Application of Dynamic Pricing to Retail and Supply Chain Management

Soulaymane Kachani

Columbia University
Kachani@ieor.columbia.edu

PLU 6000
Feb 6, 2004

University of Montreal
OUTLINE OF PRESENTATION

• The pricing challenge
• The practice of pricing
• A pricing model for retail
• A pricing model for supply chain management
• A fluid delay-based model for pricing and inventory management
• Summary
PRICING IS THE BEST LEVER FOR EARNINGS IMPROVEMENT…

Impact of price increase on operating profit

- **Improvement in price by 1% increases profitability more than 8.0%**
- **Price is the biggest profit improvement lever**
- **Price decreases are only offset by huge volume uplifts**

**Percent**
- Capture 1% price increase
- Profit increase of 8.0%

**Percent**
- Improving the lever by 1% delivers profit improvement of . . .

**Percent**
- Reducing prices by 5% requires a breakeven volume increase of 16.2%

**Percent**
- Source: Based on 2001 S&P 500 average economics
... AND SHOULD BE ON EVERY CEO AGENDA

Growth drivers

- Pricing
- New products
- New markets
- Extend product lines
- Mergers and acquisitions
• The pricing challenge
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• A pricing model for supply chains
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OVERALL OBJECTIVES OF PRICING IMPROVEMENT PROGRAMS

Achieve significant near-term improvements in profitability through enhanced price performance

Design and institutionalize comprehensive pricing management practices and processes to allow continued improvement into the future

Build systems, skills, incentives, etc. to support, enable, and sustain a high performing price management process

Achieve significant and sustainable gains in profitability through superior pricing management
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How can companies implement a consistent tactical pricing policy for increasingly dynamic markets?

Who are my customers, and what do they want?

Customer

What competing offers are they looking at?

Competition

What are my degrees of freedom to close the sale?

Cost

Tactical pricing How can I react quickly and correctly?
TACTICAL PRICING FRAMEWORK

Market impact

Determine price

Pricing algorithm

Customer

Competition

Cost

Multiple industry-specific solutions possible

Update model
CUSTOMER BEHAVIOR

- **Loyal customers**
  - Will not switch in linear region
  - Switching of loyal customers is highly non-linear
  - Switching has hysterisis (i.e., is not immediately and completely reversible)

- **Shared customers**
  - Will switch over linear region
  - Elasticities based on perceived differences in
    - Product
    - Services
    - Channel
    - Promotion
  - Switching behavior linearly dependent on cross-elasticity, price differential, and degree of awareness
  - Limited to small price band

All customers
**STATIC NON-LINEAR OPTIMIZATION AT CORE**

Profit

Margin earned/ lost with initial volume

$(p + \Delta p).V$

Net margin increase

$(p + \Delta p).(V + \Delta V)$

Margin earned/ lost with additional volume

$(p + \Delta p).\Delta V$

* Margin = Price - variable cost including channel compensation
PRICING MODEL

P: price  
C: cost  
V: volume  
b: awareness  
CC: shared customers  
E: Elasticity

\[
(P_i - C_i) \cdot \left( V_{i,0} - V_{R,ind} \cdot E_i (P_{ave} - P_{ave,0}) \pm \sum_j V_{R,comp} \cdot \beta \cdot CC_{ij} E_{ij} (P_i - P_j) \right)
\]

Full cost

Cost
- Variable cost
- Cash cost
- Full reinvestment cost

Customer

Competition
PRICING MODEL

P: price
C: cost
V: volume
b: awareness
CC: shared customers
E: Elasticity

Industry price
Awareness
Readiness
Attractiveness

\[
(P_i - C_i) \cdot \left( V_{i,0} - V_{R,ind} \cdot E_i (P_{ave} - P_{ave,0}) \pm \sum_j V_{R,comp} \cdot \beta \cdot CC_{ij} E_{ij} (P_i - P_j) \right)
\]

Customer
- Industry price
- Awareness
- Readiness
- Attractiveness

Cost

Competition
Pricing Model

\[
\left( P_i - C_i \right) \cdot \left( V_{i,0} - V_{R,ind} \cdot E_i \left( P_{ave} - P_{ave,0} \right) \right) \pm \sum_j V_{R,comp} \cdot \beta \cdot CC_{ij} \cdot E_{ij} \left( P_i - P_j \right)
\]

- **P**: price
- **C**: cost
- **V**: volume
- **b**: awareness
- **CC**: shared customers
- **E**: Elasticity

