}$$

At the origin node q, there are no priorities, and the cost of the optimal strategy is equal to

 $\omega_{2q}^{s^*}$. We will describe the computation of an optimal strategy on the example of Figure 3, We will be computed to bus lines with respective capacities set to 10 and 20, and demand for the sole O-D pair (1,5) set to 40. The set of active strategies contains the sole strategy: $s_1 = ([3, 2], [3, 5], [5, 4], [5], [])$.



Fig. 3. Computing an optimal strategy in a priority network

Loading the strategic flow $x = (x_1) = (40)$ is straightforward. From node 1, 10 units reach node 3 and the remaining 30 reach node 2. At node 2, 20 units of flow travel on arc (2,3) and 10 travel on arc (2,5), since capacity of arc (2,3) is 20. In view of priorities, only the 20 units coming from node 2 may access node 5 using arc (3, 5). Finally we obtain the strategic flows $z_j^{s_11}$ and $z_j^{s_12}$ reaching the nodes of the network :

node
$$j: 1 2 3 4 5$$

 $z_j^{s_11}: - - 20 - -$
 $z_j^{s_12}: 40 30 10 20 40$

To compute an optimal strategic answer s^* , we start from the destination 5 and set $\omega_5^{s+1} = \omega_5^{s+2} = 0$ and $E_5^{s^*} = [$]. At node 4, the optimal preference order is [5] and $\omega_4^{s+1} = \omega_4^{s+2} = 40$. At node 3, arc (3, 5) constitutes an optimal choice and thus $E_3^{s^*} = [5, 4]$. The micro-loading of strategy s^* at node 3 over the set $\{s_1 : z_3^{s_1} = 20\} \cup \{s^*\}$ yields the access probabilities $\pi_{35}^{s^*} = 1$ and $\pi_{35}^{s^*} = 0$. Similarly, the micro-loading performed successively over the sets $\{s_1 : z_3^{s_1} = 20\}$ and $\{s_1 : z_3^{s_1} = 10\} \cup \{s^*\}$ generates the probabilities $\pi_{35}^{s^*} = 1$. By applying the dynamic programming equation at node 3, there comes:

$$\omega_3^{s^{*1}} = (1)(12 + \omega_5^{s^{*2}}) + (0)(20 + \omega_4^{s^{*2}}) = 12$$

$$\omega_3^{s^{*2}} = (0)(12 + \omega_5^{s^{*2}}) + (1)(20 + \omega_4^{s^{*2}}) = 60.$$

The optimal preference order at node 2 is clearly $E_2^{s^*} = [3, 5]$. The access probabilities $\pi_{23}^{s^*}$ et $\pi_{25}^{s^*}$ are 2/3 and 1/3 for the micro-loading performed on the set $\{s_1 : z_j^{s_1} = 30\} \cup \{s^*\}$. Next we obtain :

$$\omega_2^{s^{*1}} = (1)(10 + \omega_3^{s^{*1}}) + (0)(80 + \omega_5^{s^{*2}}) = 22$$
$$\omega_2^{s^{*2}} = (2/3)(10 + \omega_3^{s^{*1}}) + (1/3)(80 + \omega_5^{s^{*2}}) = 124/3$$

The process terminates at the origin 1 with a preference order $E_1^{s^*}$ that minimizes the expected cost

$$\pi_{12}^{s}(15+\omega_{2}^{s^{*}2})+\omega_{13}^{s}(30+\omega_{3}^{s^{*}2})=(169/3)\pi_{12}^{s}+(90)\pi_{13}^{s}.$$

Since arc (1, 2) has infinite capacity, we have $E_1^{s^*} = [2]$ and $\omega_1^{s^{*1}} = \omega_2^{s^{*2}} = 169/3$. Finally, we obtain the optimal strategy $s^* = ([2], [3, 5], [5, 4], [5], [])$, with expected cost 169/3. A summary of the construction of s^* is reported in Table 3.

A detailed description of the pseudocode that implements the above procedure and returns, for a given strategic flow x and origin-destination pair (q, r), an optimal strategy together with its expected cost, is given in Hamdouch [11].

6. The quest for a strategic equilibrium

Finding a strategic equilibrium amounts to solving the variational inequality VI(C, X), where *C* is the cost function induced by the loading process, and *X* is the set of all demand-feasible strategic flow vectors. The resolution of this variational inequality poses two main difficulties. The first difficulty, due to the exponential number of strategies, can be remedied by considering a limited set of active strategies. The second difficulty rests on the implicit nature of the cost function, which is neither differentiable, separable nor monotone.

In this section, we describe two classes of algorithms, based respectively on the linearization and projection strategies. In both cases, convergence is measured with respect to a nonnegative function whose minimum value (zero) is achieved over the set of equilibria. The algorithm is stopped whenever the variational inequality (1) is approximately satisfied. A natural measure of 'satisfaction' is provided by the **relative gap function**, defined as

$$g(x) = \frac{\max_{y \in X} \langle C(x), x - y \rangle}{\langle C(x), x \rangle}.$$
(7)

6.1. A linearization approach

In static, separable traffic assignment, the popularity of the linearization strategy (Frank and Wolfe [9]) is due to its efficient handling of network structures. Indeed, at each iteration, a descent direction is derived from the computation of shortest path trees (see Sheffi [23]). In our strategic model, we mimic this procedure by constructing extremal optimal strategies \overline{x}^k that solve the linear program:

$$\min_{y \in X} \langle C(x^k), y \rangle$$

A convex combination of the vectors x^k and \overline{x}^k leads to the next iterate:

$$x^{k+1} = (1 - \theta^k)x^k + \theta^k \overline{x}^k,$$

where $\theta_k \in [0, 1]$ is some suitable stepsize. The resulting procedure for finding an equilibrium, algorithm **PSTRATEQ1**, is based on the use of the harmonic sequence $\theta_k = 1/(k+1)$.

