A bilevel programming approach to the travelling salesman problem[1]

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Abstract

We show that the Travelling Salesman Problem is polynomially reducible to a bilevel toll optimization program. Based on natural bilevel programming techniques, we recover the lifted Miller-Tucker-Zemlin constraints. Next, we derive an $O(n^2)$ multi-commodity extension whose LP relaxation is comparable to the exponential formulation of Dantzig, Fulkerson and Johnson.

Key words: Bilevel Programming, Pricing, Travelling Salesman Problem

1 Introduction

The Travelling Salesman Problem (TSP) is a well-researched problem whose interest lies beyond the Icosian game or the Knight's tour puzzle. Due to its attractive name, practical importance and theoretical "intractability", the TSP has attracted a lot of attention from the Operations Research and Mathematical Programming communities (see Lawler et al. [7]). In this short work, we uncover a new facet of this problem, namely its close relationship with a toll collection problem modelled as a bilinear bilevel program. Not only is this analogy interesting in itself, but it also yields 'tight' lower bounds for the TSP.

The classical Travelling Salesman Problem consists in determining a least cost Hamiltonian circuit in a directed graph. Several mathematical programming formulations, either exponential or polynomial in terms of the number of nodes $|\mathcal{N}| = n$, have been proposed for the TSP (see Langevin, Soumis and Desrosiers [8]). In a seminal paper, Dantzig, Fulkerson and Johnson [2] proposed an exact formulation involving an exponential number of constraints. Later, $O(n^3)$ integer programming formulations yielding the same linear relaxation bounds were obtained by Claus [1] and Wong [11], among others.

In this paper we propose $O(n^2)$ formulations whose linear programming relaxations provide lower bounds of quality comparable to that obtained by Dantzig, Fulkerson and Johnson [2]. The basic formulation is derived from a bilevel programming model of revenue maximization (see Labbé et al. [6]) which allows, in a 'natural' fashion, to derive the lifted Miller-Tucker-Zemlin constraints obtained by Desrochers and Laporte [3]. Also, we strengthen the basic formulation by introducing commodities and demonstrate, through numerical experiments, the influence of the number of commodities on the quality of the lower bound.

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2 A toll optimization problem

A bilevel program describes a hierarchical structure where the feasible set of a leader is contingent on the follower's optimal reaction to the leader's decisions. In [6], Labbé et al. formulated the Toll Optimization Problem (TOP) as a bilevel program where the leader (upper level) sets tolls on a subset of arcs of a transportation network, while the followers (lower level) travel on shortest paths with respect to a generalized cost. More precisely, let us consider a multi-commodity network where each commodity $k \in \mathcal{K}$ is associated with an origin-destination pair $(o(k), d(k))$ of a transportation network G with node set $\mathcal{N}$ and arc set $\mathcal{A}$, the latter being partitioned into the subset $\mathcal{A}_1$ of toll arcs and the complementary subset $\mathcal{A}_2$ of toll-free arcs. With each arc a of $\mathcal{A}_1$ is associated a generalized travel cost composed of a fixed part c_a representing the unit travel cost and a toll T_a. Any toll-free arc a of $\mathcal{A}_2$ bears a fixed unit travel cost d_a.

We denote by $\{n^k\}_{k\in\mathcal{K}}$ the demand for commodity k, a negative demand corresponding to supply, and introduce the nodal demands

$$b_i^k = \begin{cases} -n^k & \text{if } i = o(k), \\ n^k & \text{if } i = d(k), \\ 0 & \text{otherwise.} \end{cases}$$

At the lower level, the variable x_a^k (respectively y_a^k) represents the number of commodity k users that travel on arc $a \in \mathcal{A}_1$ (respectively on arc $a \in \mathcal{A}_2$).

For a given toll vector T, we assume that users minimize their individual generalized travel costs, i.e., users are assigned to shortest paths linking their respective departure and arrival nodes. The corresponding lower level mathematical program, parameterized in the toll vector T, is

$$\begin{aligned} \min_{x,y} \quad & \sum_{k\in\mathcal{K}} \Big(\sum_{a\in\mathcal{A}_1} (c_a + T_a) x_a^k + \sum_{a\in\mathcal{A}_2} d_a y_a^k \Big) & (1) \\ \text{subject to} \quad & \sum_{a\in i^-} (x_a^k + y_a^k) - \sum_{a\in i^+} (x_a^k + y_a^k) = b_i^k \qquad \forall k \in \mathcal{K} \quad \forall i \in \mathcal{N}, \\ & x_a^k \geq 0 \qquad \forall k \in \mathcal{K} \quad \forall a \in \mathcal{A}_1, \\ & y_a^k \geq 0 \qquad \forall k \in \mathcal{K} \quad \forall a \in \mathcal{A}_2, \end{aligned}$$

