On the existence of solutions to the dynamic user equilibrium problem

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Abstract

This paper is concerned with the existence of solutions to a dynamic network equilibrium problem modelled as an infinite dimensional variational inequality. Our results are based on properties of operators that map path flow departure rates to consistent time-dependent path flows and other link performance functions. The existence result requires the introduction of a novel concept that strengthens the familiar concept of FIFO ('First-In-First-Out').

Following Ben-Akiva [3], one can identify four main elements in any dynamic network equilibrium model (DNEP): arc performance functions, path choice criterion, a demand model, flow conservation relationships at the nodes of the network. According to this author, the extension of a static to a dynamic setting requires the determination of time-dependent arc performance functions, temporal origin-destination demand, and the assessment of queueing effects on time-varying arrival rates. In general, these relationships are not available in closed form, and consequently analytical properties of the model are difficult, if not impossible, to derive.

Several models of dynamic equilibrium have been proposed, that involve only route choice ([6] [10] [12] [16] [17]) or route choice and departure time choice ([2] [5] [10] [14] [18]). In all the abovementioned models, the issue of existence of a dynamic equilibrium has not been fully addressed. In this paper we give an existence result for the dynamic User Equilibrium – Route Choice model introduced by Friesz et al [5], thus giving a theoretical foundation for the numerical algorithms that have been proposed for this model.

The analysis is performed under the assumption that flow throughout the network obeys the FIFO ('First-In-First-Out') rule. In the absence of such a queue discipline, flow propagation can be irrealistic (see Astarita [1]). We believe that analytic models should ensure that FIFO is satisfied.

A byproduct of our analysis will be a theoretical analysis of the loading procedure (see also Xhu et al. [19]) that relates the link performance functions (flow volumes, entry rates, exit rates, departure times) to the path departure rate functions. These link performance functions constitute the control variables of the dynamic network equilibrium problem, and their evaluation lies at the heart of the DNEP.

The paper is organized as follows: the mathematical formulation of the DNEP is stated in Section 2; in Section 3 we prove that, under a strengthened FIFO condition, the dynamic network constraints are consistent and that the network flow operators are well defined over their respective domains; Section 4 is devoted to the continuity properties of the link flow, path flow and traversal time operators; from these results we derive (Section 5) the existence of a dynamic equilibrium for the route choice model with fixed departure times; in Section 6, we derive sufficient conditions that ensure the strengthened FIFO condition and hence the existence result.

1 Formulation of the dynamic network system

Let us consider a network with multiple origins and destinations, based on an underlying directed graph $G = (N, A)$ with node set N and arc set A. We denote by R the set of origin-destination couples, by P_r the set of paths associated with the origin-destination couple r in R, and let $P = \cup_{r \in R} P_r$ denote the set of all origin-destination paths. An element σ_{ap} of the link-path incidence matrix is defined by

$$\sigma_{ap} = \begin{cases} 1 & \text{if link } a \text{ belong to path } p \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Denoting by $x_a^p(t)$ the number of vehicles travelling on arc a of path p at instant t, and by $x_a(t)$ the total number of vehicles travelling on arc a at instant t, we obtain the relationships:

$$x_a(t) \quad = \quad \sum_{p \in P} \sigma_{ap} x_a^p(t).$$

Now let $u_a^p(t)$ (respectively $v_a^p(t)$) denote the inflow rate into (respectively exit rate from) link a of path p at time t. Based on these variables, the system equation for link a can be written as:

$$(x_a^p)'(t) = u_a^p(t) - v_a^p(t) \quad \forall p \in P_a \tag{2}$$

where P_a is the set of paths through link a. Now, assuming that the number of vehicles on link a at the start of the period under study is equal to zero, the number of vehicles on link a at any instant t in $[0, \overline{T}]$ can be expressed as

$$x_a^p(t) = \int_0^t [u_a^p(s) - v_a^p(s)]\, ds \quad \forall p \in P_a. \tag{3}$$

For any pair of consecutive arcs a and b of path p, flow conservation is expressed by the equation

$$v_b^p(t) = u_a^p(t). \tag{4}$$

The demand side of the model is characterized by Lipschitz continuous functions Q_r that represent the number of travelers leaving the origin node of the OD pair r at time t. The functions Q_r can be decomposed into path departure rate functions h_p according to the formula:

$$\sum_{p \in P_r} h_p(t) = Q_r(t). \tag{5}$$

The function h_p represent the control variables of the users and triggers the dynamic allocation process through the initial equation:

$$u^p_{o(p)}(t) = h_p(t), \tag{6}$$

where $o(p)$ denotes the origin node of path p.

Flow circulation within the network is governed by strictly positive, flow-dependent delay functions D_a that relate the link exit time $\tau_a(t)$ to the time of entry t:

$$\tau_a(t) = t + D_a(x_a(t)). \tag{7}$$

The equalities and inequalities (2)–(7), together with obvious nonnegative constraints on all flows involved, define a dynamical network system (DNS). Any solution of this system relates, in a coherent fashion, the state variables u^p_a, v^p_a, x^p_a and τ_a to the control variables h_p. Throughout the paper we make the assumption that the delay functions D_a are positive, strictly increasing and continuously differentiable, and that the demand functions Q_r are bounded over the period $[0, T]$. The control vector $h = (h_p)_{p \in P}$ 'lives' in the feasible set Λ of nonnegative, square-integrable vector functions that satisfy almost everywhere (a.e.), the initial condition (5). In Section 3 we will show that, if all links satisfy a strengthened FIFO condition, the state vectors x^p_a, u^p_a, v^p_a and τ_a are well defined as functions of h, i.e., the DNS is well defined over the period $[0, \overline{T}]$.

2 The DNS under the strong FIFO condition

2.1 Some key concepts and results

Before considering the existence of a dynamic network equilibrium, one must check whether the DNS is well defined. To this effect, it is natural to introduce a queue discipline that forbids vehicles overtaking each other. Unfortunately, it seems that a simple FIFO rule is not sufficient, hence the introduction of a strengthened rule, which we call 'strong FIFO'.

Definition 2.1 *A strong FIFO condition holds at link a over the time period $[0, t]$ if the function τ_a is strongly monotone with positive modulus γ_a, i.e.:*

$$t_1 \geq t_2 \Rightarrow \tau_a(t_1) - \tau_a(t_2) \geq \gamma_a(t_1 - t_2) \tag{8}$$

for all t_1, t_2 in $[0, t]$.

Sufficient conditions that ensure the strong FIFO condition will be given in Section 5. Strong FIFO implies that the functions τ^{-1} are Lipschitz continuous (see Lemma 2.1). Note that Lipschitz continuity implies absolute continuity (see Remark 4.6 in [13] and Definition 2.2 below), a key feature in the study of the flow functions x^p_a, u^p_a and v^p_a. The rest of this section is devoted to definitions and theoretical results of real analysis. The results whose proof is omitted are either standard or elementary. (See Kolmogorov and Fomin [8], Mukherjea and Pothoven [13] or Royden [15].)

Definition 2.2 *A real function f defined over the interval $[a, b]$ is absolutely continuous if, for any positive number ϵ, there exists a positive number δ such that, for every finite collection $\{(x_i, x_i')\}$ of non overlapping intervals with*

$$\sum_{i=1}^{n} |x_i' - x_i| < \delta$$

we have that

$$\sum_{i=1}^{n} |f(x_i') - f(x_i)| < \epsilon.$$

Definition 2.3 *A sequence u_n in a normed linear space X converges weakly to $u \in X$ ($u_n \rightharpoonup u$) if, for every bounded linear functional F on X, we have $F(u_n) \to F(u)$ as $n \to \infty$.*

Note that a Lipschitz continuous function is absolutely continuous.

Definition 2.4 *Let X and Y be two normed linear spaces. An operator from X to Y is weakly continuous if for every weakly converging sequence $u_n \rightharpoonup u$, we have that*

$$\|J(u_n) - J(u)\| \to 0 \text{ as } n \to \infty.$$

Theorem 2.1

(i) Any bounded and measurable function over $[a, b]$ is integrable over $[a, b]$.

(ii) Every nondecreasing function f is bounded and measurable on $[a, b]$.

(iii) Sums, differences and products of measurable functions are measurable.