Table 3. Construction of an optimal strategy for the small example

node j	$E_j^{s^*}$	$\omega_j^{s^*1}$	$\omega_j^{s^*2}$
5	[]	0	0
4	[5]	40	40
3	[5,4]	12	60
2	[3,5]	22	$\frac{141}{3}$
1	[2]	$\frac{169}{3}$	$\frac{169}{3}$

The initial vector x^0 is obtained by running the optimal strategy procedure with respect to free-flow travel times. This yields an initial working set W^0 containing a single strategy for each O-D pair, i.e., the sets of first choices correspond to shortest O-D paths.

In order to limit the size of the working set, an optimal strategy is included within the working set *W* only if its cost is significantly less than that of the current best strategy $\tilde{s} \in \arg\min_{s \in W_{qr}} C^s$, i.e,.

$$C^{s^*} + \epsilon_1 < C^{\tilde{s}}$$

for some positive quantity ϵ_1 . Also, strategies whose flow is less than some threshold value ϵ_2 are dropped from W.

A drawback of **PSTRATEQ1** is that a the stepsize is common to all origin destination pairs. This choice may have an adverse effect on convergence towards equilibrium flows for those O-D pairs that are 'almost' in equilibrium, and thus slow down overall convergence. To fix this problem algorithm **PSTRATEQ2** uses different stepsizes for each O-D pair, according to its 'equilibrium' status, i.e.,

$$\theta_k^s = 1 - \frac{C^{s_{qr}^*}}{C^s}$$

The above stepsize θ_k^s constitutes a disequilibrium measure for users travelling from q to r.

6.2. Projection algorithms

A second class of algorithms is based on the projection operator

$$p_{\alpha}(x) = \operatorname{proj}_X(x - \alpha C(x))$$

where α is a positive scalar and $p_{\alpha}(x)$ represents the Euclidean projection of the vector $x - \alpha C(x)$ onto the convex set X. A vector x^* is solution of the variational inequality VI(C, X) if and only if it is a solution of the fixed point problem :

$$x^* \in X^* \iff x^* = p_{\alpha}(x^*).$$

In our strategic model, the set *X* is product of simplices characterized by demand feasibility and flow nonnegativity constraints.

The simplest projection algorithm is obtained from the relationship $x^{k+1} = p_{\alpha}(x^k)$. The main drawback of this scheme is that theoretical convergence is based on a strong monotonicity assumption for the cost function *C*. Several variants of the base method, which converge under weaker conditions, have been proposed. One of them, due to Konnov [13], only requires *C* to be pseudomonotone⁶. Konnov's algorithm uses two projections and one convex combination:

$$p^{k} = \operatorname{proj}_{X}(x^{k} - \lambda C(x^{k})),$$

$$y^{k} = (1 - \theta)x^{k} + \theta p^{k},$$

$$x^{k+1} = \operatorname{proj}_{X}(x^{k} - \alpha C(y^{k})),$$

where $\alpha, \lambda \in \mathbb{R}^+$ and $\theta \in (0, 1)$.

Note that Konnov's method reduces to the extragradient method of Korpelevitch [14], if one sets $\theta = 1$ and $\lambda = \alpha$. While the pseudomonotonicity assumption might fail in our case, we will see that Konnov's method yields improved convergence in practice.

7. Numerical results

In this section we present numerical results on two network topologies and in assess the monotonicity of the cost function.

7.1. Numerical tests on a small network

Consider the network of Figure 4, inspired from the example in Marcotte and Nguyen [17]. In this network, subpaths 1 - 4 - 6 and 2 - 3 - 4 are served by two bus lines A and B with equal capacity of ten units of flow. The four strategies of interest are

node :	1	2	3	4	5	6
<i>s</i> ₁ :	[4]	[]	[]	[6]	[]	[]
s_2 :	[3]	[]	[4,6]	[6,5]	[6]	[]
<i>s</i> ₃ :	[]	[3,6]	[4]	[6,5]	[6]	[]
<i>s</i> ₄ :	[]	[6]	[]	[]	[]	[]

First, note that the respective costs of strategies s_1 (670) and s_4 (550) provide upper bounds on the equilibrium costs for each O-D pair. Next, let π_{23} denote the probability of accessing line *B* at node 2 for the users adopting strategy s_3 , π_{34} the probability of accessing line *B* at node 3 for the s_2 users, and π_{34} the probability of accessing line *A* at node 4 for the users using either strategy s_2 or s_3 . The access probabilities are given by the expressions

$$\pi_{23} = \min\{1, \frac{10}{x_3}\}$$

$$\pi_{34} = \begin{cases} 0 & \text{if } x_3 \ge 10, \\ \min\{1, \frac{10-x_3}{x_2}\} & \text{otherwise.} \end{cases}$$

$$\pi_{46} = \begin{cases} 0 & \text{if } x_1 = 10, \\ \frac{10-x_1}{x_2+x_3} & \text{otherwise.} \end{cases}$$

⁶ The cost function *C* is pseudomonotone if $(C(x), x - y) \le 0 \Longrightarrow (C(y), x - y) \quad \forall x, y \in X.$