where i^+ (respectively i^-) denotes the set of outgoing (respectively incoming) arcs incident to node i. A set of revenue-maximizing tolls is then obtained by solving the program

$$\begin{aligned} \max_{T} \quad & \sum_{a\in\mathcal{A}_1} T_a \sum_{k\in\mathcal{K}} x_a^k & (2) \\ \text{subject to} \quad & T_a^{\min} \leq T_a \leq T_a^{\max} \qquad \forall a \in \mathcal{A}_1, \end{aligned}$$

where it is understood that the flow variables are optimal solutions of the lower level program (1). Whenever the lower level solution is not unique, we assume that ties are broken in the leader's favour. This is in accordance with usual bilevel practice, and serves our theoretical purpose.

The membership of TOP in NP follows from a property shared with linear programming: if TOP has an optimal solution, then there exists an optimal solution which is a vertex of a suitably defined polyhedron P. In other words, TOP can be viewed as a combinatorial optimization problem defined over the vertices of P, rather than a continuous (nonconvex) mathematical program. This combinatorial problem has been shown to be NP-hard by Labbé et al [6].

3 Reformulating the TSP as a Toll Optimization Problem

In this section, we formulate the TSP as a single-commodity Toll Optimization Problem (TOP). This is achieved in two steps: first, we adapt the reduction from the Hamiltonian Path Problem to TOP proposed in [6] to the Hamiltonian Circuit Problem; next we extend this reduction technique to the TSP, taking the cost structure into account.

3.1 Equivalence of the Hamiltonian Circuit and Toll Optimization problems

Given a directed graph $G = (\mathcal{N}, \mathcal{A})$, the Hamiltonian Circuit Problem consists in determining an elementary circuit that passes through each node of $\mathcal{N}$. We define a TOP model by specifying

- its underlying graph;
- its cost structure;
- lower and upper bounds on tolls.

Since TOP involves paths rather than circuits, it is convenient to transform the Hamiltonian Circuit Problem into an equivalent Hamiltonian Path Problem. This is achieved by duplicating node 1 (its image is denoted $n+1$) and replacing, for i distinct from either the origin or destination node, arc $(i,1) \in \mathcal{A}$ by arc $(i, n+1)$. Looking for a Hamiltonian circuit in the original graph is equivalent to identifying a Hamiltonian path from node 1 to node $n+1$.

Let us identify the arc set $\overline{\mathcal{A}}_1$ of the modified graph with the set of toll arcs, and let us incorporate a *single* toll-free arc $(1, n+1)$. The resulting graph is denoted by $\overline{G} = (\overline{\mathcal{N}}, \overline{\mathcal{A}})$ with $\overline{\mathcal{N}} = \mathcal{N} \cup \{n+1\}$ and $\overline{\mathcal{A}} = \overline{\mathcal{A}}_1 \cup \{(1, n+1)\}$. We associate with the origin-destination pair $(1, n+1)$ a unit demand, with each toll arc $(i,j) \in \overline{\mathcal{A}}_1$ a fixed cost of -1, and set the cost of the toll-free arc $(1, n+1)$ to n. Finally, a lower bound of 2 is imposed on all tolls, and the upper bound is set to $+\infty$.

Our aim is to show that there exists a Hamiltonian circuit in G if and only if the optimal revenue of the above TOP is equal to $2n$. First, let us note that the revenue cannot exceed $2n$. Indeed, all arc costs (inclusive of toll) being positive, the lower level solution is an elementary path. Since fixed costs on toll arcs are set to -1 and that the cost of the shortest path is less or equal to n (there exists a toll-free path from node 1 to $n+1$ of cost n), an upper bound on the revenue associated with an optimal path using l arcs is given by

$$n - (l \times (-1)) = n + l.$$

This bound corresponds to the difference between the cost of the shortest toll-free path and the fixed cost of any path using l arcs. Hence, a revenue of $2n$ can only be generated by a shortest path using exactly n arcs, i.e., a Hamiltonian path.

Conversely, assume that there exists a Hamiltonian path p between nodes 1 and $n+1$. For each arc (i,j) on p, set the toll variables T_{ij} to their common lower bound 2, and to $n-1$ on arcs outside p. Under this cost structure, the shortest path is clearly p and the corresponding revenue is $2n$.