Lemma 2.1 *Let f be strongly monotone with modulus c over the interval $[a, b]$. Then f is invertible over $[a, b]$, its inverse f^{-1} is Lipschitz continuous (with Lipschitz constant $1/c$) and differentiable (a.e.) over the range $[f(a), f(b)]$ of f and we have: $(f^{-1})'(x) = 1/f'(f^{-1}(x))$.*

Lemma 2.2 *Let f be absolutely continuous on $[a, b]$ with $f'(x) \geq c > 0$ for almost every $x \in [a, b]$. Then f is strongly monotone with constant c over $[a, b]$.*

Lemma 2.3

(i) Let g be differentiable (a.e.) on $[a, b]$ and f be integrable on $[c, d]$, with $g([a, b]) \subset [c, d]$. Let $F(x) = \int_c^x f(t)dt$. We have that

$$\int_{g(\alpha)}^{g(x)} f(s)ds = \int_{\alpha}^{x} f(g(s))g'(s)ds \tag{9}$$

for all α and x in $[a, b]$ if and only if the function $F \circ g$ is absolutely continuous.

(ii) If g is absolutely continuous on $[a, b]$ and f is bounded by B and integrable on $[c, d]$, then $F \circ g$ is absolutely continuous and (9) holds for all α and β in $[a, b]$.

We introduce the subset $\Omega(B, T')$ of functions in $L^2[0, T']$ bounded by B:

$$\Omega(B, T') = \{u : u \in L^2[0, T'], 0 \leq u(t) \leq B \quad \forall t \in [0, T']\}.$$

Proposition 2.1 *Let J be the integral operator defined as:*

$$Ju(t) = \int_0^t u(s)\, ds. \tag{10}$$

Then J is weakly continuous on $\Omega(B, T')$

Proof First we prove that $Ju(t) \in L^2[0, T']$. By construction, $Ju(t)$ is of bounded variation on $[0, T']$, i.e., it can be expressed as the difference $Ju(t)$ and $(Ju(t))^2$ are bounded and measurable on $[0, T']$. From Theorem 2.1, both $Ju(t)$ and $(Ju(t))^2$ are bounded and measurable on $[0, T']$. Thus we get that $Ju(t) \in L^2[0, T']$. Let $\{u_k\}$ be a sequence converging weakly to u on $\Omega(B, T')$. Assume, by contradiction, that $\|Ju - Ju_k\|$ does not converge to zero, i.e., there exist $\epsilon_0 > 0$ and an infinite sequence $\{n_k\}$ such that, for every k:

$$\begin{aligned} \|Ju - Ju_{n_k}\| &\geq \epsilon_0^2 \\ |Ju(t_{n_k}) - Ju_{n_k}(t_{n_k})| &\geq \frac{\epsilon_0}{\sqrt{T'}} \end{aligned} \quad .$$

Since $[0, T']$ is compact, there exists an infinite subsequence $\{n'_k\}$ of $\{n_k\}$ such that $t_{n'_k} \to t_0 \in [0, T']$ and

$$|Ju(t_{n'_k}) - Ju_{n'_k}(t_{n'_k})| \geq \frac{\epsilon_0}{\sqrt{T'}} \quad . \tag{11}$$

But

$$|Ju(t_{n'_k}) - Ju_{n'_k}(t_{n'_k})| \leq |\int_0^{t_0} (u_{n'_k}(s) - u(s))\, ds| + |\int_{t_0}^{t_{n'_k}} (u_{n'_k}(s) - u(s))\, ds|. \tag{12}$$

The first term on the right hand side of the inequality converges to 0, since $u_{n'_k} \rightharpoonup u$. The second term also converges to 0, since $t_{n'_k} \to t_0$ and u and $u_{n'_k}$ are uniformly bounded on $[0, T']$. This is in contradiction with (11), and the operator J must therefore be weakly continuous on $\Omega(B, T')$. □

2.2 Link dynamics and the link flow operator

If the entry rate functions u_a^p are known over the period $[0, T_F]$, then the theorem below assures that the functions x_a, τ_a and v_a are well defined over their respective domain.

Theorem 2.2 *Assume that, for some finite instant T', the entry flow rate functions $(u_a^p)_{p \in P_a}$ are well defined (a.e.), nonnegative and Lebesgue integrable over the interval $[0, T']$. If strong FIFO holds over $[0, T']$ we have that:*

(i) *τ_a is strictly increasing and differentiable (a.e.) over $[0, T']$;*

(ii) *τ_a^{-1} is strictly increasing, Lipschitz continuous and differentiable (a.e.) over the interval $[\tau_a(0), \tau_a(T')]$;*

(iii) *x_a is differentiable (a.e.) on $[0, \tau_a(T')]$.*

(iv) *$(v_a^p)_{p \in P_a}$ are well defined (a.e.), nonnegative and Lebesgue integrable over the interval $[0, \tau_a(T')]$.*

Proof (i), (ii) and (iii): From the definition of strong FIFO, τ_a is strongly monotone increasing, and its inverse τ_a^{-1} exists. Let $t_{a0} = \tau_a(0)$ be the instant when the first user exits link a. The flow $x_a^p(t)$ over the period $[0, t_{a0}]$ is uniquely determined by:

$$x_a^p(t) = \int_0^t u_a^p(s)\, ds, \quad t \in [0, t_{a0}], \quad \forall p \in P_a. \tag{13}$$

From the strong FIFO condition on $[0, T']$ the inverse function τ_a^{-1} is well defined over the interval $[t_{a0}, \tau_a(T')]$ and we can write:

$$x_a^p(t) = \int_{\tau_a^{-1}(t)}^t u_a^p(s)\, ds \quad \forall t \in [t_{a0}, \tau_a(T')] \quad \forall p \in P_a. \tag{14}$$

Clearly, for given $(u_a^p(t))_{p \in P_a}$ on $[0, t_{a0}]$, $(x_a^p)_{p \in P_a}$ is well defined and differentiable (a.e.) in the first period $[0, t_{a0}]$. Thus both x_a and τ_a are well defined and differentiable (a.e.) in this period. Furthermore, from the assumption and Lemma 2.1, τ_a^{-1} is well defined, absolutely (indeed Lipschitz) continuous and differentiable (a.e.) in $[t_{a0}, t_{a1}]$. From (13)–(14) and Lemma 2.3, we know that x_a and τ_a are well defined and differentiable (a.e.) in $[t_{a0}, t_{a1}]$, where $t_{a1} = \tau_a(t_{a0})$. Denote $t_{ai} = \tau_a(t_{ai-1}), i = 1, ..., N_u$, where $t_{aN_u} = \tau_a(T')$. An induction argument on the index i completes the proof.

(iv) Recall that $(x_a^p)'(t) = u_a^p(t) - v_a^p(t)$. Together with equations (13) and (14) we obtain, from the absolute continuity of τ_a^{-1} and Lemma 2.3, an expression for the exit flow rate function v_a:

$$v_a^p(t) = \begin{cases} 0 & \text{if} \quad t \in [0, t_{a0}] \\ u_a^p(\tau_a^{-1}(t))(\tau_a^{-1})'(t) & \text{if} \quad t \in [t_{a0}, \tau_a(T')]. \end{cases} \tag{15}$$

It follows from (i) that $(\tau_a^{-1})'(t) > 0$ (a.e.). Therefore, v_a^p is well defined and nonnegative (a.e.). Moreover, we have that:

$$\begin{aligned} \int_0^t v_a^p(s)\, ds &= \int_{t_{a0}}^t u_a^p(\tau_a^{-1}(s))(\tau_a^{-1})'(s)\, ds, \quad t \in [t_{a0}, \tau_a(T')] \\ &= \int_0^{\tau_a^{-1}(t)} u_a^p(s) ds \end{aligned}$$

which shows that v_a^p is integrable over the period $[0, \tau_a(T')]$. □

Now, consider, for any given arc a, the set of bounded entry flow rate functions over a period $[0, T']$:

$$\Omega(B_a, T') = \{u : u \in L^2[0, T'], \quad 0 \le u(t) \le B_a,\ t \in [0, T']\}.$$