Fig. 4. A small test network

Assuming that all four strategies are used at equilibrium, the cost function C takes the form

$$C_{1}(x) = 670$$

$$C_{2}(x) = 150 + \frac{10 - x_{3}}{x_{2}} [110 + \frac{10 - x_{1}}{x_{2} + x_{3}} 120 + \frac{x_{3}}{x_{2} + x_{3}} (200 + 400)] + \frac{x_{2} + x_{3} - 10}{x_{2}} 500$$

$$C_{3}(x) = 80 + 110 + \frac{10 - x_{1}}{x_{2} + x_{3}} 120 + \frac{x_{3}}{x_{2} + x_{3}} (200 + 400)$$

$$C_{4}(x) = 550.$$

Setting $C_1(x) = C_2(x)$ and $C_3(x) = C_4(x)$ and solving for x, we find:

$$x^* \approx (5, 5, 5.24, 6.76).$$

In this example, the equilibrium is *not* unique. Indeed, the vector (10, 0, 0, 12) is another equilibrium solution where only two strategies, s_1 and s_4 , are active. With respect to this strategic flow assignment, no alternative strategy has a lower cost, given that the residual capacity on arc (4, 6) is zero. Now, if demand for the pair (1,6) is increased to $10 + \epsilon$,

the additional flow is pushed onto path 1-3-4-5-6 whose cost is equal to 860 > 670, and the equilibration process will lead to an equilibrium that will be a perturbation of the first one. In this sense, the solution (10, 0, 0, 12) is unstable with respect to small perturbations of the input data.

We implemented our algorithms on a working set comprising all four strategies. Algorithm **PSTRATEQ1**, based on harmonic stepsizes, converges slowly to one of the two equilibrium solutions (Tables 4 and 5), depending on the starting point. On the other hand, **PSTRATEQ2** (adaptive stepsize) always converges to the 'stable' solution (5, 5, 5.24, 6.76).

The behavior of projective methods is influenced to a great extent by the choice of parameters. For $\alpha < 0.02$, the methods converge very rapidly to one of the equilibrium solutions, depending on the initial vector choice (see Tables 7, 8 and 9). It is surprising that, for large values of α , all methods converge to the extremal and 'unstable' solution (10, 0, 0, 12) (see Table 10)! Various tests were performed by varying the parameters of the problem and of the algorithm; some of them are presented in the result tables.

#iter.		Strategi	c flow			Strategi	c costs		Relative
	x_{s_1}	x_{s_2}	x_{s_3}	x_{s_4}	C_{s_1}	C_{s_2}	C_{s_3}	C_{s_4}	gap (%)
0	0.00	10.00	0.00	12.00	670.00	380.00	310.00	550.00	27.692
1	0.00	10.00	1.33	10.67	670.00	549.33	310.00	550.00	21.744
2	0.00	10.00	2.40	9.60	670.00	684.80	353.64	550.00	19.049
5	2.31	7.69	4.62	7.38	670.00	838.54	420.77	550.00	16.076
10	4.44	5.56	6.67	5.33	670.00	1016.00	523.33	550.00	13.722
20	6.43	3.57	5.57	6.43	670.00	672.50	602.50	550.00	2.216
50	8.28	1.72	2.69	9.31	670.00	672.50	602.50	550.00	1.082
100	9.07	0.93	1.44	10.56	670.00	672.50	602.50	550.00	0.584
200	9.52	0.48	0.75	11.25	670.00	672.50	602.50	550.00	0.304
500	9.80	0.20	0.31	11.69	670.00	672.50	602.50	550.00	0.124
1000	9.90	0.10	0.15	11.85	670.00	672.50	602.50	550.00	0.062
2000	9.95	0.05	0.08	11.92	670.00	672.50	602.50	550.00	0.021
10000	10.00	0.00	0.00	12.00	670.00	860.00	790.00	550.00	0.001

Table 4. PSTRATEQ1 (small network)

Table 5. PSTRATEQ1 (small network, second starting point)

#iter.		Strateg	ic flows			Strateg	ic costs		Relative
	x_{s_1}	x_{s_2}	<i>x</i> _{<i>s</i>₃}	x_{s_4}	C_{s_1}	C_{s_2}	C_{s_3}	C_{s_4}	gap (%)
0	2.00	8.00	4.00	8.00	670.00	769.50	406.00	550.00	14.408
1	3.14	6.86	5.14	6.86	670.00	857.27	460.86	550.00	13.418
2	4.00	6.00	6.00	6.00	670.00	931.33	502.00	550.00	12.729
5	5.64	4.36	6.55	5.45	670.00	858.77	580.55	550.00	7.146
10	4.50	5.50	5.25	6.75	670.00	739.73	526.00	550.00	4.023
20	5.08	4.92	5.08	6.92	670.00	623.69	553.69	550.00	1.939
50	5.04	4.96	5.57	6.43	670.00	732.68	551.71	550.00	1.154
100	5.11	4.89	5.32	6.68	670.00	668.95	555.43	550.00	0.528
200	5.01	4.99	5.18	6.82	670.00	656.52	550.47	550.00	0.257
500	4.98	5.02	5.24	6.76	670.00	672.04	549.24	550.00	0.115
1000	4.99	5.01	5.24	6.76	670.00	669.93	549.62	550.00	0.022
2000	5.00	5.00	5.25	6.75	670.00	670.51	550.05	550.00	0.017
10000	5.00	5.00	5.24	6.76	670.00	670.01	550.01	550.00	0.002

 Table 6. PSTRATEQ2 (small network)

