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Systems and Replication

Review and Analysis of Solutions of the Three Point Perspective Pose Estimation Problem

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Abstract. In this paper, the major direct solutions to the three point perspective pose estimation problems are reviewed from a unified perspective beginning with the first solution which was published in 1841 by a German mathematician, continuing through the solutions published in the German and then American photogrammetry literature, and most recently in the current computer vision literature. The numerical stability of these three point perspective solutions are also discussed. We show that even in case where the solution is not near the geometric unstable region, considerable care must be exercised in the calculation. Depending on the order of the substitutions utilized, the relative error can change over a thousand to one. This difference is due entirely to the way the calculations are performed and not due to any geometric structural instability of any problem instance. We present an analysis method which produces a numerically stable calculation.

1 Introduction

Given the perspective projection of three points constituting the vertices of a known triangle in 3D space, it is possible to determine the position of each of the vertices. There may be as many as four possible solutions for point positions in front of the center of perspectivity and four corresponding solutions whose point positions are behind the center of perspectivity. In photogrammetry, this problem is called the three point space resection problem.

This problem is important in photogrammetry as well as in computer vision, because it has a variety of applications, such as camera calibration (Tsai 1987), object recognition, robot picking, and robot navigation (Linnainmaa et al. 1988; Horaud 1987; Lowe 1987; Dhome 1988) in computer vision and the determination of the location in space from a set of landmarks appearing in the image in photogrammetry (Fischler and Bolles 1981). Three points is the minimal information to solve such a problem. It was solved by a direct solution first by a German mathematician in 1841 (Grunert 1841) and then refined by German photogrammatrists in 1904

and 1925 (Müller 1925). Then it was independently solved by an American photogrammatrist in 1949 (Merritt 1949).

The importance of the direct solution became less important to the photogrammetry community with the advent of iterative solutions which could be done by computer. The iterative solution technique which was first published by Church (1945, 1948), needs a good starting value which constitutes an approximate solution. In most photogrammetry situations scale and distances are known to within 10% and angle is known to within 15°. This is good enough for the iterative technique which is just a repeated adjustment to the linearized equations. The technique can be found in many photogrammetry books such as Wolf (1974) or the Manual of Photogrammetry (Slama 1980).

The exact opposite is true for computer vision problems. Most of the time approximate solutions are not known so that the iterative method cannot be used. This makes the direct solution method more important in computer vision. Indeed, in 1981 the computer vision community independently derived its first direct solution (Fischler and Bolles 1981). And the community has produced a few more direct solutions since then.

In this paper, first, we give a consistent treatment of all the major direct solutions to the three point pose estimation problem. There is a bit of mathematical tedium in describing the various solutions, and perhaps it is worthwhile to put them all in one place so that another vision researcher can be saved from having to redo the tedium himself or herself. Then, we compare the differences of the algebraic derivations and discuss the singularity of all the solutions. In addition to determining the positions of the three vertices in the 3D camera coordinate system, it is desirable to determine the transformation function from a 3D world coordinate system to the 3D camera coordinate system, which is called the absolute orientation in photogrammetry. Though many solutions to the absolute orientation can be found in photogrammetry literature (Schut 1960; Schonemann 1966; 1970; Wolf 1974; Slama 1980; Horn 1988; and Haralick et al. 1989) we present a simple linear

solution in Appendix I to make t
perspective pose estimation solut

Second, we run experiments to
merical stability of each of these
to evaluate some analysis metho
Appendix II to improve the num
of the calculation. It is well-kno
ing errors accumulate with incre
of calculation and significantly m
kinds of operations. Furthermor
the order of using these equation
final solution affects the accurac
results. The results show that th
be improved by a factor of about
accumulation of rounding errors
gated into the calculation of the
tation problem. As a result the
very serious at the final stage. In
better sensors and higher image
numerical stability will play a n
role in the errors of computer vi

Finally, we summarize the resu
of thousands experiments which
merical behaviors of the six dif
techniques, the effect of the ord
manipulation, and the effectiven
methods. These results show t
sis techniques in Appendix II a
determining equation order man
source codes and documentatio
experiments in the paper is a
CDROM. The interested reade
request to the Intelligent Systems
the University of Washington.

2 The Problem Definition

Grunert (1841) appears to have
one to solve the problem. The so
is outlined by Müller (1925). Th
be set up in the following way whi
in Figure 1.