Note that the marginal benefit raised on any given toll arc decreases with increasing toll value. For instance, if the toll is set at its lower bound 2, the arc cost is equal to 1; this leads to a benefit/cost ratio of 2, while a toll set at 3 results in a ratio of 1.5. Intuitively, it is to the leader's advantage to set the toll as low as possible, thus inducing the users to travel on the longest possible path, e.g., a Hamiltonian path. Of course, all tolls on the arcs that are not part of the Hamiltonian path should be set at a value high enough to discourage their use. The argument relies heavily on the negativity of the fixed costs and on the lower bound on tolls, which prevents the benefit/cost ratio of growing infinite.

3.2 A TOP formulation of the TSP

By perturbing slightly the cost and bound structure in the TOP formulation of the Hamiltonian Path Problem, one may preserve the 'benefit/cost ratio' principle, while favoring the least cost paths. Accordingly, the users will travel on a long (Hamiltonian) path of lowest cost, i.e., an optimal Travelling Salesman tour. In the perturbed problem, the fixed cost of toll arc (i,j) is set to $-1 + c_{ij}/L$ and the lower bound on the corresponding toll is set at $2 - c_{ij}/L$, where L is an upper bound on the cost of an optimal tour, for instance $L = n \times \max_{(i,j)\in\mathcal{A}}\{c_{ij}\}$. This yields the bilevel program

$$
\begin{array}{llll}
\text{TOP-TSP:} & \max\limits_{T} & \sum\limits_{(i,j)\in\overline{\mathcal{A}}_1} T_{ij}x_{ij} & \\
& \text{subject to} & T_{ij} \geq 2 - \dfrac{c_{ij}}{L} & \forall (i,j) \in \overline{\mathcal{A}}_1, \\
& \min\limits_{x,y} & \sum\limits_{(i,j)\in\overline{\mathcal{A}}_1} ((-1 + \dfrac{c_{ij}}{L}) + T_{ij})x_{ij} + nx_{1(n+1)} & \\
& \text{subject to} & \sum\limits_{j|(j,i)\in\overline{\mathcal{A}}} x_{ji} - \sum\limits_{j|(i,j)\in\overline{\mathcal{A}}} x_{ij} = \begin{cases} -1 & \text{if } i = 1, \\ 1 & \text{if } i = n+1, \\ 0 & \text{otherwise,} \end{cases} & \\
& & x_{ij} \geq 0 & \forall (i,j) \in \overline{\mathcal{A}}.
\end{array}
$$

We now show that the optimal solution of TOP-TSP yields tolls that induce an assignment of lower level flows to a shortest Hamiltonian path. To this aim, let $p(T)$ denote an optimal path for a given toll vector T. If $p(T)$ uses l arcs, we have that

$$\sum_{(i,j)\in p(T)} (-1 + T_{ij} + \frac{c_{ij}}{L}) \leq n, \tag{3}$$

which yields the inequalities

$$n + l \geq \text{ revenue } = \sum_{(i,j)\in p(T)} T_{ij} \geq \sum_{(i,j)\in p(T)} (2 - \frac{c_{ij}}{L}) = 2l - \frac{1}{L}\sum_{(i,j)\in p(T)} c_{ij} > 2l - 1.$$

It follows that a revenue larger than $2n - 1$ can only be achieved by inducing the follower to travel on a path of length n, i.e., a Hamiltonian path $p_H(T)$. In order that the cost of this path be competitive with the cost of the toll-free path, the inequality (3) must hold. For a given Hamiltonian path $p_H(T)$, the maximal revenue is obtained when the equality

$$\sum_{(i,j)\in p_H(T)} T_{ij} = 2n - \frac{1}{L}\sum_{(i,j)\in p_H(T)} c_{ij}$$

holds. This can easily be achieved by setting the tolls on the arcs of p_H at their lower bound, and the remaining tolls at L. The optimal path corresponds, clearly, to a Hamiltonian path p^* of minimal cost $\sum_{(i,j)\in p^*} c_{ij}$ in the graph $\overline{G}$. In the original graph G, p^* corresponds to a Hamiltonian tour of minimum cost, i.e., a solution of the Travelling Salesman Problem. One possible choice for the toll vector T is, as indicated above

$$T^*_{ij} = \begin{cases} 2 - \dfrac{c_{ij}}{L} & \text{if } (i,j) \in p^*, \\ L & \text{if } (i,j) \notin p^*. \end{cases} \tag{4}$$

The choice of the constant L ensures that any alternative path has a cost higher than p^*.