#iter.		Strategi	ic flows			Strategi	c costs		Relative
	x_{s_1}	x_{s_2}	x_{s_3}	x_{s_4}	C_{s_1}	C_{s_2}	C_{s_3}	C_{s_4}	gap (%)
0	0.00	10.00	0.00	12.00	670.00	380.00	310.00	550.00	27.692
1	0.00	10.00	5.24	6.76	670.00	1045.02	310.00	550.00	34.023
2	3.59	6.41	8.19	3.81	670.00	1339.71	482.25	550.00	26.715
3	6.79	3.21	8.66	3.34	670.00	1254.74	636.09	550.00	16.459
5	8.51	1.49	5.88	6.12	670.00	762.98	692.98	550.00	6.854
10	9.13	0.87	2.09	9.91	670.00	719.18	649.18	550.00	1.849
40	5.00	5.00	5.30	6.70	670.00	681.27	549.93	550.00	0.426
70	5.00	5.00	5.24	6.76	670.00	670.02	549.99	550.00	0.001
100	5.00	5.00	5.24	6.76	670.00	670.00	550.00	550.00	0.000

Table 7. Projection algorithm (small network, $\alpha = 0.01$)

#iter.		Strateg	ic flows			Strategi	c costs		Relative
	x_{s_1}	<i>x</i> _{<i>s</i>₂}	x_{s_3}	x_{s_4}	C_{s_1}	C_{s_2}	C_{s_3}	C_{s_4}	gap (%)
0	0.00	10.00	0.00	12.00	670.00	380.00	310.00	550.00	27.692
1	0.00	10.00	1.20	10.80	670.00	532.40	310.00	550.00	22.276
2	0.00	10.00	2.40	9.60	670.00	684.80	310.00	550.00	19.049
3	0.07	9.93	3.60	8.40	670.00	833.43	313.55	550.00	25.644
5	2.28	7.72	5.77	6.23	670.00	1014.06	419.26	550.00	22.838
10	7.01	2.99	5.72	6.28	670.00	695.23	625.23	550.00	3.660
20	7.17	2.83	3.05	8.95	670.00	629.30	559.30	550.00	2.425
50	5.00	5.00	5.24	6.76	670.00	669.90	550.11	550.00	0.008
100	5.00	5.00	5.24	6.76	670.00	670.00	550.00	550.00	0.000

Table 8. Konnov's algorithm (small network, $\alpha = 0.01$, $\lambda = 0.05$, $\theta = 0.001$)

#iter.		Strateg	ic flows			Strategi	c costs		Relative
	x_{s_1}	<i>x</i> _{<i>s</i>₂}	<i>x</i> _{s3}	<i>x</i> _{<i>s</i>₄}	C_{s_1}	C_{s_2}	C_{s_3}	C_{s_4}	gap (%)
0	2.00	8.00	0.00	12.00	670.00	380.00	310.00	550.00	31.512
1	0.55	9.45	1.20	10.80	670.00	491.94	336.40	550.00	21.167
2	0.00	10.00	2.27	9.73	670.00	668.05	310.00	550.00	18.339
3	0.00	10.00	3.47	8.53	670.00	820.45	310.00	550.00	25.423
5	2.10	7.90	5.69	6.31	670.00	1011.52	410.74	550.00	23.528
10	6.95	3.05	5.76	6.24	670.00	693.93	623.93	550.00	3.617
20	7.03	2.97	3.19	8.81	670.00	628.37	558.37	550.00	2.417
50	5.00	5.00	5.24	6.76	670.00	669.89	550.10	550.00	0.008
100	5.00	5.00	5.24	6.76	670.00	670.00	550.00	550.00	0.000

At the 'stable' solution x^* , the Jacobian matrix of C takes the form

$$J = C'(x^*) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 45.7 & 196.11 & 206 & 0 \\ 48 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since the eigenvalues of $J + J^t$ are nonnegative, the cost function *C* is monotone in the vicinity of x^* . This property might largely explain the nice behavior of the solution algorithms close to x^* . When demand is decreased from 10 to 9, convergence of **PSTRATEQ1** slows down (Table 11), while both **PSTRATEQ2** algorithm (Table 12)

#iter.		Strateg	ic flows			Strategi	c costs		Relative
	x_{s_1}	x_{s_2}	x_{s_3}	x_{s_4}	C_{s_1}	C_{s_2}	C_{s_3}	C_{s_4}	gap (%)
0	0.00	10.00	10.00	2.00	670.00	1650.00	310.00	550.00	49.662
1	4.90	5.10	10.02	1.98	670.00	1650.00	545.21	550.00	27.438
2	9.80	0.20	8.87	3.13	670.00	849.42	779.42	550.00	13.479
3	10.00	0.00	7.67	4.33	670.00	860.00	790.00	550.00	12.164
5	10.00	0.00	5.27	6.73	670.00	860.00	790.00	550.00	8.691
7	10.00	0.00	2.87	9.13	670.00	860.00	790.00	550.00	4.931
8	10.00	0.00	1.67	10.33	670.00	860.00	790.00	550.00	2.933
9	10.00	0.00	0.47	11.53	670.00	860.00	790.00	550.00	0.849
10	10.00	0.00	0.00	12.00	670.00	860.00	790.00	550.00	0.000

Table 9. Extragradient algorithm (small network, $\alpha = 0.01$)

Table 10. Projection algorithm (small network, $\alpha = 1$)

#iter.		Strategi	ic flows			Strategi	c costs		Relative
	x_{s_1}	x_{s_2}	x_{s_3}	x_{s_4}	C_{s_1}	C_{s_2}	C_{s_3}	C_{s_4}	gap (%)
0	0.00	10.00	1.00	11.00	670.00	507.00	310.00	550.00	23.098
1	0.00	10.00	12.00	0.00	670.00	1650.00	350.00	550.00	47.343
2	10.00	0.00	12.00	0.00	670.00	1650.00	750.00	550.00	15.287
3	10.00	0.00	0.00	12.00	670.00	860.00	790.00	550.00	0.000