Let the unknown positions of t
of the known triangle be

$$p_1, p_2, \text{ and } p_3;\ p_i = \begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix},$$

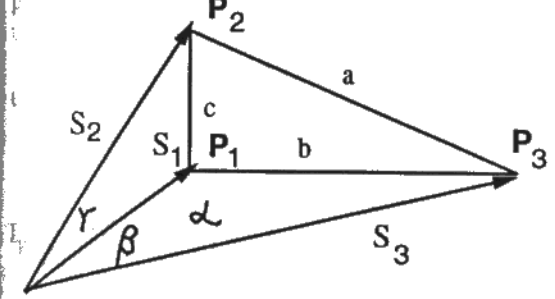

Fig. 1. Illustrates the geometry of the three point space resection problem. The triangle side lengths a, b and c are known and the unit vectors j_1, j_2, and j_3 are known. The problem is to determine the lengths s_1, s_2, and s_3 from which the 3D vertex point positions p_1, p_2, and p_3 can be immediately determined.

Let the known side lengths of the triangle be

$$a = \|p_2 - p_3\|$$
$$b = \|p_1 - p_3\|$$
$$c = \|p_1 - p_2\|.$$

We take the origin of the camera coordinate frame to be the center of perspectivity and the image projection plane to be a distance f in front of the center of perspectivity. Let the observed perspective projection of p_1, p_2, p_3 be q_1, q_2, q_3, respectively;

$$q_i = \begin{pmatrix} u_i \\ v_i \end{pmatrix}, \quad i = 1,2,3.$$

By the perspective equations,

$$u_i = f\frac{x_i}{z_i}$$
$$v_i = f\frac{y_i}{z_i}.$$

The unit vectors pointing from the center of perspectivity to the observed points p_1, p_2, p_3 are given by

$$\frac{1}{\sqrt{u_i^2 + v_i^2 + f^2}} \begin{pmatrix} u_i \\ v_i \\ f \end{pmatrix}, \quad i = 1,2,3$$

respectively. The center of perspectivity together with the three points of the 3D triangle form a tetrahedron. Let the angles at the center of perspectivity opposite sides a, b, γ. These angles are given by

$$\cos\alpha = j_2 \cdot j_3$$
$$\cos\beta = j_1 \cdot j_3$$
$$\cos\gamma = j_1 \cdot j_2$$

where j_1, j_2, and j_3 are unit vecto

$$j_1 = \frac{1}{\sqrt{u_1^2 + v_1^2 + f^2}} \begin{pmatrix} u \\ v \\ f \end{pmatrix}$$

$$j_2 = \frac{1}{\sqrt{u_2^2 + v_2^2 + f^2}} \begin{pmatrix} u \\ v \\ f \end{pmatrix}$$

$$j_3 = \frac{1}{\sqrt{u_3^2 + v_3^2 + f^2}} \begin{pmatrix} u \\ v \\ f \end{pmatrix}$$

Let the unknown distances c p_1, p_2, p_3 from the center of pe s_1, s_2, and s_3, respectively. Then $1,2,3$. To determine the position p_1, p_2, p_3 with respect to the cam frame, it is sufficient to determi s_3 since

$$p_i = s_i j_i, \quad i = 1,2,3$$

3 The Solutions

There are six solutions presente (1841), Finsterwalder (1937), M Fischler and Bolles (1981), Linn (1988), and Grafarend et al. (1 tively. In this section we first giv tion of the Grunert solution to s problem can be solved. Then, we are the major differences among derivation of these solutions befor detailed derivations for the rest Finally, we will give comparisons derivation among six solutions.

Grunert's Solution

Grunert proceeded in the followin law of cosines,

$$s_2^2 + s_3^2 - 2s_2 s_3 \cos\alpha =$$

$$s_1^2 + s_3^2 - 2s_1 s_3 \cos\beta = b^2 \quad (2)$$

$$s_1^2 + s_2^2 - 2s_1 s_2 \cos\gamma = c^2 \quad (3)$$

Let

$$s_2 = us_1 \text{ and } s_3 = vs_1. \quad (4)$$

Then

$$s_1^2(u^2 + v^2 - 2uv\cos\alpha) = a^2$$
$$s_1^2(1 + v^2 - 2v\cos\beta) = b^2$$
$$s_1^2(1 + u^2 - 2u\cos\gamma) = c^2.$$