3.3 A mixed integer formulation for the TSP

In this section, we reformulate TOP-TSP as a Mixed Integer Program (MIP) by performing standard operations on a bilevel program and obtain, as a consequence, the lifted version of the Miller-Tucker-Zemlin formulation of the TSP proposed by Desrochers and Laporte [3]. First, we replace the lower-level shortest path program by its optimality conditions, to derive the equivalent single level program:

$$
\begin{aligned}
\text{MIP:} \qquad & \max_{T,x,y} && \sum_{(i,j)\in\overline{\mathcal{A}}_1} T_{ij}x_{ij} \\
& \text{subject to} && \sum_{j|(j,i)\in\overline{\mathcal{A}}} x_{ji} - \sum_{j|(i,j)\in\overline{\mathcal{A}}} x_{ij} = \begin{cases} -1 & \text{if} \quad i = 1, \\ 1 & \text{if} \quad i = n+1, \\ 0 & \text{otherwise,} \end{cases} \\
& && \mu_j - \mu_i \leq -1 + T_{ij} + \frac{c_{ij}}{L} \qquad \forall (i,j) \in \overline{\mathcal{A}}_1, \\
& && \mu_{n+1} - \mu_1 \leq n, \\
& && (-1 + T_{ij} + \frac{c_{ij}}{L} + \mu_i - \mu_j)\, x_{ij} = 0 \qquad \forall (i,j) \in \overline{\mathcal{A}}_1, \\
& && (n + \mu_1 - \mu_{n+1})\, x_{1(n+1)} = 0, \\
& && T_{ij} \geq 2 - \frac{c_{ij}}{L} \qquad \forall (i,j) \in \overline{\mathcal{A}}_1, \\
& && x_{ij} \geq 0 \qquad \forall (i,j) \in \overline{\mathcal{A}}
\end{aligned}
\tag{5}
$$

where, without loss of generality, we set $\mu_1 = 0$, i.e., μ_i represents the shortest distance from node 1 to node i. At an optimal solution, we obviously have $\mu_{n+1} = n$ and $x_{1(n+1)} = 0$.

Based on the construction of Labbé et al [6] and on the fact that the flow variables x_{ij} are binary-valued for an (extremal) optimal path, we are able to propose a stronger linearization of the complementarity constraints without any need for additional variables. First, note that the choice of T^* specified in (4) implies

$$(T_{ij} - 1 + \frac{c_{ij}}{L})x_{ij} = x_{ij}$$

at the optimal solution. This allows to rewrite the complementarity term (5) as

$$(1 + \mu_i - \mu_j)x_{ij} = 0.$$

If $x_{ij} = 1$, this is equivalent to the reverse inequalities

$$\mu_j - \mu_i \leq x_{ij}, \tag{6}$$
$$\mu_j - \mu_i \geq x_{ij}. \tag{7}$$

Symmetrically, if $x_{ji} = 1$, we obtain

$$\mu_i - \mu_j \leq x_{ji} \tag{8}$$

and

$$\mu_i - \mu_j \geq x_{ji}. \tag{9}$$

Since x_{ij} and x_{ji} cannot be both nonzero, we can merge (7) and (8) into the single constraint

$$\mu_i - \mu_j \leq x_{ji} - x_{ij} + M(1 - x_{ij} - x_{ji}), \tag{10}$$

for some suitably large constant M. Let us consider the case where $i \neq n+1$ and $j \neq 1$. Since $\mu_i - \mu_j$ cannot exceed $n-2$, we can set $M = n-2$ to derive the valid inequality:

$$\mu_i - \mu_j \leq (n-2) + (1-n)x_{ij} + (3-n)x_{ji}. \tag{11}$$

Considering constraints (6) and (9) instead of (7) and (8) leads to the same constraint. Let us now consider the case where $i = 1$ or $j = n+1$. If $x_{1j} = 1$ and since $\mu_1 = 0$, there holds

$$\begin{aligned} \mu_j - 0 &\leq x_{1j}, && (12)\\ \mu_j - 0 &\geq x_{1j}. && (13) \end{aligned}$$

Similarly, if $x_{j(n+1)} = 1$ and since $\mu_{n+1} = n$, we obtain the inequalities

$$\begin{aligned} n - \mu_j &\leq x_{j(n+1)}, && (14)\\ n - \mu_j &\geq x_{j(n+1)}. && (15) \end{aligned}$$