**Competition**
- Industry price
- Intangibles
- Price differential (net of competitive response)

**Cost**

**Customer**

**Intangibles**

**Industry price**

**Price differential**
**Update model**

- Update parameters to reduce difference between predicted and actual
- Several approaches possible (e.g., Kalman filters)

---

**Pricing algorithm**

**Market impact**

**Determine price**

---

Customer  
Competition  
Cost
IMPACT OF NEW PRICING POLICY

EBITDA impact, $ million

- 16% gain
- Average 1.5 c/gal increase
- Less than 3% volume loss

Current pricing policy: 166
Optimized for consumers: 9
Optimized around competitors: 18
New pricing policy: 193
INTRA-DAY SEGMENTATION

<table>
<thead>
<tr>
<th></th>
<th>Customers</th>
<th>Elasticity (light)</th>
<th>Elasticity (peak)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loyal</td>
<td>55%</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Shared, Mobil</td>
<td>29%</td>
<td>16%</td>
<td>12%</td>
</tr>
<tr>
<td>Shared, Texaco</td>
<td>14%</td>
<td>8%</td>
<td>6%</td>
</tr>
</tbody>
</table>

Diagram showing customer distribution and elasticity over different times of the day.
IMPACT OF IMPROVED NEW PRICING POLICY

EBITDA impact, $ million

- 22% gain
- Average 2.1 c/gal increase
- Less than 5% volume loss

<table>
<thead>
<tr>
<th>Current pricing policy</th>
<th>Optimized for consumers</th>
<th>Optimized around competitors</th>
<th>Accounting for time segments</th>
<th>New pricing policy</th>
</tr>
</thead>
<tbody>
<tr>
<td>166</td>
<td>9</td>
<td>18</td>
<td>9</td>
<td>202</td>
</tr>
</tbody>
</table>

Accounting for time segments
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BACKGROUND

Scope of the engagement
• Find improvement opportunities in key account management for a leading manufacturer (with 75% of market share) through better price management, without changing the existing mix-structure
• Leverage manufacturer-retailer relationship to develop win-win situations

Purpose of the analysis
• Improve manufacturer category profitability by helping the retailer improve its own category results through better pricing policies
• Identify optimal category price structures for selected categories within the retailer scope of action*

Analyses performed
• Multiple regression analyses to determine own-price and cross-price elasticities for each SKU,** using weekly price, volume, and promotional activity data for 2 sample stores
• Margin optimization process for each category in both stores, incorporating retailer list prices and manufacturer unit costs per SKU

End-products and impact
• Own-price and cross-price elasticities for the top 5 SKUs in each analyzed category
• Optimal pricing schemes, resulting in 10% margin improvement for the retailer and 6% for the manufacturer

* This analysis was limited to margin changes only by the retailer. To simulate changes involving the manufacturer price list, competitive reaction must be incorporated
** Stock keeping unit
When plotting raw price and volume data, no apparent correlation exists between the two variables. Prices and volumes have different time series properties. A simple log price vs. log volume regression in most cases will not be of much use. The complete data set for each regression requires competitor product prices, promotional activity dummy variables, and other qualitative variables, such as seasonality or stock-outs.
Econometric analysis has several advantages when it comes to estimating elasticity.* The general form of the log-price equation we used is:

$$\log Q_i = b_0 + b_i \log P_i + \sum_j c_j \log P_j + \sum_k d_k D_k$$

where:
- $Q_i$ is the volume sold and $P_i$ is the price of target SKU $i$
- $P_j$ is the price of competitor SKU $j$
- $b_i$ and $c_j$ are own-price and cross-price elasticities
- $D_k$ are dummy variables accounting for promotional activities, store location, seasonality, etc. with their corresponding coefficients $d_k$

A product's sales volume ($Q_i$) at a given point in time can be explained in terms of its own price ($P_i$), other competing product prices ($P_j$), relevant promotional activities, and other events, such as stock-outs and seasonal patterns ($D_k$).

Make sure the correlation among explanatory ("right-hand") variables is low, especially between continuous (i.e., price) and binary (i.e., "catalog") variables; keep only one of the highly correlated explanatory variables.

Use alternative model specifications* (linear demand function, deviation-from-mean model, etc.) to improve model fit.

Given its complexity, it is critical to involve client team members in this process. Client team members should be able to present model assumptions and results to management and thus step away from a conceptual black-box perspective.