Table 11. PSTRATEQ1 (small network, $d_{16} = 9$)

#iter.		Strateg	gic flows			Strategi	c costs		Relative
	x_{s_1}	x_{s_2}	x_{s_3}	x_{s_4}	C_{s_1}	C_{s_2}	C_{s_3}	C_{s_4}	gap (%)
0	0.00	9.00	0.00	12.00	670.00	380.00	310.00	550.00	28.742
1	0.00	9.00	1.71	10.29	670.00	480.79	310.00	550.00	23.475
2	0.00	9.00	3.00	9.00	670.00	662.22	310.00	550.00	18.243
5	1.64	7.36	5.45	6.55	670.00	914.53	388.55	550.00	21.088
10	3.94	5.06	7.50	4.50	670.00	1116.17	499.00	550.00	17.153
20	5.88	3.12	6.92	5.08	670.00	674.65	592.46	550.00	2.384
50	4.82	4.18	6.21	5.79	670.00	709.07	541.43	550.00	1.670
100	5.09	3.91	6.23	5.77	670.00	659.21	554.53	550.00	0.659
200	4.98	4.02	6.23	5.77	670.00	683.81	549.07	550.00	0.480
500	5.02	3.98	6.21	5.79	670.00	671.81	550.76	550.00	0.094
1000	5.00	4.00	6.20	5.80	670.00	672.02	550.05	550.00	0.066
2000	5.00	4.00	6.20	5.80	670.00	671.23	549.90	550.00	0.022
10000	5.00	4.00	6.19	5.81	670.00	670.32	550.00	550.00	0.001

and the projective methods (Table 13) converge rapidly to the equilibrium. Note that the projective methods diverge, as should be expected, when α becomes too large (see Table 14).

7.2. Numerical tests on 'Sioux Falls' network

The second set of numerical tests has been performed on a simplified version of the Sioux Falls network used by Suwansirikul et al. [26]. It is composed of 24 nodes, 41 arcs, 4 OD pair and 5 transit lines (see Tables 15 and 16). A typical strategy for this network

#iter.		Strateg	gic flows			Strategi	c costs		Relative
	x_{s_1}	x_{s_2}	<i>x</i> _{<i>s</i>₃}	x_{s_4}	C_{s_1}	C_{s_2}	C_{s_3}	C_{s_4}	gap (%)
0	0.00	9.00	0.00	12.00	670.00	380.00	310.00	550.00	28.742
1	0.00	9.00	5.24	6.76	670.00	977.80	310.00	550.00	31.063
2	2.83	6.17	8.19	3.81	670.00	1316.76	445.99	550.00	27.811
3	5.86	3.14	8.91	3.09	670.00	1306.18	591.38	550.00	15.771
5	7.53	1.47	6.87	5.13	670.00	717.81	647.81	550.00	5.549
10	7.59	1.41	3.87	8.13	670.00	640.92	570.92	550.00	2.381
20	5.59	3.41	6.39	5.61	670.00	644.15	574.15	550.00	2.354
50	5.00	4.00	6.19	5.81	670.00	669.97	550.01	550.00	0.001
100	5.00	4.00	6.19	5.81	670.00	670.00	550.00	550.00	0.000

Table 12. PSTRATEQ2 (small network, $d_{12} = 9$)

Table 13. Projection algorithm (small network, $\alpha = 0.01$ and $d_{12} = 9$)

#iter.		Strategic flows				Strateg	ic costs		Relative
	x_{s_1}	x_{s_2}	x_{s_3}	x_{s_4}	C_{s_1}	C_{s_2}	C_{s_3}	C_{s_4}	gap (%)
0	0.00	9.00	0.00	12.00	670.00	380.00	310.00	550.00	28.742
1	0.00	9.00	1.20	10.80	670.00	408.22	310.00	550.00	25.956
2	0.00	9.00	2.40	9.60	670.00	577.56	310.00	550.00	20.531
3	0.00	9.00	3.60	8.40	670.00	746.89	382.36	550.00	21.737
5	1.51	7.49	5.91	6.09	670.00	995.87	601.07	550.00	24.592
10	6.34	2.66	6.63	5.37	670.00	671.07	536.75	550.00	2.633
20	5.00	4.00	5.48	6.52	670.00	606.75	550.00	550.00	3.271
50	5.00	4.00	6.19	5.81	670.00	670.00	550.00	550.00	0.000

Table 14. Extragradient algorithm (small network, $\alpha = 0.1$ and $d_{12} = 9$)

#iter.		Strategic flows			Strategic costs				Relative
	x_{s_1}	x_{s_2}	x_{s_3}	x_{s_4}	C_{s_1}	C_{s_2}	C_{s_3}	C_{s_4}	gap (%)
0	0.00	9.00	0.00	12.00	670.00	380.00	310.00	550.00	28.742
1	0.00	9.00	12.00	0.00	670.00	1650.00	350.00	550.00	46.299
2	9.00	0.00	12.00	0.00	670.00	1650.00	710.00	550.00	13.195
5	0.00	9.00	12.00	0.00	670.00	1650.00	350.00	550.00	46.299
10	9.00	0.00	12.00	0.00	670.00	1650.00	710.00	550.00	13.195
20	9.00	0.00	0.00	12.00	670.00	380.00	310.00	550.00	43.467
50	9.00	0.00	12.00	0.00	670.00	1650.00	710.00	550.00	13.195
100	9.00	0.00	0.00	12.00	670.00	380.00	310.00	550.00	43.467
200	9.00	0.00	0.00	12.00	670.00	380.00	310.00	550.00	43.467