Using Theorem 2.2 we can prove that, under the strong FIFO condition, the link dynamics correctly defines a link flow operator x_a over the domain $(\Omega(B_a, T'))^{|P_a|}$:

$$x_a : u_a \to x_a(u_a), \quad \forall u_a \in (\Omega(B_a, T'))^{|P_a|} \tag{16}$$

where $|P_a|$ is the number of paths through link a. Indeed, for a given $u_a = (u_a^p)_{p \in P_a}$, we can define $x_a(u_a)$ recursively as follows:

$$x_a(t) = \begin{cases} x_{a0}(t) = \sum_{p \in P_a} \int_0^t u_a^p(s)\, ds & \text{if} \quad t \in [0, t_{a0}] \\ x_{a1}(t) = \sum_{p \in P_a} \int_{\tau_{a0}^{-1}(t)}^t u_a^p(s)\, ds & \text{if} \quad t \in [t_{a0}, t_{a1}] \\ \quad \vdots & \ \vdots \qquad \vdots \\ x_{aN_u}(t) = \sum_{p \in P_a} \int_{\tau_{aN_u-1}^{-1}(t)}^t u_a^p(s)\, ds & \text{if} \quad t \in [t_{aN_u} - 1, t_{aN_u}] \end{cases} \tag{17}$$

where $t_{a0} = D_a(0)$, $\tau_{a0}(t) = t + D_a(x_{a0}(t))$ for t in $[0, t_{a0}]$, $t_{ai} = \tau_{ai-1}(t_{ai-1})$, $\tau_{ai}(t) = t + D_a(x_{ai}(t))$ for $t \in [t_{ai-1}, t_{ai}]$, $i = 1, ..., aN_u$ and $t_{aN_u} = \tau_{aN_u-1}(T')$.

Clearly, for given $u_a(t)$ in $[0, t_{a0}]$, $(x_{a0}(t), \tau_{a0}(t))$ is well defined in the first period. Given the link entry flow rate u_a for the periods $[t_{ai-1}, t_{ai}]$, $i = 1, ..., aN_u$, $(x_{ai}(t), \tau_{ai}(t))$ is well defined by $\tau_{ai-1}^{-1}(t)$. The latter information only depends on the information $(x_{ai-1}(t), \tau_{ai-1}(t))$ at the preceding time period. Therefore the operator x_a is well defined. Note that this operator is exactly the dynamic link performance function proposed by Ben-Akiva [2]. If all links satisfy the strong FIFO condition on the appropriate time period, then the DNS is well defined, and so are the link flow operators x_a. □

Theorem 2.3 *Assume that $h \in \Lambda$ and that the strong FIFO condition is satisfied at all links of the transportation network. Then the flow operator $x(h) = (x_a(h))_{a \in A}$ is well defined.*

Proof Let $\xi = \min\{t_{a0} = \tau_a(0) : a \in A\} > 0$ and $h \in \Lambda$. For every link $a \in A$, $u_a = (u_a^p(h(t)))_{p \in P_a}$ is well defined over the interval $[0, \xi]$. From Theorem 2.2, we have a unique solution $x_a(h) = \sum_{p \in P_a} x_a^p(t)$ where $x_a^p(t) = \int_0^t u_a^p(s)\, ds$. The functions τ_a^{-1} and v_a^p are well defined over $[\tau_a(0), \tau_a(\xi)]$ for every link $a \in A$ with $\tau_a(\xi) \geq 2\xi$ and $v_a^p(t) = u_a^p(\tau_a^{-1}(t))(\tau_a^{-1})'(t)$. Note that by obtaining the nonnegative and Lebesgue integrable exit functions $v_a^p(t)$ for all $t \in [\tau_a(0), \tau_a(\xi)]$, $p \in P_a$ and $a \in A$, we have determined $u_a^p(t)$, $p \in P_a$ on $[0, 2\xi]$ for all links. Applying Theorem 2.2 again, we obtain unique solutions $x_a(t)$ on $[0, 2\xi]$ and $v_a^p(t)$ on $[\tau_a(0), \tau_a(2\xi)]$ with $\tau_a(2\xi) \geq 3\xi$ for all $a \in A$ and $p \in P_a$. We know that v_a^p is nonnegative and Lebesgue integrable. Repeat the above process and let $\overline{T}'$ be the (finite) instant when every vehicle having entered the network during the period $[0, T]$ has reached its destination.

In at most $n = \lceil \overline{T}'/\xi \rceil$ steps, we have determined nonnegative and Lebesgue integrable functions u_a for all $t \in [0, n\xi]$ and all links. By Theorem 2.2, the flow $x_a(t)$ is well defined over $[0, \tau_a(n\xi)]$. This shows that, for given $h \in \Lambda$, x_a is uniquely determined, i.e., the operator x_a is well defined over the set Λ, as a function of h. □

3 Continuity of the network flow operator

Our objective in this section is to show that the network flow operator x is weakly continuous over Λ, as a function of h. This strong result will be instrumental in analyzing the existence of solutions to both the DNS and the DNEP, and also performing sensitivity analysis for the DNS.

3.1 Strong FIFO and the DNS

Under the strong FIFO condition, the link dynamics possesses several nice properties that are stated in the next theorem.

Theorem 3.1 *Assume that, almost everywhere over a period $[0, T']$,* (a) *for every $p \in P_a$, the flow entry rate function $u_a^p(t)$ is well defined, nonnegative, Lebesgue integrable and bounded from above by some number B_a* (b) *the strong FIFO condition with constant γ_a holds over $[0, T']$* (c) *the link traversal function D_a is nonnegative, nondecreasing, continuously differentiable and Lipschitz continuous with constant L_{D_a}. Then:*

(i) *τ_a^{-1} is Lipschitz continuous with constant $1/\gamma_a$ over $[\tau_a(0), \tau_a(T')]$ and $(\tau_a^{-1})'(t) \leq 1/\gamma_a$ (a.e.);*

(ii) *for every* $p \in P_a$, $x_a^p(t) \leq B_a T'$, $(x_a^p)'(t) \leq B_a$ *and* x_a^p *is Lipschitz continuous with constant* $(1 + 1/\gamma_a)B_a$ *over its domain;*

(iii)
$$\gamma_a \leq \tau_a'(t) \leq 1 + L_{D_a}|P_a|B_a; \tag{18}$$

(iv) *for every* $p \in P_a$, $v_a^p(t) \leq B_a/\gamma_a$ *over its domain.*

Proof (i) See the proof of Lemma 2.1

(ii) For any two instants t and t' in $[0, t_{a0}]$, with $t_{a0} = \tau_a(0)$, we have:

$$\begin{aligned} |x_a^p(t) - x_a^p(t')| &= |\int_t^{t'} u_a^p(s)\, ds| \\ &\leq B_a|t - t'| \\ &\leq (1 + 1/\gamma_a)B_a|t - t'|. \end{aligned}$$

Since τ_a^{-1} is Lipschitz continuous with constant $1/\gamma_a$ on $[t_{a0}, \tau_a(T')]$, we obtain, for $t \in [t_{a0}, \tau_a(T')]$:

$$\begin{aligned} |x_a^p(t) - x_a^p(t')| &= |\int_{\tau_a^{-1}(t)}^{t} u_a^p(s)\, ds - \int_{\tau_a^{-1}(t')}^{t'} u_a^p(s)\, ds| \\ &\leq |\int_{\tau_a^{-1}(t)}^{\tau_a^{-1}(t')} u_a^p(s)\, ds| + |\int_t^{t'} u_a^p(s)\, ds| \\ &\leq (1 + 1/\gamma_a)B_a|t - t'|. \end{aligned}$$

Therefore x_a^p is Lipschitz continuous on $[0, \tau_a(T')]$, with constant $(1 + 1/\gamma_a)B_a$. Since u_a^p is bounded by B_a, (13) and (14) directly yield:

$$0 \leq x_a^p(t) \leq T'B_a.$$

Moreover, since v_a^p is nonnegative, the system equation (2) yields:

$$\begin{aligned} (x_a^p)'(t) &= u_a^p(t) - v_a^p(t) \\ &\leq u_a^p(t) \\ &\leq B_a. \end{aligned}$$

(iii) First, we note that $(x_a(t))' = \sum_{p \in P_a} (x_a^p)'(t) \leq |P_a|B_a$. The right part of the inequality of (18) follows from the definition of τ_a and the Lipschitz continuity of D_a. By the definition of strong FIFO, the derivative of τ_a satisfies (a.e.): $\tau_a'(t) \geq \gamma_a$.