Considering the constraints (12) and (15), we obtain

$$\mu_j \leq x_{1j} + (n-1)x_{j(n+1)} + M(1 - x_{1j} - x_{j(n+1)}). \tag{16}$$

Noticing that $\mu_j \leq n-2$ if $x_{1j} = 0$ and $x_{j(n+1)} = 0$, we derive

$$\mu_j \leq (n-2) + (3-n)x_{1j} + x_{j(n+1)}. \tag{17}$$

If we consider constraints (13) and (14), and following the same development, we obtain

$$\mu_j \geq (n-3)x_{j(n+1)} - x_{1j} + 2. \tag{18}$$

Finally, since the optimal toll vector induces a lower-level tour, one can replace the flow conservation constraints of MIP by the equivalent degree constraints (19) and (20). This yields the mixed-integer formulation

$$\begin{aligned}
\text{MIP-TSP:} \quad & \max_x \sum_{(i,j)\in\overline{\mathcal{A}}_1} (2 - \frac{c_{ij}}{L})x_{ij} \equiv \min_x \sum_{(i,j)\in\overline{\mathcal{A}}_1} c_{ij}x_{ij} \\
\text{subject to} \quad & \sum_{j|(i,j)\in\overline{\mathcal{A}}_1} x_{ij} = 1 \qquad \forall i \in \overline{\mathcal{N}}, && (19)\\
& \sum_{j|(j,i)\in\overline{\mathcal{A}}} x_{ji} = 1 \qquad \forall i \in \overline{\mathcal{N}}, && (20)\\
& \mu_i - \mu_j \leq (n-2) + (1-n)x_{ij} + (3-n)x_{ji} \qquad \forall (i,j) \in \overline{\mathcal{A}}_1, && (21)\\
& \mu_j \leq (n-2) + (3-n)x_{1j} + x_{j(n+1)} \qquad \forall j \in \{2,\dots,n\}, && (22)\\
& \mu_j \geq (n-3)x_{j(n+1)} - x_{1j} + 2 \qquad \forall j \in \{2,\dots,n\}, && (23)\\
& x_{ij} \in \{0,1\} \qquad \forall (i,j) \in \overline{\mathcal{A}}_1.
\end{aligned}$$

This single-commodity reformulation of the TSP is nothing but the lifted formulation of MTZ derived by Desrochers and Laporte [3]. The constraint (21) corresponds to the lifted subtour elimination constraint of MTZ, and defines a facet of the MTZ polytope. Constraints (22) and (23), which are obtained by lifting the bound constraints on the 'potentials' μ_i, have been proved to be facet-defining for MTZ by Driscoll [4].

3.4 A multi-commodity extension

In this section, we introduce a multi-commodity TOP reformulation of the TSP, where a tour is partitioned into contiguous simple paths, each path being associated with a commodity $k \in \mathcal{K}$. More precisely, let us denote by $o(k)$ the origin node of path (commodity) k and by $d(k)$ the destination node of path k. Assuming that we know the ordering of the commodities on some optimal tour, we can write: $o(k+1) = d(k)$ for $k = 1, \ldots, |\mathcal{K}| - 1$ and $d(|\mathcal{K}|) = o(1)$. To construct a TOP, we associate with each commodity k a variable $y^k_{o(k),d(k)}$ associated with the toll-free arc $(o(k), d(k))$. In contrast with the single-commodity case, the *fixed* cost of toll arcs are not known a priori. Let μ^k_i denote the potential associated with commodity k. In order that the formulation be valid, the following properties must hold:

1. for each commodity index k, the potential μ^k_i is equal to the number of arcs from the origin $o(k)$ to the destination $d(k)$ in the optimal tour;
2. the costs w^k of the toll-free arcs must sum up to n;
3. each arc must be used once and only once.

Following the same line of reasoning as for the single-commodity case, we derive the formulation:

$|\mathcal{K}|$-TOP-TSP:

$$\begin{aligned}
\max_{T,x,y,w} \quad & \sum_{k\in\mathcal{K}} \sum_{(i,j)\in\mathcal{A}} T_{ij} x^k_{ij} \\
\text{subject to} \quad & T_{ij} \geq 2 - \frac{c_{ij}}{L} && \forall (i,j) \in \mathcal{A}, \\
& \sum_{k\in\mathcal{K}} w^k = n, \\
& \sum_{k\in\mathcal{K}} \sum_{j|(i,j)\in\mathcal{A}} x^k_{ij} \leq 1 && \forall i \in \mathcal{N}, \\
& w^k \geq 0 && \forall k \in \mathcal{K},
\end{aligned}$$

$$\begin{aligned}
\min_{x,y} \quad & \sum_{k\in\mathcal{K}} \sum_{(i,j)\in\mathcal{A}} ((-1 + \frac{c_{ij}}{L}) + T_{ij}) x^k_{ij} + \sum_{k\in\mathcal{K}} w^k y^k_{o(k),d(k)}, \\
\text{subject to} \quad & \sum_{j|(j,i)\in\mathcal{A}} x^k_{ji} - \sum_{j|(i,j)\in\mathcal{A}} x^k_{ij} = \begin{cases} y^k_{o(k),d(k)} - 1 & \text{if } i = o(k), \\ -y^k_{o(k),d(k)} + 1 & \text{if } i = d(k), \\ 0 & \text{otherwise}, \end{cases} \\
& x^k_{ij} \geq 0 \qquad \forall (i,j) \in \mathcal{A} \quad \forall k \in \mathcal{K}, \\
& y^k_{o(k),d(k)} \geq 0 \qquad \forall k \in \mathcal{K}.
\end{aligned}$$