Table 15. Demand for the Sioux Falls network

O-D pair	Notation	Demand	Path with infinite capacity	Cost
(1,24)	OD1	35	$p_1 = \{1, 3, 12, 13, 24\}$	120
(1,22)	OD2	25	$p_2 = \{1, 3, 12, 13, 22\}$	140
(7,24)	OD3	20	$p_3 = \{7, 18, 22, 24\}$	130
(7,22)	OD4	20	$p_4 = \{7, 18, 22\}$	100

Table 16. Transit lines data for the Sioux Falls network

Transit lines (L_i)	Capacity
10-15-17	5
15-20-22	10
4-11-14-23	15
5-9-10-11	20
8-16-17-19-22	25

Table 17. A strategy for the Sioux Falls network (OD 1-24)

node :	1	2	3	4	5	6
preference set :	[3]	[]	[4]	[11, 5]	[9, 6]	[8]
node :	7	8	9	10	11	12
preference set :	[]	[16, 9]	[10, 16]	[11, 15, 16, 17]	[14, 12]	[13]
node :	13	14	15	16	17	18
preference set :	[24]	[23, 15]	[20, 17, 19]	[17, 18]	[19, 20]	[22]
node :	19	20	21	22	23	24
preference set :	[22, 20]	[22, 23, 21]	[24]	[24]	[24]	[]

Table 18. PSTRATEQ1 on Sioux Falls network

# iter.	N	Number of strategies				Gap contribution (%)			
	OD1	OD2	OD3	OD4	OD1	OD2	OD3	OD4	total
0	1	1	1	1	11.141	5.307	12.439	0.000	28.887
1	2	2	2	1	2.979	4.441	7.188	0.046	14.654
2	3	3	3	2	1.791	3.212	4.661	0.340	9.994
5	3	4	3	2	1.090	3.195	3.910	0.819	9.013
10	3	4	3	3	0.780	1.310	1.664	0.213	3.967
20	3	4	3	3	0.313	0.676	0.889	0.101	1.979
50	3	4	3	3	0.057	0.265	0.362	0.060	0.744
100	3	4	3	3	0.059	0.131	0.166	0.030	0.386
200	3	4	4	3	0.011	0.068	0.086	0.012	0.176
500	3	2	2	3	0.015	0.002	0.004	0.005	0.026
1000	2	2	2	3	0.002	0.001	0.002	0.003	0.008
2000	2	2	2	3	0.000	0.001	0.001	0.001	0.003
10000	2	2	1	2	0.000	0.000	0.000	0.001	0.001

is presented in Table 17. The tolerance parameters ϵ_1 and ϵ_2 have been set to 10^{-4} and 0.05 for algorithm **PSTRATEQ1** and to 10^{-4} and 0.01 for algorithm **PSTRATEQ2**. Tables 18 and 19 contain, for each OD pair, the number of strategies generated and the contribution of each OD pair to the gap function.

While the behavior of both algorithms is satisfactory, with a relative gap of 1% reached in less than 40 iterations, **PSTRATEQ1** is clearly outperformed by **PSTRATEQ2**, with a gap of 1% reached after less than 20 iterations, and a gap of

# iter.	N	umber of	f strategi	es		Gap contribution (%)				
	OD1	OD2	OD3	OD4	OD1	OD2	OD3	OD4	total	
0	1	1	1	1	11.141	5.307	12.439	0.000	28.887	
1	2	2	2	1	4.941	5.837	7.978	1.151	19.907	
2	2	2	3	2	3.872	5.047	5.105	0.064	14.087	
5	3	3	3	2	1.554	3.199	1.169	0.101	6.022	
10	3	4	3	2	1.011	1.361	0.182	0.069	2.623	
20	3	4	2	3	0.220	0.250	0.037	0.057	0.562	
50	3	2	2	3	0.026	0.002	0.039	0.087	0.155	
100	3	2	2	3	0.001	0.005	0.026	0.069	0.101	
200	2	2	2	3	0.000	0.001	0.013	0.028	0.042	
500	2	2	2	3	0.000	0.000	0.001	0.003	0.004	
1000	2	2	1	2	0.000	0.000	0.000	0.000	0.000	

Table 19. PSTRATEQ2 (Sioux Falls network)

Table 20. PSTRATEQ1 (Sioux Falls network, second data set)

# iter.	N	Number of strategies				Gap contribution (%)			
	OD1	OD2	OD3	OD4	OD1	OD2	OD3	OD4	total
0	1	1	1	1	11.579	12.157	11.887	4.942	40.564
1	2	2	2	2	2.655	8.551	11.079	0.887	23.161
2	3	3	3	2	7.496	6.978	7.320	3.510	25.304
5	3	4	3	2	0.931	6.657	4.188	1.240	13.016
10	3	5	3	2	0.137	2.183	1.793	0.032	4.146
20	3	5	3	3	0.169	1.456	1.278	0.350	3.253
50	3	5	3	3	0.136	0.541	0.414	0.104	1.194
100	3	5	3	3	0.066	0.244	0.207	0.062	0.578
200	3	5	3	3	0.021	0.124	0.125	0.031	0.301
500	3	5	3	3	0.002	0.059	0.035	0.016	0.111
1000	2	3	2	3	0.005	0.025	0.010	0.008	0.047
2000	2	3	2	3	0.003	0.006	0.002	0.004	0.015
10000	2	2	2	1	0.000	0.000	0.000	0.001	0.001