(iv) Follows directly from (15) and (i). □

The continuity of the link flow operator relies on analytical properties of integral operators that are stated below. For ease of presentation, we drop the indices a and p and present two new definitions and two preliminary results.

Definition 3.1 *Let* u *and* $\{u_k\} \in \Omega(B, T')$. *We say that the sequence* u_k *converges to* u *in the integral sense on* $\Omega(B, T')$ *if* $\int_0^t u_k(s)\, ds \to \int_0^t u(s)\, ds$ *for every* $t \in [0, T']$ *and we write:* $u_k \xrightarrow{I} u$ *on* $\Omega(B, T')$.

Clearly, if u_k converges weakly to u ($u_k \rightharpoonup u$), then $u_k \xrightarrow{I} u$ on $\Omega(B,T')$.

Definition 3.2 *Let F be an operator defined on $\Omega(B,T')$. We say that F is pseudo weakly continuous on $\Omega(B,T')$ if $F(u_k) \to F(u)$ whenever $u_k \xrightarrow{I} u$ on $\Omega(B,T')$.*

Using the above definition and Proposition 2.1, we obtain that the integral operator J defined by (10) is pseudo weakly continuous on $\Omega(B,T')$.

Proposition 3.1 *Let r be a continuous real function from $[0,T']$ into $[0,T']$ and consider the functional*

$$\psi(t,u) = \int_0^{r(t)} u(s)\, ds. \tag{19}$$

If $u_k \xrightarrow{I} u$ on $\Omega(B,T')$ then $\{\psi(t,u_k)\}$ converges uniformly to $\psi(t,u)$ on $[0,T']$.

Proof For any $\epsilon > 0$ and fixed t, let $K(t)$ be an index such that:

$$|\psi(t,u_k) - \psi(t,u)| \leq \epsilon$$

for all $k \geq K(t)$. Since r is continuous on $[0,T']$ and $\Omega(B,T')$ is bounded, for every instant t there is an open neighborhood $N(t) \subset [0,T']$ such that for any t' in $N(t)$ we have $|\psi(t,u_k) - \psi(t',u_k)| \leq \epsilon$ and $|\psi(t,u) - \psi(t',u)| \leq \epsilon$. Therefore we have $|\psi(t',u) - \psi(t',u_k)| \leq 3\epsilon$ for all t' in $N(t)$ and for all $k \geq K(t)$. By the Heine-Borel covering theorem (see page 42 of [15]), there exists a finite subcover of the compact set $[0,T']$ by elements $N(t)$. Therefore, for any $\epsilon > 0$, there exists a number K_ϵ such that, for $k \geq K_\epsilon$ and t in $[0,T']$:

$$|\psi(t,u) - \psi(t,u_k)| \leq 3\epsilon,$$

i.e., $\psi(t,u_k)$ converges uniformly to $\psi(t,u)$ on $[0,T']$. □

3.2 Pseudo weak continuity of the link flow operator

We are now in position to study the properties of the link flow operator x. To simplify the notation we will temporarily omit the subscript 'a'. Consider a sequence $\{u_k\} = \{(u_k^p)_{p\in P}\}$ pseudo weakly converging to u on $\Omega(B,T'))^{|P|}$. We will show that, for sufficient large k, there is a bijection between the 'breakpoints' in the link flow nested formulations (17) of u_k and those of u. As a consequence, $N_{u_k} = N_u$. For given u_k, the function $x(u_k(t))$ can be expressed as:

$$x(u_k(t)) = \begin{cases} x_{0k}(t) = \sum_{p\in P} \int_0^t u_k^p(s)\, ds & \text{if } \ t \in [0,t_0] \\ x_{1k}(t) = \sum_{p\in P} \int_{\tau_{0k}^{-1}(t)}^t u_k^p(s)\, ds & \text{if } \ t \in [t_0, t_{1k}] \\ \quad\vdots & \ \vdots \qquad \vdots \\ x_{N_{u_k}}(t) = \sum_{p\in P} \int_{\tau_{N_{u_k}-1}^{-1}(t)}^t u_k^p(s)\, ds & \text{if } \ t \in [t_{N_{u_k}-1}, t_{N_{u_k}}] \end{cases}$$

where $\tau_{0k}(t) = t + D(x_{0k}(t))$, $t \in [0,t_0]$, $t_{ik} = \tau_{(i-1)k}(t_{(i-1)k})$, $\tau_{ik}(t) = t + D(x_{ik}(t))$, $t \in [t_{(i-1)k}, t_{ik}]$, $i = 1,...,N_{u_k}$ and $t_{N_{u_k}} = \tau_{N_{u_k}-1}(T')$.

Proposition 3.2 *Let $f_k \in L^2[0,T']$ be a sequence of differentiable functions with uniformly bounded derivatives that converges towards a continuous function $f \in L^2[0,T']$. Then $|f_k(t) - f(t)|$ converges uniformly to 0 on $[0,T']$.*

Proof (i) Convergence.

Suppose that $|f_k(t_1) - f(t_1)|$ does not converge to 0 for $t_1 \in [0, T']$, i.e., there exists a positive number ϵ_1 and a sequence $\{k'\}$ such that, for all k':

$$|f_{k'}(t_1) - f(t_1)| > \epsilon_1. \tag{20}$$

From the continuity of f, there exists a neighbourhood $N(t_1)$ of t_1 such that

$$|f(t_1) - f(t)| \leq \frac{\epsilon_1}{3} \quad \forall t \in N(t_1). \tag{21}$$

Since f_k is differentiable with uniformly bounded derivative, there exists a neighbourhood $N_1(t_1)$ of t_1 such that

$$|f_{k'}(t) - f_{k'}(t_1)| \leq \frac{\epsilon_1}{3}. \quad \forall t \in N_1(t_1).$$

However, for every $t \in N(t_1) \cap N_1(t_1)$,

$$|f_{k'}(t) - f(t)| \geq |f_{k'}(t_1) - f(t_1)| - |f(t_1) - f(t)| - |f_{k'}(t) - f_{k'}(t_1)|,$$

in contradiction with the convergence of $f_k \to f$ on $L^2[0, T']$.

(ii) Uniform convergence.

Assume, by contradiction, that there exists a positive number ϵ_2 and an infinite sequence $\{t(k')\}$ such that

$$|f_{k'}(t(k')) - f(t(k'))| > \epsilon_2. \tag{22}$$

Since $\{t(k')\}$ is bounded, there exists a subsequence $\{t(k'')\}$ converging to $t_2 \in [0, T']$. For sufficiently large values of the index k'', we have the inequalities

$$\begin{aligned} |f_{k''}(t(k'')) - f_{k''}(t_2)| &\leq \frac{\epsilon_2}{3} \\ |f(t_2) - f(t(k''))| &\leq \frac{\epsilon_2}{3} \\ |f_{k''}(t_2) - f(t_2)| &\leq \frac{\epsilon_2}{3}. \end{aligned}$$

The addition of the above inequalities contradicts (22). □

Theorem 3.2 *Assume that*

(i) *there exists a finite instant T' such that $u = (u^p)_{p \in P}$ and $u_k = (u_k^p)_{p \in P}$ are well defined, nonnegative, Lebesgue integrable and bounded from above by B on $[0, T']$;*

(ii) *the strong FIFO condition holds, with constant γ;*

(iii) *the arc traversal time function D is nonnegative, nondecreasing, differentiable and Lipschitz continuous with constant L_D;*

(iv) *$u_k^p \xrightarrow{I} u^p$, $\forall p$ on $\Omega(B, T')$.*

Then the operator sequence $\{x(u_k)\}$ converges to $x(u)$, i.e., x is pseudo weakly continuous on $(\Omega(B, T'))^{|P|}$ and $v_k^p \xrightarrow{I} v^p$ on $\Omega(B/\gamma, \tau(T'))$, where v^p is the exit flow rate corresponding the entry flow rate u^p.