### Margin Optimization Model's Process

**Input**
- Daily sales data
  - Prices
  - Volume
  - Cost of goods sold
- Promotional activity log
  - Inclusion in catalogs
  - Temporary exhibit
  - Special event
- Manufacturer margin
- Industry constraints
  - Market share
  - Price/brand positioning
  - Average category pricing

**Working Steps**
- Define the objective function(s)
- Enter demand and cost functions
- Add industry constraints
- Solve the model using a non-linear optimization software
- Stress-test the results
- Discuss recommendations with the retailer and validate results through pilot tests

**Output**

<table>
<thead>
<tr>
<th>SKU</th>
<th>Price change</th>
<th>Volume change</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-3.7%</td>
<td>19%</td>
</tr>
<tr>
<td>2</td>
<td>+6.6%</td>
<td>-25%</td>
</tr>
<tr>
<td>3</td>
<td>+1.2%</td>
<td>-5%</td>
</tr>
<tr>
<td>4</td>
<td>+5.0%</td>
<td>-23%</td>
</tr>
<tr>
<td>5</td>
<td>+0.2%</td>
<td>1%</td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td>-4%</td>
</tr>
</tbody>
</table>

- **Retailer margin ($ million)**
- **Manufacturer margin (MUS$)**
## OBJECTIVE FUNCTIONS

**Retailer**

Category gross margin \[=\] Category revenues* \(\times\) (selected store) \(-\) Category CGS

<table>
<thead>
<tr>
<th>SKU#</th>
<th>Price</th>
<th>Volume</th>
</tr>
</thead>
<tbody>
<tr>
<td>10111</td>
<td>$399</td>
<td>10,103</td>
</tr>
<tr>
<td>10213</td>
<td>$249</td>
<td>24,815</td>
</tr>
<tr>
<td>10207</td>
<td>$389</td>
<td>53,201</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

Volume calculated from the elasticity model assuming a "neutral" setting (i.e., all dummy variables set to zero)

**Manufacturer**

Category gross margin \[=\] Category revenues (selected store) \(-\) Manufacturing cost*

<table>
<thead>
<tr>
<th>SKU#</th>
<th>Price</th>
<th>Volume</th>
</tr>
</thead>
<tbody>
<tr>
<td>10111</td>
<td>$345</td>
<td>10,103</td>
</tr>
<tr>
<td>10112</td>
<td>$357</td>
<td>12,250</td>
</tr>
<tr>
<td>10115</td>
<td>$266</td>
<td>7,843</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

The objective function includes a sub-sample of the category mix

**EXAMPLE**

The retailer focuses on overall category contribution, whereas the manufacturer maximizes its own product mix contribution.

*If possible, trade spend and support should be added to the retailer’s category revenues and to the manufacturing cost. Trade spend and support includes rappel, volume discount, year-end bonuses, indirect discounts, fixed-trade spend, and cost-to-serve variable expenses.*
MATHEMATICAL PROBLEM SETTING

Optimize:
\[
\max \sum_{i} Q_i (P_i - C_i) \quad \text{and/or} \quad \max \sum_{i} Q_i (C_i - M_i)
\]

Subject to:
\[
\log Q_i = b_0 + b_i \log P_i + \sum_{j} c_j P_j \quad \forall i
\]
\[
Q_{\text{MAX}} \geq \sum_{i} Q_i \geq Q_{\text{MIN}}
\]
\[
P_1 > \$4.99
\]
\[
P_2 > P_1 \leq P_4
\]
\[
\frac{Q_4}{\sum_{i} Q_i} \geq 30\%
\]

Where: \(Q\) is volume in units sold, \(P\) is price per unit, \(C\) is retailer cost per unit (and manufacturer list price), \(M\) is manufacturer unit cost including trade spend, \(b_s\) are own-price elasticity estimates, and \(c_s\) are cross-price elasticity estimates.
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Observation:
- A newly produced unit of good incurs a sojourn time before being sold.

This sojourn time is similar to a travel time incurred in a transportation network.

Sojourn time incurred:
- This sojourn time depends on:
  - unit price,
  - level of inventory, and
  - competitors’ prices

Time the unit is produced

$10,000 / car

Time the unit is sold

This sojourn time is similar to a travel time incurred in a transportation network.
Introduction

Observation:
- A newly produced unit of good incurs a sojourn time before being sold
  - This sojourn time depends on unit price, competitors’ prices and level of inventory
  - This sojourn is similar to a travel time incurred in a transportation network

Contribution:
- Propose and study a dynamic pricing model:
  - Incorporates the delay of price and level of inventory in affecting demand
  - Includes pricing, production and inventory decisions in a multi-product environment

Approach:
- A transportation fluid dynamics model that incorporates:
  - Price/Inventory level delay function
  - Production and sales dynamics
  - Production capacity constraints

Goals:
- Apply analytical methodologies and solution algorithms borrowed from the transportation setting to inventory control and supply chain
- Capture a variety of insightful phenomena that are harder to capture using current models in the literature
Pricing theory has been extensively studied by researchers from a variety of fields:

- **Economics** (see for example R. Wilson (1993))
- **Marketing** (see for example G. Lilien et. Al (1992))
- **Revenue management and supply chain management** (see for example G. Bitran and S. Mondschein (1997), LMA. Chan et. al (2000), and J. McGill and G. Van Ryzin (1999))
- **Telecommunications** (see for example, F. P. Kelly (1994), F. P. Kelly et al. (1998), and, I. Paschalidis and J. Tsitsiklis (1998))
- **The book by Zipkin (1999), and references therein, provide a thorough review of inventory models.**
Assumptions and Notations

- We consider:
  - Stackelberg leader (Monopoly is a special case)
  - Many products
  - Common capacity
  - No substitution between products
  - Holding costs
  - No setup costs
  - Non-perishable products
  - Unit price is a function of inventory $p_i(I_i)$ (e.g. linear, hyperbolic)
  - Deterministic model.

- Average delay to sell a unit of good
  \[
  A_i(I_i(t)) = T_i(I_i(t), p_i(I_i(t), p_{i,1}(p_i(.)), p_{i,2}(p_i(.)), \ldots, p_{i,J(i)}(p_i(.)))
  \]
  Average time needed to sell, at time $t$, a unit of product $i$, given an inventory $I_i(t)$, a unit price $p_i(I_i(t))$, and competitors’ prices $p_{i,j}(p_i(.))$, $j \in \{1, \ldots, J(i)\}$.
  - Provide a methodology to estimate such a function in practice.
  - Establish connection with the travel functions derived in the transportation context.
### Assumptions and Notations

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U_i(t)$</td>
<td>cumulative production flow of product $i$ during interval $[0, t]$;</td>
</tr>
<tr>
<td>$u_i(t)$</td>
<td>production flow rate of product $i$ at time $t$;</td>
</tr>
<tr>
<td>$V_i(t)$</td>
<td>cumulative sales flow of product $i$ during interval $[0, t]$;</td>
</tr>
<tr>
<td>$v_i(t)$</td>
<td>sales flow rate of product $i$ at time $t$;</td>
</tr>
<tr>
<td>$I_i(t)$</td>
<td>inventory (number of units of product) $i$ at time $t$;</td>
</tr>
<tr>
<td>$p_i(I_i(t))$</td>
<td>sales price of one unit of product $i$ given an inventory $I_i(t)$;</td>
</tr>
<tr>
<td>$T_i(I_i(t), p_i(\cdot), (p_{i,j}^c(p_i(\cdot)), j \in {1, \ldots, J(i)}))$</td>
<td>time needed to sell both a unit of product $i$ produced at time $t$ and all inventory $I_i(t)$, given a unit price $p_i(I_i(t))$ and competitors prices $p_{i,j}^c(p_i(\cdot))$;</td>
</tr>
<tr>
<td>$D_i(\cdot)$</td>
<td>product delay function, a function of the inventory $I_i(t)$, for example, we can choose $D_i(I_i(t)) = T_i(I_i(t), p_i(\cdot), (p_{i,j}^c(p_i(\cdot)), j \in {1, \ldots, J(i)}))$;</td>
</tr>
<tr>
<td>$s_i(t)$</td>
<td>exit time of a production flow of product type $i$ entering at time $t$;</td>
</tr>
<tr>
<td></td>
<td>($s_i(t) = t + D_i(I_i(t))$);</td>
</tr>
<tr>
<td>$c_i(t)$</td>
<td>production cost of product $i$ at time $t$;</td>
</tr>
<tr>
<td>$h_i(t)$</td>
<td>inventory cost of product $i$ at time $t$;</td>
</tr>
<tr>
<td>$[0, T]$</td>
<td>production period. After time $T$, the company ceases producing;</td>
</tr>
<tr>
<td>$[0, T_{\infty}]$</td>
<td>analysis period. It is the interval of time from the instant when the first unit of product is produced to the first instant when all products have been sold.</td>
</tr>
</tbody>
</table>
Link dynamics equations
The link dynamics equations express the relationship between the flow variables of a link. They are given by:

\[
\frac{dI_i(t)}{dt} = u_i(t) - v_i(t), \quad \forall i \in \{1, \ldots, n\}.
\]
A Continuous-Time Fluid Dynamics Delay Model for Dynamic Pricing and Inventory Management

Link dynamics equations
The link dynamics equations express the relationship between the flow variables of a link. They are given by:

\[
\frac{dI_i(t)}{dt} = u_i(t) - v_i(t), \quad \forall i \in \{1, ..., n\}.
\]

Flow propagation equations
Flow propagation equations are used to describe the flow progression over time.

\[
V_i(t) = \int_{\omega \in W} u_i(\omega) d\omega, \quad \forall i \in \{1, ..., n\},
\]

where \( W = \{\omega : s_i(\omega) \leq t\} \).