Table 21. PSTRATEQ2 (Sioux Falls network, second data set)

# iter.	N	Number of strategiesGap contribution (%)							
	OD1	OD2	OD3	OD4	OD1	OD2	OD3	OD4	total
0	1	1	1	1	10.823	11.311	11.465	4.492	38.090
1	2	2	2	2	3.064	8.723	8.727	1.397	21.911
2	2	3	2	2	4.146	4.767	5.435	0.334	14.682
5	2	5	3	2	1.547	4.387	2.939	0.688	9.561
10	3	5	3	2	0.376	1.221	0.251	0.317	2.164
20	3	5	2	3	0.083	0.186	0.045	0.085	0.400
50	3	3	2	3	0.023	0.021	0.096	0.058	0.197
100	3	3	2	3	0.004	0.025	0.064	0.033	0.125
200	2	3	2	3	0.001	0.022	0.031	0.008	0.062
500	2	3	2	2	0.000	0.005	0.005	0.000	0.010
1000	2	3	2	1	0.000	0.000	0.000	0.000	0.000

	U	1
Path	Sioux Falls	Sioux Falls (*)
p_1	62%	76%
p_2	0%	63%
p_3	0%	15%
p_4	44%	100%

Table 22. Usage rates of uncapacitated links

.1% reached after 100 iterations. Note also that the size of the working set is larger under **PSTRATEQ1** than under **PSTRATEQ2**. A comparison of the strategic flows obtained by both algorithms reveals no significant differences. Indeed, at equilibrium, strategic costs agree to four significant digits.

A second test problem involved demands increased to 40, 50, 30 and 40, and transit line capacities reduced by 3 units. The travel costs of arcs (12, 13) and (18, 22) are set to 80 and 90 respectively. As expected (see Tables 20 and 21), the number of strategies active at equilibrium increases, as more 'stress' is put on the network. This is illustrated in Table 22, where we observe the increase in the use of uncapacitated ('long') paths. However, this has no significant effect on the convergence behavior of the algorithms.

8. Conclusion

In this paper, we have analyzed a network equilibrium model where users take into account in their travel behavior the possibility that an arc be unavailable upon arrival, due to its finite capacity. Such approach radically differs from the standard path-based approach, and allows for equilibria that meet Wardrop's condition of equal (expected) travel costs.

Difficulties of theoretical and computational nature arise when dealing with this complex model. While our implementation cannot yet deal with large scale networks, we believe it is possible to do so by taking advantage of the fact that only a small number of capacity constraints will be active at equilibrium.

In a companion paper, we adapt the strategic approach to a time-discretized model of traffic equilibrium where the FIFO condition is automatically enforced. At term, we aim for a model that deals both with time-dependent, fixed schedule networks, a situation that, in our opinion, cannot be addressed by traditional path-based or hyperpath-based approaches.

A. Loading Procedure

for $j \in N$ (in topological order) do $\overline{W} := \{s \in W : E_j^s \neq \emptyset\}$

for $s \in \overline{W}$ do $\overline{E}_{j}^{s} := E_{j}^{s}$ Priority $(j, s) \rightarrow z_{j}^{s1}, z_{j}^{s2}$ endfor [set of active strategies]

[ordered set of available preferences] [priority and no-priority strategic flow] $\overline{W}^1 := \{s \in \overline{W} : \, z_j^{s1} > 0\}$ $\overline{W}^2 := \{s \in \overline{W} : z_i^{s^2} > 0\}$ for $k \in j^+$ do $\bar{u}_{jk} := u_{jk}$ endfor for i = 1 to 2 do for $s \in \overline{W}^i$ do $\bar{z}_j^s := z_j^{si} \\ \rho_j^{si} := 1$ endfor while $\overline{W}^i \neq \emptyset$ do $K := \emptyset$ for $k \in j^+$ do $d_k := 0$ $W_k := \emptyset$ endfor for $s \in \overline{W}^i$ do $k := \overline{E}_{i}^{s}(1)$ $K := \check{K} \cup \{k\}$ $W_k := W_k \cup \{s\}$ $d_k := d_k + \bar{z}_i^{si}$ endfor $\nu := \max\{d_k / \bar{u}_{jk} : k \in K\}$ $\beta^{i} := \min\{1/\nu, 1\}$ for $k \in K$ do for $s \in W_k$ do $z_k^s := z_k^s + \beta^i \bar{z}_j^{si}$ $v_{jk}^s := v_{jk}^s + \beta^i \bar{z}_j^{si}$ $\bar{z}_j^{si} := \bar{z}_j^{si} - \beta^i \bar{z}_j^{si}$ if $z_j^s > 0$ then $\pi_{jk}^{s} := \pi_{jk}^{s} + \beta^{i} \frac{\overline{z}_{j}^{s}}{\overline{z}_{j}^{s}}$ else $\pi_{jk}^{s} := \pi_{jk}^{s} + \beta^{i} \rho_{j}^{si}$ endif $\rho_{j}^{si} := (1 - \beta^{i}) \rho_{j}^{si}$ dfor endfor $\bar{u}_{jk} := \bar{u}_{jk} - \beta^i d_k$ if $\bar{u}_{jk} = 0$ then for $s \in \overline{W}^1 \cup \overline{W}^2$ do $\overline{E}_j^s := \overline{E}_j^s - \{k\}$ endfor endif