Proof From the assumptions and Theorem 2.1, all functions involved are measurable and bounded, hence square integrable. Let us consider the interval $M_0 = [0, t_0]$. Since $u_k^p \xrightarrow{I} u^p$ on $\Omega(B, T')$, by Proposition 2.1, we have $x_{0k}^p \to x_0^p$ on M_0, i.e.,

$$\Delta_{0k}^p = \int_{M_0} |x_0^p(s) - x_{0k}^p(s)|^2 \, ds \to 0.$$

By Proposition 3.1, we have that $x_{0k}(t_0) \to x_0(t_0)$. But:

$$|\tau_0(t_0) - \tau_{0k}(t_0)| \leq L_D |x_0(t_0) - x_{0k}(t_0)|.$$

Thus $\tau_{0k}(t_0) \to \tau_0(t_0)$, i.e., $t_{1k} \to t_1$. Hence, for any $\delta > 0$, there exists an index K_0 such that for all $k \geq K_0$, we have

$$t_{1k} \in N_1 = (t_1 - \delta, t_1 + \delta). \tag{23}$$

Now consider the interval $M_1 = [t_0, t_1 - \delta]$ and set $t' = \tau_0(t)$. We have:

$$\begin{aligned}
|x_{1k}^p(t') - x_1^p(t')|^2 &\leq 2 \left| \int_{\tau_{0k}^{-1}(t')}^{\tau_0^{-1}(t')} u_k^p(s) \, ds \right|^2 + 2 \left| \int_{t'}^{\tau_0^{-1}(t')} (u^p(s) - u_k^p(s)) \, ds \right|^2 \\
&\leq 2B^2 |\tau_0^{-1}(t') - \tau_{0k}^{-1}(t')|^2 + 2 |\psi_0^p(t') - \psi_{0k}^p(t')|^2
\end{aligned}$$

where $\psi_0^p(t') = \int_{t'}^{\tau_0^{-1}(t')} u^p(s) \, ds$ and $\psi_{0k}^p(t') = \int_{t'}^{\tau_0^{-1}(t')} u_k^p(s) \, ds$. Thus:

$$\begin{aligned}
\Delta_{1k}^p &= \int_{M_1} |x_{1k}^p(t') - x_1^p(t')|^2 dt' \\
&\leq 2B^2 \int_{M_1} |\tau_0^{-1}(t') - \tau_{0k}^{-1}(t')|^2 dt' + 2 \int_{M_1} |\psi_0^p(t') - \psi_{0k}^p(t')|^2 dt' \\
&= 2\Delta_{1k}^{p1} + 2\Delta_{1k}^{p2}.
\end{aligned} \tag{24}$$

For $k \geq K_0$, the first term of (24) satisfies:

$$\begin{aligned}
\Delta_{1k}^{p1} &= B^2 \int_{\tau_0^{-1}(M_1)} |\tau_0^{-1}(\tau_0(t)) - \tau_{0k}^{-1}(\tau_0(t))|^2 \tau_0'(t) \, dt \\
&\leq B^2 (1 + L_D |P| B) \int_{\tau_0^{-1}(M_1)} |\tau_{0k}^{-1}(\tau_{0k}(t)) - \tau_{0k}^{-1}(\tau_0(t))|^2 \, dt \\
&\qquad \text{(since } 0 < \tau_0'(t) \leq 1 + L_D |P| B\text{)} \\
&\leq \frac{B^2 (1 + L_D |P| B)}{\gamma^2} \int_{\tau_0^{-1}(M_1)} |\tau_{0k}(t) - \tau_0(t)|^2 \, dt \\
&\qquad \text{(from the Lipschitz continuity of } \tau_{0k}^{-1}\text{)} \\
&\leq \frac{L_D^2 B^2 (1 + L_D |P| B)}{\gamma^2} \int_{\tau_0^{-1}(M_1)} |x_{0k}(t) - x_0(t)|^2 \, dt \\
&\qquad \text{(from the Lipschitz continuity of } D\text{)} \\
&\leq \frac{L_D^2 B^2 (1 + L_D |P| B)}{\gamma^2} \int_{M_0} |x_{0k}(t) - x_0(t)|^2 \, dt \\
&\qquad \text{(since } \tau_0^{-1}(M_1) \subset M_0\text{)} \\
&= \alpha \Delta_{0k}
\end{aligned}$$

where $\alpha = L_D^2 B^2 (1 + L_D |P| B) / \gamma^2$.

Let us estimate the second term of (24): $\Delta_{1k}^{p2} = \int_{M_1} |\psi_0^p(t') - \psi_k^p(t')|^2 dt'$. From Proposition 3.1, ψ_k^p converges uniformly to ψ_0^p on M_1 and $\Delta_{1k}^{p2} \to 0$. We conclude that for sufficiently large k, we have $\Delta_{1k}^p \leq \alpha\Delta_{0k} + \Delta_{1k}^{p2}$. Hence $\Delta_{1k}^p \to 0$ when $k \to +\infty$, i.e., $x_{1k}^p \to x_1^p$ on M_1 for every p. Moreover, we have that $x_{1k} \to x_1$ on M_1, i.e., $\Delta_{1k} = \int_{M_1} |x_1(s) - x_{1k}(s)|^2\, ds \to 0$. On the other hand, the Lipschitz continuity of x and Proposition 3.2 imply, for sufficiently large k, the relationships:

$$\begin{aligned}
|\tau_{1k}(t_1-\delta) - t_2| &\leq |(t_1-\delta) - t_1| + L_D|x_{1k}(t_1-\delta) - x_1(t_1)| \\
&\leq \delta + L_D(|x_1(t_1-\delta) - x_1(t_1)| + |x_{1k}(t_1-\delta) - x_1(t_1-\delta)|) \\
&\leq \delta + L_D|P|B(1+\frac{1}{\gamma})\delta + L_D\delta \\
&= [1 + L_D + L_D|P|B(1+1/\gamma)]\delta \\
&= \beta\delta \\
&= \delta_2.
\end{aligned}$$

As stated in the first part of this proof we have, for $k \geq K_0$ and $t_{1k} \in N_1$:

$$t_{1k} = \tau_{1k}(t_0) \leq t_1 + \delta.$$

Thus, for sufficiently large k we have:

$$[t_1+\delta, t_2-\delta_2] \subset [\tau_{1k}(t_0), \tau_{1k}(t_1-\delta)].$$

Letting $M_2 = [t_1+\delta, t_2-\delta_2]$ we get $\tau_1^{-1}(M_2) \subset M_1$. Furthermore:

$$\begin{aligned}
|x_{1k}(t_{1k}) - x_1(t_1)| &= |\sum_{p\in P}(\int_{\tau_{1k}^{-1}(t_{1k})}^{t_{1k}} u_k^p(s)\, ds - \int_{\tau_1^{-1}(t_1)}^{t_1} u^p(s)\, ds)| \\
&= |\sum_{p\in P}(\int_{t_0}^{t_{1k}} u_k^p(s)\, ds - \int_{t_0}^{t_1} u^p(s)\, ds)| \\
&\leq |\sum_{p\in P}(\int_{t_0}^{t_1} (u_k^p(s) - u^p(s))\, ds)| + B|P|\cdot|t_{1k} - t_1|.
\end{aligned}$$

Since, for every $p \in P$, $u_k^p \xrightarrow{I} u^p$ on $\Omega(B, T')$ and $t_{1k} \to t_1$, there follows: $x_{1k}(t_{1k}) \to x_1(t_1)$. Also:

$$\begin{aligned}
|t_{2k} - t_2| &= |\tau_{1k}(t_{1k}) - \tau_1(t_1)| \\
&\leq |t_{1k} - t_1| + L_D|x_{1k}(t_{1k}) - x_1(t_1)|
\end{aligned}$$

implies that $t_{2k} \to t_2$. We conclude that there exists $K_1 \geq K_0$ such that for every $k \geq K_1$:

$$\tau_1^{-1}(M_2) \subset M_1 \quad \text{and} \quad t_{2k} \in N_2 = [t_2-\delta_2, t_2+\delta_2]. \tag{25}$$