If the product exit time functions \( s_i(\cdot) \) are continuous, and if the strict First-In-First-Out (FIFO) property is satisfied, then

\[
V_i(t) = \int_{n}^{s_i^{-1}(t)} u_i(\omega) d\omega, \quad \forall i \in \{1, ..., n\}.
\]
Link dynamics equations
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If the product exit time functions \(s_i(\cdot)\) are continuous, and if the strict First-In-First-Out (FIFO) property is satisfied, then

\[
V_i(t) = \int_{0}^{s_i^{-1}(t)} u_i(\omega) d\omega, \quad \forall i \in \{1, \ldots, n\}.
\]

Boundary equations

\[
U_i(0) = 0, \quad V_i(0) = 0, \quad I_i(0) = 0, \quad \forall i \in \{1, \ldots, n\}.
\]
Non-negativity and Capacity Constraints

\[ u_i(.) \geq 0, \forall i \in \{1, \ldots, n\}, \quad CFR(.) \geq 0. \]

\[ \sum_{i=1}^{n} u_i(t) \leq CFR(t). \]
Non-negativity and Capacity Constraints

\[ u_i(.) \geq 0, \forall i \in \{1, \ldots, n\}, \ CFR(.) \geq 0. \]
\[ \sum_{i=1}^{n} u_i(t) \leq CFR(t). \]

Objective function

The objective of the company is to maximize its profits. Profits are obtained by subtracting production costs and inventory costs from sales. The objective function can be expressed as:

\[ \sum_{i=1}^{n} \int_{0}^{T_{\infty}} p_i(I_i(t))v_i(t) - c_i(t)u_i(t) - h_i(t)I_i(t)dt. \]

In general, the DPM is a continuous-time non-linear program. In this system of equations, the known variables are the product delay functions \( A_i(.) \) and the shared capacity \( CFR(.) \). The unknown variables are \( u_i(t), U_i(t), v_i(t), V_i(t), I_i(t), s_i(t) \) and the parameters of \( p_i(I_i) \).

Feasibility conditions are similar to the Dynamic Network Loading (DNL) Problem in the dynamic traffic assignment context.

Extensive work done on the DNL problem, especially at CRT in Montreal.
Conclusions and Future Steps

Discretized DPM Model

Solution Algorithm

Objective Function:

\[ Obj = -\text{Min} \sum_{i=1}^{n} \left( k_i \left[ \sum_{j=0}^{N-1} u_{ij}u_{ij+1} + \sum_{j=0}^{N} u_{ij}^2 \right] + \sum_{j=0}^{N} g_{ij}u_{ij} \right) \]

Constraints:

\[ \sum_{i=0}^{n} u_{ij} \leq CFR_j, \quad \forall j \in \{0,1,\ldots,N\} \]

\[ u_{ij} \geq 0, \quad \forall i \in \{1,2,\ldots,n\}, \forall j \in \{0,1,\ldots,N\} \]

where \( g_{ij} = -\delta \left( p_i^{\max} - c_{ij} - \frac{h_{ij} + h_{ij+1}}{\delta} \right), \)

\[ k_i = \frac{\varepsilon_i \delta^2}{2}, \text{ and } \varepsilon_i = \frac{p_i^{\max} - p_i^{\min}}{C_i} \]

Approach:

\[ C_{ij} = -\frac{\partial Obj}{\partial u_{ij}} = 2k_iu_{ij} + k_i(u_{ij+1} + u_{ij-1}) + g_{ij} \]
\[ C_{ij} = 2k_i u_{ij} + k_i (u_{ij+1} + u_{ij-1}) + g_{ij} \]
Step 0: ($k=0$) for every $j \in \{0, \ldots, N\}$, for every $i \in \{1, \ldots, n\}$

$$u^0_{ij} = \frac{CFR_j}{n}$$

$k = 1$

Step $k$: for every $j \in \{0, \ldots, N\}$:

$$m^k_{ij} = k_i(u^k_{ij-1} + u^k_{ij+1}) + g_{ij}$$

$$C^k_{ij} = 2k_iu^k_{ij} + m^k_{ij}$$

We order the $m_{ij}$'s in non-decreasing order

$$m^k_{\text{order } 1, j} \leq m^k_{\text{order } 2, j} \leq \ldots \leq m^k_{\text{order } n, j}$$