The arguments developed in Section 3.2 to validate the TOP-TSP model can be extended to the multi-commodity formulation. For commodity k, let $p^k(T)$ denote the shortest path associated with a toll vector T. If $p^k(T)$ uses l^k arcs, we have that

$$\sum_{(i,j)\in p^k(T)} (-1 + T_{ij} + \frac{c_{ij}}{L}) \leq w^k,$$

which yields the inequalities

$$w^k + l^k \geq \sum_{(i,j)\in p^k(T)} T_{ij} \geq \sum_{(i,j)\in p^k(T)} (2 - \frac{c_{ij}}{L}) = 2l^k - \frac{1}{L} \sum_{(i,j)\in p^k(T)} c_{ij}.$$

Summing over the commodity indices, we obtain

$$n + l \geq \sum_k \sum_{(i,j)\in p^k(T)} T_{ij} \geq \sum_k \sum_{(i,j)\in p^k(T)} (2 - \frac{c_{ij}}{L}) = 2l - \frac{1}{L} \sum_k \sum_{(i,j)\in p^k(T)} c_{ij} > 2l - 1.$$

where l is the total number of toll arcs used by *all* commodities. These inequalities, together with arguments similar to those of Section 3.2, allow us to affirm that an optimal tour p^* can be recovered by setting

$$T^*_{ij} = \begin{cases} 2 - \frac{c_{ij}}{L} & \text{if } (i,j) \in p^*, \\ L & \text{otherwise.} \end{cases}$$

Writing down the lower-level optimality conditions, eliminating toll-free variables $y^k_{o(k),d(k)}$ (which must vanish at the optimal solution) and the associated w^k variables, linearizing the complementarity constraints and determining tight bounds on L, we derive a mixed-integer formulation. Note that, in contrast with the single-commodity case, the value of $\mu^k_{o(k)}$ and $\mu^k_{d(k)}$ cannot be set a priori since the number of nodes on each commodity path is unknown a priori. However, we can either fix the value of $\mu^k_{o(k)}$ to zero or the value of $\mu^k_{d(k)}$ to $n - |\mathcal{K}| + 1$. Fixing the values of $\mu^k_{o(k)}$ to zero yields the single-level problem

$$\begin{aligned}
\max_{T,x,y} \quad & \sum_{k\in\mathcal{K}} \sum_{(i,j)\in\mathcal{A}} (2 - \frac{c_{ij}}{L}) x^k_{ij} \\
\text{subject to} \quad & \sum_{k\in\mathcal{K}} \sum_{j|(i,j)\in\mathcal{A}} x^k_{ij} = 1 \qquad \forall i \in \mathcal{N}, \\
& \sum_{j|(j,i)\in\mathcal{A}} x^k_{ij} - \sum_{j|(i,j)\in\mathcal{A}} x^k_{ji} = \begin{cases} -1 & \text{if } i = o(k), \\ 1 & \text{if } i = d(k), \\ 0 & \text{otherwise,} \end{cases} && (24) \\
& \mu^k_i - \mu^k_j \leq (n - |\mathcal{K}| - 1) + (|\mathcal{K}| - n) x^k_{ij} + (|\mathcal{K}| + 2 - n) x^k_{ji} \\
& \qquad\qquad \forall i \text{ and } j \text{ distinct from } o(k) \text{ or } d(k), \forall k \in \mathcal{K}, && (25) \\
& \mu^k_j \leq (n - |\mathcal{K}| - 1) + x^k_{jd(k)} - (n - |\mathcal{K}| - 2) x^k_{o(k)j} \\
& \qquad\qquad \forall j \text{ distinct from } o(k) \text{ or } d(k), \forall k \in \mathcal{K}, && (26) \\
& \mu^k_j \geq 2 - 2x^k_{jd(k)} - x^k_{o(k)j} \\
& \qquad\qquad \forall j \in \mathcal{N}, j \text{ distinct from } o(k) \text{ or } d(k), \forall k \in \mathcal{K}, && (27) \\
& x^k_{ij} \in \{0,1\} \qquad \forall (i,j) \in \mathcal{A} \quad \forall k \in \mathcal{K}.
\end{aligned}$$