Consider the interval $M_2 = [t_1+\delta_1, t_2-\delta_2]$. Let $t'' = \tau_1(t')$. Using the same arguments as before, we obtain:

$$\begin{aligned}
\Delta_{2k}^p &= \int_{M_2} |x_{2k}^p(t'') - x_2^p(t'')|^2\, dt'' \\
&\leq 2B^2\int_{M_2} |\tau_1^{-1}(t'') - \tau_{1k}^{-1}(t'')|^2\, dt'' + 2\int_{M_2} |\psi_1^p(t'') - \psi_{1k}^p(t'')|^2\, dt'' \\
&= 2\Delta_{2k}^{p1} + 2\Delta_{2k}^{p2}
\end{aligned}$$

where $\psi_1^p(t'') = \int_{t''}^{\tau_1^{-1}(t'')} u^p(s)\, ds$ and $\psi_{1k}^p(t'') = \int_{t''}^{\tau_1^{-1}(t'')} u_k^p(s)\, ds$. We can also demonstrate that $\Delta_{2k}^p \to 0$ for every p and $\Delta_{2k} \to 0$. For sufficient large k, we have:

$$\begin{aligned} |\tau_{2k}(t_1+\delta) - t_2| &\leq \beta\delta = \delta_2 \\ |\tau_{2k}(t_2-\delta_2) - t_3| &\leq \beta\delta_2 = \delta_3. \end{aligned}$$

Letting $M_3 = [t_2 + \delta_2, t_3 - \delta_3]$ we get: $\tau_2^{-1}(M_3) \subset M_2$. We can also show that $t_{3k} \to t_3$. Thus we conclude that there exists $K_2 \geq K_1$ such that for any $k \geq K_2$, $\tau_2^{-1}(M_3) \subset M_2$ and $t_{3k} \in N_3 = [t_3 - \delta_3, t_3 + \delta_3]$.

We pursue the above process until $k = N_u$. The analysis of this recursive scheme shows that, for sufficiently large k, the number of partitions for u_k is equal to that for u, i.e., $N_{u_k} = N_u$. If δ satisfies $\max_{1\leq i\leq N_u} \beta^{i-1}\delta < D(0)/2$, we have $\bigcap_{i=0}^{N_u} M_i = \emptyset$. For any such positive constant δ, there exists K such that for any $k \geq K$, we have

$$\begin{aligned} \Delta_k^p &= \int_0^{\tau(T')} |x^p(t) - x_k^p(t)|^2 dt \\ &\leq \sum_{i=0}^{N_u} \Delta_{ik}^p + 2\sum_{i=1}^{N_u} \delta_i \\ &\leq \sum_{i=0}^{N_u} \Delta_{ik}^p + 2\delta\sum_{i=1}^{N_u} \beta^{i-1} \\ &\leq \sum_{i=0}^{N_u} \Delta_{ik}^p + m\delta \end{aligned}$$

where m is some positive constant and $\Delta_{ik}^p \to 0$ when $k \to \infty$. Finally, since δ is an arbitrarily small positive number, we conclude that $\Delta_k^p \to 0$ when $k \to +\infty$, i.e., $\|x_k^p - x^p\| \to 0$. We also note that $\|x_k - x\| \to 0$ and

$$\int_0^t (v_k^p(s) - v^p(s))\, ds = \int_0^t (u_k^p(s) - u^p(s))\, ds - (x_k^p(t) - x^p(t)) \tag{26}$$

for any t in $[0, \tau(T')]$. Since, for every $p \in P$, $x_k^p \to x^p$ and $u_k^p \xrightarrow{I} u^p$ on $\Omega(B, T')$, thus $v_k^p \xrightarrow{I} v^p$ on $\Omega(B/\gamma, \tau(T'))$. This completes the proof. □

Theorem 3.3 (weak continuity) *Assume that*

(i) *there exists a finite instant T such that the path flow departure rate functions h_p, $p \in P$, are well defined on Λ;*

(ii) *all links satisfy the strong FIFO condition with uniform constant γ on $[0, T]$;*

(iii) *the link traversal time functions D_a are nonnegative, nondecreasing, continuously differentiable and Lipschitz continuous (a.e.).*

Then the flow operator $x(h) = (x_a(h))_{a\in\mathcal{A}}$ is well defined and weakly continuous on Λ.

Proof The assumptions imply the existence of a number $\bar{B} > 0$ such that $h_p(t) \leq \bar{B}$ for all t in $[0, T]$ and all h_p in Λ. Let $h_k \rightharpoonup h$ on Λ and $\xi = \min\{t_{a0} : a \in A\} > 0$. It is clear that, for all link $a \in A$ and $p \in P_a$, $u_a^p(h(t))$ and $u_a^p(h_k(t))$ are well defined on the interval $[0, \xi]$, and $u_a^p(h_k) \xrightarrow{I} u_a^p(h)$ on $\Omega(\bar{B}, \xi)$. By Theorem 3.2, the functions $x_a(h_k)$ and $x_a(h)$ are well defined on $[0, \xi]$ while the functions $v_a^p(h_k)$ and $v_a^p(h), p \in P_a$, are well defined on $[0, \tau_a(\xi)]$ for all links $a \in A$ with $\tau_a(\xi) \geq 2\xi$. We have that $x_a(h_k) \to x_a(h)$ and $v_a^p(h_k) \xrightarrow{I} v_a^p(h)$ on $\Omega(\bar{B}/\gamma, \tau_a(\xi))$. Let $B_1 = \bar{B}/\gamma$. In obtaining the flow exit functions $v_a^p(h(t))$, $t \in [t_{a0}, \tau_a(\xi)]$, $p \in P_a, a \in A$, we simultaneously determined $u_a^p(h(t))$ and $u_a^p(h_k(t))$ on $[0, 2\xi]$. Applying again Theorem 3.2, we determine uniquely $x_a(h(t))$, $x_a(h_k(t))$, $\tau_a(h(t))$, $\tau_a(h_k(t))$ on $[0, 2\xi]$ and $v_a^p(h_k(t))$, $v_a^p(h(t))$ on $[0, \tau(2\xi))$ with $\tau_a(2\xi) \geq 3\xi$. Moreover we have that $v_a^p(h_k) \xrightarrow{I} v_a^p(h)$ on $\Omega(B_1/\gamma, \tau_a(2\xi))$. Let T' be the instant when the last vehicle exits the network. Since T' is finite, in at most $n = \lceil T'/\xi \rceil$ steps, we have determined $u_a^p(h(t))$ and $u_a^p(h_k(t))$ for all t in $[0, n\xi]$ and $u_a^p(h_k) \xrightarrow{I} u_a^p(h)$ on $\Omega(B_n, \tau_a(n\xi))$ with $\tau(n\xi) \geq (n+1)\xi$. Finally, we obtain the unique solution $x_a(h(t))$, $x_a(h_k(t))$ for t in $[0, T']$, and $x_a(h_k) \to x_a(h)$ when $h_k \rightharpoonup h$ on Λ. This completes the proof. □

4 The dynamic user equilibrium problem

In this section, we address the existence of a solution to the dynamic user equilibrium problem.

4.1 The path traversal time

Let $S_p(t, h)$ denote the traversal time for path p given that departure from the origin occurs at time t and that the path flow departure rate is $h(t)$ for t in $[0, T]$. Given a flow departure rate function $h \in \Lambda$, the traversal time $S_p(t, h)$ for users travelling on path $p = (a_{p1}, a_{p2}, ..., a_{pn_p})$ and leaving the origin node at time t is

$$S_p(t, h) = D_{p1}(x_{p1}(t)) + \sum_{i=2}^{n_p} D_{pi}(x_{pi}(\tau_{pi-1}(t))) \tag{27}$$

where $\tau_{pi}(t) = \tau_{pi-1}(t) + D_{pi}(x_{pi}(\tau_{pi-1}(t)))$ and $\tau_{p1}(t) = t + D_{p1}(x_{p1}(t))$. If all links satisfy the strong FIFO condition, then a similar condition holds for all paths of the network, i.e., $S_p(t, h) > S_p(t', h)$ whenever $t > t'$. Hence, from Theorem 2.1, $S_p(t, h)$ is measurable and square integrable. The corresponding path travel time operator is

$$S_p(h) = S_p(t, h) \tag{28}$$

and the network traversal time operator is defined as:

$$S(h) = (S_p(h))_{p \in \mathcal{P}} \quad . \tag{29}$$

Theorem 4.1 *Assume that there exists a finite instant T' such that*

(i) *all path flow departure rates $h_p(t)$ are well defined over Λ;*

(ii) *all links satisfy the strong FIFO condition with a uniform constant γ over $[0, T']$;*

(iii) *the link traversal time functions D_a are nonnegative, nondecreasing, differentiable and Lipschitz continuous (a.e.) over $[0, T']$.*

Then the network traversal time operator S is weakly continuous over Λ.