[set of priority active strategies] [set of no-priority active strategies] [residual capacity of arc (j, k)] [loading over \overline{W}^1 and \overline{W}^2] [unassigned strategic flow] [proportion of unsigned strategic flow] [active strategies residual first choices] [residual demand for first choice node k] [set of active strategies having node k as residual first choice] [first residual choice of strategy s] [construction of K] [construction of W_k] [construction of d_k] [proportion of strategic assigned flow] [update of node strategic flow] [update of arc strategic flow] [update of unassigned strategic flow] [update of access probability] [update of proportion of unassigned flow] [update of residual capacity of arc (j, k)] [disactivation of saturated arc (j, k)] [update of residual preference set]

endfor if $\beta^i = 1$ then [end of loading for \overline{W}^i] $\overline{W}^i = \emptyset$ endif endwhile endfor for $s \in W$ do $\mathbf{for} = k \in E^s_j$ do $C^s := C^s + c_{jk}\tau^s_j\pi^s_{jk}$ [update of cost C^s] $\tau^s_k := \tau^s_k + \tau^s_j\pi^s_{jk}$ [update of probability τ^s_k] endfor endfor

endfor

References

- 1. Aubin, J.P.: Optima and equilibria: An introduction to nonlinear analysis. Springer-Verlag, 1993
- 2. Border, K.: Fixed point theorems with applications to economics and game theory. Cambridge University Press, Cambridge, 1985
- Cepeda, M.: Modèle d'équilibre dans les réseaux de transport en commun: le cas des capacités explicites des services. Ph.D dissertation, Département d'informatique et de recherche opérationnelle, Université de Montréal, 2002
- 4. Chriqui, C., Robillard, P.: Commun bus line, Transportation Sci. 9, 115–121 (1975)
- Cominetti, R., Correa, J.: Common-lines and passenger assignment in congested transit networks. Transportation Sci 35, 250–267 (2001)
- De Cea, J., Fernández, J.E.: Transit assignment to minimal routes: an efficient new algorithm. Traffic Eng. Contr 30, 491–494 (1989)
- De Cea, J., Fernández, J.E.: Transit assignment for congested public transport systems: an equilibrium model. Transportation Sci 27, 133–147 (1993)
- 8. Dial, R.B.: Transit pathfinder algorithm. Highway Res. Record 205, 67–85 (1967)
- 9. Frank, M., Wolfe, P.: An algorithm for quadratic programming. Naval Res. Logistic Q. 3, 95-110 (1956)
- Gendreau, M.: Une étude approfondie d'un modèle d'équilibre pour l'affectation des passagers dans les réseaux de transport en commun. Ph.D dissertation, Département d'informatique et de recherche opérationnelle, Université de Montréal, 1984
- Hamdouch, Y.: Affectation statique et dynamique des usagers dans un réseau de transport avec capacités rigides. Ph.D dissertation, Département d'informatique et de recherche opérationnelle, Université de Montréal, 2002
- 12. Hearn, D.W.: Bounding flows in traffic assignment models. Research report 80-4, Department of Industrial and Systems Engineering, University of Florida, Gainesville, 1980
- Konnov, I.: Combined relaxation methods for finding equilibrium points and solving related problems. Mathematika 37, 44–51 (1993)
- Korpelevitch, G.M.: The extragradient method for finding saddle points and other problems. Matekon 13, 35–49 (1977)
- 15. Le Clercq, F.: A public transport assignment method. Traffic Eng. Contr. 14, 91-96 (1972)
- Lawphongpanich, S., Hearn, D.W.: Simplicial decomposition of the asymmetric traffic assignment problem. Transportation Res. 18B, 123–133 (1984)
- Marcotte, P., Nguyen, S.: Hyperpath formulations of traffic assignment problems. In: Equilibrium and Advanced Transportation Modelling. P. Marcotte, S. Nguyen (eds.), Kluwer Academic Publisher, 1998, pp. 175–199
- Marcotte, P., Nguyen, S., Schoeb, A.: A strategic flow model of traffic assignment in capacitated networks. Forthcoming in Operations Research, 2000
- Nguyen, S., Pallottino, S.: Equilibrium traffic assignment for large scale transit networks. Eur. J. Oper. Res. 37, 176–186 (1988)

- Nguyen, S., Pallottino, S.: Hyperpaths and shortest hyperpaths. In: B. Simeone (ed.), Combinatorial Optimization, volume 1403 of Lecture Notes in Mathematics, Springer-Verlag, Berlin, 1989, pp. 258–271
- Nguyen, S., Pallottino, S., Malucelli, P.: A modeling framework for the passenger assignment on a transport network with time-tables. Transportation Sci. 55, 238–249 (2001)
- 22. Patriksson, M.: The Traffic Assignment Problem-Models and Methods. Topics in Transportation, VSP BV, The Netherlands, 1994
- 23. Sheffi, U.: Urban transportation networks: equilibrium analysis with mathematical programming methods. Prentice Hall, New Haven, Connecticut, 1985
- Spiess, M.: On optimal route choice strategies in transit networks. Publication 286, Centre de recherche sur les transports, Université de Montréal, 1983
- Spiess, H., Florian, M.: Optimal strategies: a new assignment model for transit networks. Transportation Res. B 23, 83–102 (1989)
- Suwansirikul, C., Friesz, T.L., Tobin, R.L.: Equilibrium decomposition optimization: a heuristic for the continuous equilibrium network design problem. Transportation Sci 21, 254–263 (1987)
- Wu, J.H., Florian, M., Marcotte, P.: Transit equilibrium assignment: a model and solutions algorithms. Transportation Sci 28, 193–203 (1994)
- 28. Wardrop, J.G.: Some theoretical aspects of road traffic research. Proc. Institution of Civil Engineers, Part II, 325–378 (1952)