On the other hand, had we fixed $\mu^k_{d(k)}$ to $n - |\mathcal{K}| + 1$, then constraints (26) and (27) would have been replaced by

$$\mu^k_j \leq (n - |\mathcal{K}| - 1) + x^k_{jd(k)} + 2x^k_{o(k)j} \;, \tag{28}$$

$$\mu^k_j \geq (n - |\mathcal{K}| - 2) x^k_{jd(k)} - x^k_{o(k)j} + 2. \tag{29}$$

One can reduce the number of variables by observing that each commodity uses distinct paths and never visit twice the same node. Hence the variables μ^k_i can be set to arbitrary values (in particular 0) at the nodes outside its path, or simply discarded. Finally, since an arc is used by at most one commodity, we can lift constraints (25–27) to obtain the streamlined formulation

$$\begin{aligned}
\max_{T,x,y} \quad & \sum_{k\in\mathcal{K}}\sum_{(i,j)\in\mathcal{A}} (2-\frac{c_{ij}}{L})x_{ij}^k \\
\text{subject to} \quad & \sum_{k\in\mathcal{K}}\sum_{j|(i,j)\in\mathcal{A}} x_{ij}^k = 1 \qquad \forall i\in\mathcal{N} \\
& \sum_{j|(j,i)\in\mathcal{A}} x_{ji}^k - \sum_{j|(i,j)\in\mathcal{A}} x_{ij}^k = \begin{cases} -1 & \text{if } i=o(k),\\ 1 & \text{if } i=d(k),\\ 0 & \text{otherwise,}\end{cases} & (30)\\
& \mu_i-\mu_j \le (n-|\mathcal{K}|-1)+(|\mathcal{K}|-n)\sum_{k\in\mathcal{K}} x_{ij}^k + (|\mathcal{K}|+2-n)\sum_{k\in\mathcal{K}} x_{ji}^k \\
& \qquad\qquad \forall\ i \text{ and } j \text{ neither an origin or destination node,} & (31)\\
& \mu_j \le (n-|\mathcal{K}|-1)+\sum_{k\in\mathcal{K}} x_{jd(k)}^k - (n-|\mathcal{K}|-2)\sum_{k\in\mathcal{K}} x_{o(k)j}^k \\
& \qquad\qquad \forall\ j \text{ neither an origin or destination node,} & (32)\\
& \mu_j \ge 2-2\sum_{k\in\mathcal{K}} x_{jd(k)}^k - \sum_{k\in\mathcal{K}} x_{o(k)j}^k \\
& \qquad\qquad \forall\ j \text{ neither an origin or destination node,} & (33)\\
& x_{ij}^k\in\{0,1\} \qquad \forall (i,j)\in\mathcal{A}, \forall k\in\mathcal{K}.
\end{aligned}$$

Alternatively, we could have lifted constraints (28) and (29).

Remark: The a priori knowledge of the ordering of commodities in $|\mathcal{K}|$ might seem a strong assumption. However, in the symmetric case, we can select any three nodes to construct a 3-TOP-TSP formulation of the original problem. Indeed, the symmetric cost structure implies that all six permutations are equivalent. Moreover, in the Euclidean case, it has been proved by Flood [5] that the optimal tour visits vertices of the convex envelope in the natural cyclic order, e.g., clockwise. Hence, if the nodes in $|\mathcal{K}|$ belong to the set of such vertices, only one $|\mathcal{K}|$-TOP-TSP, has to be solved to find a solution to the original Travelling Salesman Problem.

4 Numerical experiments

In order to assess the quality of our formulations, we solved the linear programming relaxations of TOP-TSP and 3-TOP-TSP. Our testbed is composed of three problem classes: random Euclidean, symmetric and asymmetric. The last two sets of problems are taken from the TSPLIB library [9].

The random problems consist of Euclidean TSPs of size 50 (complete graphs) where nodes, rounded to the nearest integer, are generated according to the procedure proposed by Desrochers and Laporte [3]:

$$\begin{aligned}
c_{ij} &= [(x_i-x_j)^2+(y_i-y_j)^2]^{1/2},\\
(x_i,y_i) &\sim U[0,50]^2 \qquad \text{for } i<j.
\end{aligned}$$

In Table 1 we present the linear relaxations for the TOP-TSP formulation (equivalent to the lifted version MTZ$^+$ of Miller, Tucker and Zemlin's formulation), the 3-TOP-TSP formulation, and the classical bound of Dantzig, Fulkerson and Johnson's (DFJ). In Table 1, the heading 'GAP' refers to the difference between the value of the DFJ relaxation and the value of the relaxation lower bound, expressed in percentages.