Proof We only need to prove the weak continuity of every operator $S_p(h)$ on Λ. From the assumptions, Theorem 3.3 holds and $x(h) = (x_a(h))_{a \in A}$ is well defined and weakly continuous over Λ. Since $\tau_{p1}(h(t)) = t + D_{p1}(x_{p1}(t))$, $\tau_{p1}(h)$ is weakly continuous over Λ. Moreover, $\tau_{p2}(h(t)) = \tau_{p1}(t) + D_{p2}(x_{p2}(\tau_{p1}(t)))$. Thus $\tau_{p2}(h)$ is also weakly continuous over Λ. A similar argument shows that $\tau_{p3}(h), ..., \tau_{pn_p}(h)$ are weakly continuous over Λ. The weak continuity of S_p follows from the continuity of D_a and the weak continuity of x_a. □

4.2 Definition of the dynamic user equilibrium model

In [16] and [17], a dynamic user equilibrium for the route choice problem with predetermined departure times (RC equilibrium) has been characterized as follows.

Definition 4.1 (RC equilibrium) *Let $h \in \Lambda$. Then h and the user cost function $S(t, h)$ form an RC equilibrium if, for almost every instant t and for every two paths associated with a given origin-destination pair, one has:*

$$S_p(t, h) > S_q(t, h) \Rightarrow h_p(t) = 0 \tag{30}$$

Smith and Wisten [17] have have characterized an RC equilibrium $h^* \in \Lambda$ as a solution of the variational inequality:

$$\sum_{r,s} \sum_{p \in P_{rs}} \int_0^T S_p(t, h^*)(h_p(s) - h_p^*(s))\, ds \geq 0 \quad \forall h(t) \in \Lambda \tag{31}$$

or, in compact form:

$$\langle S(h^*), h^* - h \rangle \leq 0 \qquad \forall h \in \Lambda. \tag{32}$$

4.3 Existence of solutions to the dynamic user equilibrium problem

In this section we adapt the technique developed in [11] to prove the existence of at least one solution to the variational inequality (32). The following definition will be required in the proof of the existence theorem.

Definition 4.2 (Lions [9]) *Let Λ be a subset of a reflexive Banach space V, V^* its dual space and J a mapping from Λ into V^*. We say that J is pseudomonotone (in the Lions sense) if J is bounded on Λ and, for all sequences $\{h_k\}$ of Λ converging weakly to h and such that:*

$$\limsup_k \langle J(h_k), h_k - h \rangle \leq 0$$

we have:

$$\liminf_k \langle J(h_k), h_k - y \rangle \geq \langle J(h), h - y \rangle \quad \forall y \in \Lambda.$$

Note that the former definition does not correspond to the standard definition of pseudomonotonicity for point-to-set mappings (see Karamardian and Schaible [7]). In Lions [9] it is shown that if J is pseudomonotone (in the Lions sense) on a nonempty, closed, convex and bounded subset Λ of a reflexive Banach space, then there exists at least one point h^* satisfying the variational inequality:

$$\langle S(h^*), h^* - h \rangle \leq 0 \quad \forall h \in \Lambda.$$

Theorem 4.2 (existence of an RC equilibrium) *Let T' be a finite instant such that, for all t in $[0, T']$, all path flow departure rate functions h_p are well defined on Λ, all links satisfy the strong FIFO condition with constant γ and the link traversal functions D_a are nonnegative, nondecreasing, differentiable and Lipschitz continuous (a.e.). Then the solution set of the infinite dimensional variational inequality* (32) *is nonempty.*

Proof We proceed to check the assumptions of Lions' existence theorem.

(i) By construction, the set Λ is closed, convex and bounded.

(ii) From Theorem 4.1, S is weakly continuous over Λ.

(iii) If S is not bounded, then there exists a sequence $\{h_k\}$ in Λ such that $\|S(h_k)\| \to +\infty$. But, since Λ is weakly compact, there is a subsequence $\{h_{n_k}\}$ converging weakly to h in Λ. From the weak continuity of S, we have $\|S(h_k) - S(h)\| \to 0$, a contradiction. We conclude that $S(h)$ must be bounded (a.e.).

(iv) Pseudomonotonicity of S: Let $\{h_k\}$ be a sequence in Λ converging weakly to h in Λ, and y an arbitrary element of Λ. We have:

$$\begin{aligned}\left[\int_0^T \langle S(h_k(t)) - S(h(t)), h_k(t) - y(t)\rangle_2 \, dt\right]^2 &\leq \|S(h_k) - S(h)\|^2 \int_0^T \|h_k(t) - y(t)\|_2^2 \, dt \\ &\leq \|S(h_k) - S(h)\|^2 \hat{D}^2,\end{aligned}$$

where $\hat{D}$ is the diameter of the bounded set Λ. From (ii), we get:

$$\lim_{k\to\infty} \langle S(h_k) - S(h), h_k - y\rangle = \lim_{k\to\infty} \int_0^T \langle S(h_k(t)) - S(h(t)), h_k(t) - y(t)\rangle_2 \, dt = 0$$

and:

$$\begin{aligned}\lim_{k\to\infty} \langle S(h_k), h_k - y\rangle &= \lim_{k\to\infty} [\langle S(h), h_k - y\rangle + \langle S(h_k) - S(h), h_k - y\rangle] \\ &= \langle S(h), h - y\rangle.\end{aligned}$$

This implies, clearly, that S is pseudomonotone (in the Lions sense) over Λ. □

5 A sufficient condition for strong FIFO

The FIFO and strong FIFO conditions are specified with respect to a time interval. A link satisfying the FIFO condition during period $[0, T_1]$ might not satisfy it over a longer period $[0, T_2]$. From the definition of link dynamics we have, during the initial period $[0, t_{a0}]$:

$$x_a^p(t) = \int_0^t u_a^p(s) \, ds.$$

and the strong FIFO condition will always prevail in the time interval $[0, t_{a0}]$, provided that the link traversal time functions D_a are positive. In practice, we are interested in the strong FIFO condition over the domain $[0, T']$ of link entry flow rates. Friesz et al. [5] have proved that the FIFO condition is met when the functions D_a are affine. We strengthen this result by showing that strong FIFO is satisfied as well.

Theorem 5.1 *Let T' be a finite instant such that, for all t in $[0, T']$, the function u_a^p ($p \in P_a$) is well defined, nonnegative, bounded from above by B_a and Lebesgue integrable. Assume further that $D_a(x_a) = \alpha_a x_a + \beta_a$, where α_a is nonnegative and β_a is positive. Then*

$$v_a^p(t) \leq 1/\alpha_a \quad \forall t \in [0, T']. \tag{33}$$

and strong FIFO holds with constant γ_a, i.e.

$$\tau_a'(t) \geq \alpha_a \sum_{p \in P_a} u_a^p(t) + \gamma_a, \quad \text{with } \gamma_a > 0. \tag{34}$$

Proof We will adapt the recursive technique devised by Friesz et al. in [5]. In the initial period $[0, t_{a0}]$, x_a is absolutely continuous and differentiable (a.e.) and so is τ_a, with:

$$\begin{aligned} \tau_a'(t) &= 1 + D_a'(x_a(t)) \sum_{p \in P_a} u_a^p(t) \\ &= 1 + \alpha_a \sum_{p \in P_a} u_a^p(t) \\ &> \alpha_a \sum_{p \in P_a} u_a^p(t) + \frac{1}{1 + \alpha_a |P_a| B_a}. \end{aligned}$$