Equilibration approach

Find $l_j : C^k_{\text{order } 1, j} = \ldots = C^k_{\text{order } l_j, j} = \alpha^k_{\text{order } l_j, j} \leq C^k_{\text{order } l_j+1, j} \leq \ldots \leq C^k_{\text{order } n, j}$

$$u^k_{\text{order } 1, j} > 0, \ldots, u^k_{\text{order } l_j, j} > 0, \sum_{i=1}^{l_j} u^k_{\text{order } i, j} = CFR_j, \quad u^k_{\text{order } l_j+1, j} = \ldots = u^k_{\text{order } n, j} = 0$$
Step k (continued): for every \( j \in \{0, \ldots, N\} \):

Let \( \alpha^k_{\text{order}(i, j)} = \frac{CFR_j + \sum_{m=1}^{i} \frac{m^k_{\text{order}(m, j)}}{2k^k_{\text{order}(m, j)}}}{\sum_{m=1}^{i} \frac{1}{2k^k_{\text{order}(m, j)}}} \)

Let \( l_j = \arg \min \{ i \in \{1, \ldots, n-1\} : \alpha^k_{\text{order}(i, j)} \leq m^k_{\text{order}(i+1, j)} \text{, if it exists} \}

If \( i > l_j \), \( u^k_{\text{order}(i, j)} = 0 \)

If \( i \leq l_j \), \( u^k_{\text{order}(i, j)} = \frac{a^k_{\text{order}(l_j, j)} - m^k_{\text{order}(i, j)}}{2k_{\text{order}(i, j)}} \).

**Convergence criterion:**

If \( u^k_{ij} = 0 \Rightarrow C^k_{ij} \geq \alpha^k_{\text{order}(l_j, j)} - \epsilon \), stop. Otherwise \( k = k + 1 \), go to step \( k \).

**Main result:** The iterative relaxation algorithm converges to the unique optimal solution.
Small Case Example

Discretized DPM Model

**Inputs:**
5 products, 10 discretization intervals

**Price/Inventory Relationship Parameters**

<table>
<thead>
<tr>
<th>$p_i^{\text{max}}$</th>
<th>$p_i^{\text{min}}$</th>
<th>Product 1</th>
<th>Product 2</th>
<th>Product 3</th>
<th>Product 4</th>
<th>Product 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>10.1</td>
<td>4.1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10.2</td>
<td>4.2</td>
<td></td>
<td></td>
<td></td>
<td></td>
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</tr>
<tr>
<td>10.3</td>
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<td>4.5</td>
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</table>

**Shared Capacity Flow Rate Function**

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<tr>
<td>$CFR_j$</td>
<td>19</td>
<td>21</td>
<td>23</td>
<td>25</td>
<td>27</td>
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<td>31</td>
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## Small Case Example

### Inputs (continued):

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<tbody>
<tr>
<td>$c_{3j}$</td>
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<td>4.4405</td>
<td>4.6481</td>
<td>4.8231</td>
<td>4.9772</td>
<td>5.1167</td>
<td>5.2449</td>
<td>5.3642</td>
<td>5.4762</td>
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<table>
<thead>
<tr>
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<th>4</th>
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<th>6</th>
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</tr>
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<tbody>
<tr>
<td>$h_{1j}$</td>
<td>1.6037</td>
<td>1.5135</td>
<td>1.4865</td>
<td>1.4236</td>
<td>1.4107</td>
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<td>1.3504</td>
<td>1.3013</td>
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<tr>
<td>$h_{2j}$</td>
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<td>1.2862</td>
<td>1.2482</td>
<td>1.1591</td>
<td>1.1409</td>
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<tr>
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<td>0.4532</td>
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<td>0.2085</td>
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### Discretized DPM Model

**Intermediary computations:**

#### Modified Margins

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<tbody>
<tr>
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<td><strong>0.1500</strong></td>
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#### Minimum Equilibrium Costs

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<tr>
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<td>$\alpha_{2j}$</td>
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### Optimal Production Flow Rates

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<tr>
<td>$u_{1j}$</td>
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<td>5.5334</td>
<td>2.0110</td>
<td>7.0729</td>
<td>1.8772</td>
</tr>
</tbody>
</table>

**Optimal Profit:** $\$405.0681$
Discretized DPM Model

Joint pricing and inventory management:

Optimal Profit as a function of the slope (i.e. \(s+s^2)/C\))

Optimal Profit: $419.4365

Optimal slope: 0.0551
In what follows, we express $p_i^l$ as $p_i^l(I_i(t), I_i^{-1}(t))$. This assumes:

- Knowledge of inventories of all players
- $\{p_i^l(I_i(t), p_i^{-1}), l=1,\ldots,K\}$ is “invertible”
Best Response Problem for Retailer $l$
Best Response Problem Retailer $l$

Max \( \sum_{i=1}^{n} \int_{0}^{T_{\infty}} p_{i}^{l}(I_{i}^{l}(t), I_{-i}^{l}(t)) v_{i}^{l}(t) - c_{i}^{l}(t) u_{i}^{l}(t) - h_{i}^{l}(t) I_{i}^{l}(t) \, dt \)

\( I_{i}^{l}(t) = U_{i}^{l}(t) - V_{i}^{l}(t) \quad \forall i \in \{1, \ldots, n\} \)

such that \( \sum_{i=1}^{n} u_{i}^{l}(t) \leq CFR^{l}(t) \)

\( u_{i}^{l}(.) \geq 0 \quad \forall i \in \{1, \ldots, n\} \)

\( I_{i}^{l}(0) = U_{i}^{l}(0) = V_{i}^{l}(0) = 0 \quad \forall i \in \{1, \ldots, n\} \)

If the product exit time functions $s_{i}^{l}(.)$ continuous, if strict FIFO holds, then

\( s_{i}^{l}(w) = w + D_{i}^{l}(I_{i}^{l}(w), I_{-i}^{l}(w)) \)

\( V_{i}^{l}(t) = \int_{0}^{s_{i}^{l-1}(t)} u_{i}^{l}(w) \, dw \quad \forall i \in \{1, \ldots, n\} \)
Discretized DPM Model

**Pricing function:**

\[ p_{i}^{\prime}(I_{i}(t)) = p_{i0}^{\prime}\max -\varepsilon_{i}^{\prime}I_{i}^{\prime}(t) + \sum_{k \neq i} \phi_{i}^{k}I_{i}^{k}(t) \]

\[ \varepsilon_{i}^{\prime} > 0 \]

**Discretization:** intervals of length \( \delta/M \) where \( M \) : discretization accuracy

For every discretization interval \( j \in \{0, 1, \ldots, (N+1)M - 1\} \), and

for every \( t \in [j\delta/M, (j+1)\delta/M) \):

\[ CFR(t) = CFR_{j}, \quad u_{i}(t) = u_{ij}, \quad c_{i}(t) = c_{ij}, \quad \text{and} \quad h_{i}(t) = h_{ij} \]

**Decision variables:**

Production levels: \( u_{ij} \) for every product \( i \) and for every discretization interval index \( j \)

For simplicity of the presentation, we consider \( M=1 \)
Best Response – Retailer $l$

$$
\text{Obj}^l = - \text{Min} \sum_{i=1}^{n} (k_i^l \left[ \sum_{j=0}^{N-1} u_{ij}^l u_{ij+1}^l + \sum_{j=0}^{N} u_{ij}^{l\,2} \right] + \sum_{j=0}^{N} g_{ij}^l u_{ij}^l

- \sum_{k \neq l} l_i^k \left[ \sum_{j=0}^{N-1} u_{ij}^l u_{ij+1}^k + \sum_{j=0}^{N} u_{ij}^l u_{kj}^k \right]) \right)
$$

such that

$$
\sum_{i=0}^{n} u_{ij}^l \leq \text{CFR}_j^l, \; \forall j \in \{0,1,\ldots,N\}
$$

$$
u_{ij}^l \geq 0, \; \forall i \in \{1,2,\ldots,n\}, \; \forall j \in \{0,1,\ldots,N\}
$$

$$
p_i^l (I_i (t)) = p_i^{l\,\text{max}} - \varepsilon_i^l I_i^l (t) + \sum_{k \neq l} \Phi_i^k I_i^k (t)
$$

$$
k_i^l = \frac{\varepsilon_i^l \delta^2}{2}
$$

$$
l_i^k = \frac{\Phi_i^k \delta^2}{2}
$$

$$
g_{ij}^l = -\delta \left( p_i^{l\,\text{max}} - c_{ij}^l - \frac{h_{ij}^l + h_{ij+1}^l}{2} \right)
$$
Best response model:

- The best response problem is a strictly convex quadratic problem.
- There exists a solution to the best response problem, and this solution is unique.