In Tables 2 and 3, 'GAP' is computed with respect to the optimal solution (OPT). For 3-TOP-TSP, the nodes $o(k)$, $k = 1, 2, 3$ have been selected such as to maximize the area of the triangle with vertices $o(1)$, $o(2)$ and $o(3)$.

Next, we consider Euclidean (symmetric) instances on complete graphs involving up to 200 nodes, drawn from the TSPLIB [9] library. Numerical results are reported in Table 2.

Finally, we consider three asymmetric instances from the TSPLIB [9] library that were also considered by Sherali and Driscoll [10]. Table 3 provides the linear relaxation values corresponding to the TOP-TSP, Sherali and Driscoll (SD) and 3-TOP-TSP formulations. These are compared with the optimal solution values (OPT). As before, GAP denotes the difference between the optimal value and the lower bound.

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1-TOP (MTZ^+)	GAP (%) vs. DFJ	3-TOP	GAP (%) vs. DFJ	DFJ
271.83	3.00	279.67	0.21	280.25
215	11.52	235	3.29	243
238.5	6.10	250	1.57	254
272.92	4.41	282	1.23	285.5
240	11.11	253	6.30	270
268.33	1.17	270	0.55	271.5
275.5	0.90	277	0.36	278
266.75	6.57	283.5	0.70	285.5
238.85	7.24	247.5	3.88	257.5
247.5	5.63	262	0.00	262
Average	5.76		1.81	
Std dev.	3.63		2.05	

Table 1: 50-node problems

Problem	1-TOP-TSP (MTZ$^+$)	GAP (%) vs. OPT	3-TOP-TSP	GAP(%) vs. OPT	DFJ	GAP (%) vs. OPT	OPT
gr17	1684	19.23	2085	0.00	2085	0.00	2085
gr21	2707	0.00	2707	0.00	2707	0.00	2707
gr24	1224.5	3.73	1272	0.00	1272	0.00	1272
bayg29	1544	4.10	1608	0.12	1608	0.12	1610
bays29	1954	3.27	1986.5	1.66	2013.5	0.32	2020
dantzig42	622	11.02	668	4.43	674	3.58	699
gr48	4768	5.51	4957.75	1.75	4959	1.72	5046
eil51	416.5	2.23	421.5	1.06	422.5	0.82	426
berl52	7163	5.03	7542	0.00	7542	0.00	7542
brazil58	20895	17.72	24200.5	4.70	25354.5	0.16	25395
st70	623.5	7.63	649	3.85	671	0.59	675
eil76	534	0.74	535	0.56	537	0.19	538
pr76	98994	8.47	104967.5	2.95	105120	2.81	108159
gr96	493.5	10.61	493.66	10.58	507.5	8.08	552.09
kroA100	19378.5	8.94	20686.5	2.80	20936.5	1.62	21282
kroB100	20339.5	8.14	21081	4.79	21834	1.39	22141
kroC100	19702.5	3.79	19994	2.37	20472.5	0.03	20479
kroD100	19949.1	6.32	20951.5	1.61	21141.5	0.72	21294
kroE100	20621.7	6.55	20911.1	5.24	21799.5	1.22	22068
eil101	619	1.59	623	0.95	627.5	0.24	629
gr120	6662.5	4.03	6810	1.90	6911.25	0.44	6942
bier127	112274	5.08	112478	4.91	117431	0.72	118282
gr137	658	5.80	692.5	0.86	693.5	0.72	698.53
kroA150	24839.9	6.35	25287.2	4.66	26299	0.85	26524
kroB150	24694	5.50	25220	3.48	25732.5	1.52	26130
kroA200	27052.5	7.88	27784.5	5.39	29065	1.03	29368
kroB200	27346.5	7.10	27981	4.95	29165	0.92	29437
Average		6.53		2.80		1.10	
Std. Dev.		4.40		2.45		1.64	

Table 2: Euclidean TSPLIB problems

Problem	1-TOP-TSP (MTZ$^+$)	GAP(%) vs. OPT	SD-LB	GAP(%) vs. OPT	3-TOP-TSP	GAP(%) vs. OPT	OPT
br17	22.0	43.59	27.7	28.97	39.0	0.00	39
p43	108.0	98.08	432.3	92.31	5602.0	0.32	5620
ry48p	13577.8	5.85	13602.5	5.68	14346.5	0.52	14422

Table 3: Asymmetric TSPLIB problems