Moreover we have that $v_a^p(t) = 0 \leq 1/\alpha_a$ over that time period. Now assume that, at the ith period $[t_{i-1}, t_i]$, x_a and τ_a are well defined (a.e.), absolutely continuous, differentiable, and that (33) and (34) hold with $\gamma_a = 1/(1 + \alpha_a |P_a| B_a)^i$. We want to prove that x_a, τ_a are also absolutely continuous and (33)–(34) hold for the $(i+1)$st time period $[t_i, t_{i+1}]$, with $\gamma_a = 1/(1 + \alpha_a |P_a| B_a)^{i+1}$, $t_i = \tau_a(t_{i-1})$ and $t_{i+1} = \tau_a(t_i)$. Since (34) holds and τ_a is absolutely continuous in the ith period, then τ_a^{-1} is well defined (by Lemma 2.1) and (Lipschitz continuous) absolutely continuous in the $(i+1)$st period (by Lemma 2.2). Using (2) and (14), we observe that

$$\begin{aligned} v_a^p(t) &= \frac{u_a^p(\tau_a^{-1}(t))}{\tau_a'(\tau_a^{-1}(t))} \\ &\leq \frac{u_a^p(\tau_a^{-1}(t))}{\alpha_a \sum_{p \in P_a} u_a^p(\tau_a^{-1}(t))} \\ &\leq 1/\alpha_a, \end{aligned} \tag{35}$$

i.e., (33) holds. From the knowledge of τ_a^{-1} in the $(i+1)$st period x_a and τ_a are well defined and absolutely continuous in the $(i+1)$st period. Furthermore, based on (2) and (14) we obtain:

$$\begin{aligned} \tau_a'(t) &= 1 + D_a'(x_a(t)) x_a'(t) \\ &= 1 + \alpha_a \sum_{p \in P_a} u_a^p(t) - \frac{\alpha_a \sum_{p \in P_a} u_a^p(\tau_a^{-1}(t))}{\tau_a'(\tau_a^{-1}(t))} \\ &\geq \alpha_a \sum_{p \in P_a} u_a^p(t) + \frac{\frac{1}{(1+\alpha_a |P_a| B_a)^i}}{\alpha_a \sum_{p \in P_a} u_a^p(\tau_a^{-1}(t)) + \frac{1}{(1+\alpha_a |P_a| B_a)^i}} \\ &\geq \alpha_a \sum_{p \in P_a} u_a^p(t) + \frac{1}{(1 + \alpha_a |P_a| B_a)^{i+1}} \end{aligned}$$

i.e., (34) holds with $\gamma_a = 1/(1 + \alpha_a |P_a| B_a)^{i+1}$. Since $t_i - t_{i-1} \geq t_{a0} = D_a(0) > 0$ for all i and T' is finite, the recursion must be finite. □

Corollary 5.1 *Let T' be a finite instant such that $h_p(t)$ is bounded by $\bar{B}$ for all p in P and all t in $[0, T']$ and assume that, for all a in A, $D_a(x_a) = \alpha_a x_a + \beta_a$, where α_a is nonnegative and β_a is positive. Then every link entry rate u_a^p is bounded by $B_a = \max\{\bar{B}, 1/\alpha\}$, where $\alpha = \min_{a\in A}\{\alpha_a\}$ and satisfies the strong FIFO condition with constant $(1+\max_{a\in A}\{\alpha_a|P_a|B_a\})^{-1}$ on the appropriate time period.*

Proof From (35), the link exit flow rate $v_a^p(t)$ is uniformly bounded by $1/\alpha_a$. Thus the link entry flow rate u_a^p is bounded by:

$$B_a = \max\{\bar{B}, 1/\alpha\}.$$

To demonstrate the strong FIFO condition, we only need to increase the system time index and apply repeatedly the argument used in the proofs of Theorem 2.3 and Theorem 5.1. □

Our final result deals with the case of nonlinear traversal time functions and states that strong FIFO will be satisfied if the derivative of the link traversal time function D_a is not too high.

Theorem 5.2 *Let T' be a finite instant such that, for all t in T' the functions u_a^p, $p \in P_a$ are well defined, nonnegative and Lebesgue integrable, and $\sum_{p\in P_a} u_a^p(t)$ is bounded from above by B_a ($B_a \geq 1$). Let the functions D_a be nonnegative, nondecreasing and differentiable with respect to x_a. If $D_a'(x_a) < 1/(B_a + \eta)$ for some positive number η, then the strong FIFO condition on arc a with constant $\eta/(B_a + \eta)$*

Proof We will prove, by induction, that

$$\sum_{p\in P_a} v_a^p(t) \leq B_a \tag{36}$$

and

$$\tau_a'(t) \geq \frac{\eta}{B_a + \eta}. \tag{37}$$

The second inequality implies that strong FIFO holds for link a.

In the initial period $[0, t_{a0}]$, x_a is absolutely continuous and differentiable (a.e.) and so is τ_a. We have

$$\tau_a'(t) = 1 + D_a'(x_a(t)) \sum_{p\in P_a} u_a^p(t) \geq 1 \geq \eta/(B_a + \eta)$$

and $\sum_{p\in P_a} v_a^p(t) = 0 \leq B_a$, trivially. Now suppose that, in the ith period $[t_{i-1}, t_i]$, x_a and τ_a are well defined, absolutely continuous and differentiable (a.e.) and that (37) and (36) hold. We want to prove that (37)–(36) are also satisfied in the $(i+1)$st period $[t_i, t_{i+1}]$, where $t_i = \tau_a(t_{i-1})$ and $t_{i+1} = \tau_a(t_i)$. Since, in the ith period, (37) holds and τ_a is absolutely continuous in the ith period, from Lemmas 2.1 and 2.2, τ_a^{-1} is well defined and absolutely (indeed Lipschitz) continuous in the $(i+1)$st period. Using Lemma 2.3, (2) and (14), we get:

$$\begin{aligned}
\sum_{p\in P_a} v_a^p(t) &= \frac{\sum_{p\in P_a} u_a^p(\tau_a^{-1}(t))}{\tau_a'(\tau_a^{-1}(t))} \\
&\leq \frac{[\sum_{p\in P_a} u_a^p(\tau_a^{-1}(t))](B_a + \eta)}{[\sum_{p\in P_a} u_a^p(\tau_a^{-1}(t))] + \eta} \\
&\leq B_a,
\end{aligned}$$

i.e., (36) holds. From the knowledge of τ_a^{-1} in the $(i+1)$st period, x_a and τ_a are well defined, absolutely continuous and differentiable (a.e.) in the $(i+1)$st period and:

$$\begin{aligned}
\tau_a'(t) &= 1 + D_a'(x_a(t))x_a'(t) \\
&\geq 1 + \min\{0, \frac{\sum_{p\in P_a}(u_a^p(t) - v_a^p(t))}{B_a+\eta}\} \\
&\geq \frac{[\sum_{p\in P_a} u_a^p(t)] + \eta}{B_a+\eta} \\
&\geq \frac{\eta}{B_a+\eta},
\end{aligned}$$

i.e., (37) holds. Finally, since $t_i - t_{i-1} \geq t_{a0} = D_a(0) > 0$ for every i and T' is finite, the recursion is finite. □

Corollary 5.2 *Let $\sum_{p\in P} h_p$ be bounded from above by $\bar{B}$ on $[0,T]$ and let:*

$$D_a'(x_a) \leq \frac{1}{|P_a|^n\bar{B}+\eta} \tag{38}$$

where $\eta > 0$, $n = T'/\min_{a\in A} D_a(0)$ and T' is the exit time of the last vehicle having entered the network before or at instant T. Then every total link entry flow is bounded by $|P_a|^n\bar{B}$ and satisfies the strong FIFO condition with constant $\eta/(|P_a|^n\bar{B}+\eta)$.

Proof The proof is similar to that of Corollary 5.1, i.e., we increase the time index and repeatedly make use of Theorem 5.2 and the argument used in the proof of Theorem 2.3. In contrast with the affine case, the bound on the total exit flow $\sum_{p\in P_a} v_a^p(t)$ now depends on the bound on the total entry flow rate $\sum_{p\in P_a} u_a^p(t)$. From the assumptions, we have: $D_a'(x_a) \leq 1/(|P_a|^n\bar{B}+\eta)$ and thus the total entry flow rate $\sum_{p\in P_a} u_a^p(t)$ is bounded by $B_a = |P_a|^n\bar{B}$ over $[0,T']$. Thus $\tau_a'(t) \geq \eta/(|P_a|^n\bar{B}+\eta)$ for t in $[0,T']$ and the result follows. □

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