Nash equilibrium:

- If $\epsilon_i > \sum_{k=1}^{n} |\phi_i|^k$, there exists a Nash Equilibrium, and this equilibrium is unique.
Solution Algorithm

Main ideas behind the solution algorithm

- Non-separability by retailer is overcome using an iterative learning algorithm: outer loop
  - We start with initial production policies for every retailer
  - At each iteration, retailers solve the QP using information from past iteration about other retailers: inner loop

- In the inner loop, non-separability by time period and shared capacity constraint among products are overcome using an iterative relaxation algorithm
Solution Algorithm - **Inner Loop**: For each retailer

**Objective Function:**

\[
Obj = - \text{Min} \sum_{i=1}^{n} \left( k_i \left[ \sum_{j=0}^{N-1} u_{ij} u_{ij+1} + \sum_{j=0}^{N} u_{ij}^2 \right] + \sum_{j=0}^{N} g_{ij} u_{ij} \right)
\]

**Constraints:**

\[
\sum_{i=0}^{n} u_{ij} \leq CFR_j, \ \forall j \in \{0,1,...,N\}
\]

\[
u_{ij} \geq 0, \ \forall i \in \{1,2,...,n\}, \ \forall j \in \{0,1,...,N\}
\]

where \( g_{ij} = -\delta \left( p_i^\text{max} - c_{ij} - \frac{h_{ij} + h_{ij+1}}{2} \delta \right) - \sum_{k \text{ competitors}} l_i^k (u_{ij}^k + u_{ij+1}^k) \)

**Approach:**

\[
C_{ij} = -\frac{\partial Obj}{\partial u_{ij}} = 2k_i u_{ij} + k_i (u_{ij+1} + u_{ij-1}) + g_{ij}
\]
Solution Algorithm - **Inner Loop**: For each retailer

**Step 0**: \(k=0\) for every \(j \in \{0, \ldots, N\}\), for every \(i \in \{1, \ldots, n\}\)

\[
u_{ij}^0 = \frac{CFR_j}{n}
\]

**k = 1**

**Step k**: for every \(j \in \{0, \ldots, N\}\):

\[
m_{ij}^k = k_i(u_{ij}^k + u_{ij+1}^{k-1}) + g_{ij}
\]

\[
C_{ij}^k = 2k_iu_{ij}^k + m_{ij}^k
\]

We order the \(m_{ij}\)'s in non-decreasing order

\[
m_{\text{order } (1,j)}^k \leq m_{\text{order } (2,j)}^k \leq \ldots \leq m_{\text{order } (n,j)}^k
\]

**Equilibration approach**

Find \(l_j\) such that:

\[
C_{\text{order } (1,j)}^k = \ldots = C_{\text{order } (l_j,j)}^k = \alpha_{\text{order } (l_j,j)}^k \leq C_{\text{order } (l_{j+1},j)}^k \leq \ldots \leq C_{\text{order } (n,j)}^k
\]

\[
u_{\text{order } (1,j)}^k > 0, \ldots, \nu_{\text{order } (l_j,j)}^k > 0, \sum_{i=1}^{l_j} \nu_{\text{order } (i,j)}^k = CFR_j, \nu_{\text{order } (l_{j+1},j)}^k = \ldots = \nu_{\text{order } (n,j)}^k = 0
\]
Solution Algorithm - **Inner Loop**: For each retailer

Step k (continued): for every $j \in \{0, \ldots, N\}$:

Let $\alpha^k_{order(i,j)} = \frac{CFR_j + \sum_{m=1}^{i} \frac{m^k_{order(m,j)}}{2k_{order(m,j)}}}{\sum_{m=1}^{i} \frac{1}{2k_{order(m,j)}}}$

Let $l_j = \left\{ \begin{array}{ll} \arg\min\{i \in \{1,\ldots,n-1\} : \alpha^k_{order(i,j)} \leq m^k_{order(i+1,j)} \text{, if it exists} \} & \text{if it exists} \\ n, \text{ otherwise} & \text{otherwise} \end{array} \right.$

If $i > l_j$, $u^k_{order(i,j)} = 0$

If $i \leq l_j$, $u^k_{order(i,j)} = \frac{a^k_{order(l,j),j} - m^k_{order(i,j)}}{2k_{order(i,j)}}$.

**Convergence criterion:**

If $u_{ij}^k = 0$ $\Rightarrow C_{ij}^k \geq \alpha^k_{order(l_j,j)} - \varepsilon$, stop. Otherwise $k = k + 1$, go to step k.

**Main results:**

- The Iterative Relaxation Algorithm converges to the unique optimal solution of the inner-loop problem
- The Iterative Learning&Relaxation Algorithm converges to the the unique Nash Equilibrium
OUTLINE OF PRESENTATION

• The pricing challenge
• The practice of pricing
• A pricing model for retail
• A pricing model for supply chain management
• A fluid delay-based model for pricing and inventory management
• Summary
Questions?

Thank You