Hence,

$$s_1^2 = \frac{a^2}{u^2 + v^2 - 2uv\cos\alpha} = \frac{b^2}{1 + v^2 - 2v\cos\beta} = \frac{c^2}{1 + u^2 - 2u\cos\gamma} \quad (5)$$

from which

$$u^2 + \frac{b^2 - a^2}{b^2}v^2 - 2uv\cos\alpha + \frac{2a^2}{b^2}v\cos\beta - \frac{a^2}{b^2} = 0 \quad (6)$$

$$u^2 - \frac{c^2}{b^2}v^2 + 2v\frac{c^2}{b^2}\cos\beta - 2u\cos\gamma + \frac{b^2 - c^2}{b^2} = 0. \quad (7)$$

From equation (6)

$$u^2 = -\frac{b^2 - a^2}{b^2}v^2 + 2uv\cos\alpha - \frac{2a^2}{b^2}v\cos\beta + \frac{a^2}{b^2}.$$

This expression for u^2 can be substituted into equation (7). This permits a solution for u to be obtained in terms of v.

$$u = \frac{\left(-1+\frac{a^2-c^2}{b^2}\right)v^2 - 2\left(\frac{a^2-c^2}{b^2}\right)\cos\beta v + 1 + \frac{a^2-c^2}{b^2}}{2(\cos\gamma - v\cos\alpha)} \quad (8)$$

This expression for u can then be substituted back into the equation (6) to obtain a fourth order polynomial in v.

$$A_4v^4 + A_3v^3 + A_2v^2 + A_1v + A_0 = 0 \quad (9)$$

where

$$A_4 = \left(\frac{a^2 - c^2}{b^2} - 1\right)^2 - \frac{4c^2}{b^2}\cos^2\alpha$$

$$A_3 = 4\left[\frac{a^2 - c^2}{b^2}\left(1 - \frac{a^2 - c^2}{b^2}\right)\cos\beta - \left(1 - \frac{a^2 + c^2}{b^2}\right)\cos\alpha\cos\beta + 2\frac{c^2}{b^2}\cos^2\alpha\cos\beta\right]$$

$$A_2 = 2\left[\left(\frac{a^2 - c^2}{b^2}\right)^2 - 1 + 2\left(\frac{a^2}{b^2}\right) + 2\left(\frac{b^2 - c^2}{b^2}\right)\cos^2\alpha - 4\left(\frac{a^2 + c^2}{b^2}\right)\cos\alpha\cos\beta + 2\left(\frac{b^2 - a^2}{b^2}\right)\cos^2\gamma\right]$$

$$A_1 = 4\left[-\left(\frac{a^2 - c^2}{b^2}\right)\left(1 + \frac{a^2 - c^2}{b^2}\right)\cos\beta + \frac{2a^2}{b^2}\cos^2\gamma\cos\beta - \left(1 - \left(\frac{a^2 + c^2}{b^2}\right)\right)\cos\right.$$

$$A_0 = \left(1 + \frac{a^2 - c^2}{b^2}\right)^2 - \frac{4a^2}{b^2}\cos$$

This fourth order polynomia have as many as four real ro tion (8), to every solution for v corresponding solution for u. T ues for u and v it is an easy matt a value for s_1 from equation (for s_2 and s_3 are immediately d equation (4). Most of the tir solutions (Wolfe et al. 1991).

Outline of the Differences of Alge

As we can find in the Grune procedure to solve the problem three unknown variables s_1, s_2, quadratic equations (1), (2) ar variables u and v, and then fur variables u and v into one variab we find the solutions of v and back into equation (5) and equat s_1, s_2, and s_3. Though all six follow the outline mentioned a several differences from the alge

point of view. We classify the differences from the following aspects.

Change of variables

Linnainmaa et al. use $s_2 = u + \cos\gamma s_1$ and $s_3 = v + \cos\beta s_1$ instead of $s_2 = us_1$ and $s_3 = vs_1$ which are used by others.

Different pairs of equations

There are three unknowns in the three equations (1), (2), and (3). After the change of variables is used, any two pairs of equations can be used to eliminate the third variable. For example, Grunert uses the pair of equations (1) and (2) and the pair of equations (2) and (3) and Merritt uses the pair of equations (1) and (2) and the pair of equations (1) and (3).

Approaches of further variables reduction

When reducing two variables into one variable, Grunert and Merritt use substitution. Fischler and Bolles and Linnainmaa et al. use directly elimination to reduce the variables. Finsterwalder and Grafarend et al. introduce a new variable λ before reducing the variables.

The flow chart shown in Figure 2 gives a summary of the differences of algebraic derivation of six solutions in a unified frame. In the flow chart we start from the three equations (1), (2), and (3), make different change of variables, use different pairs of equations, do further variable reduction by different approaches, if necessary, solve the new variable, then we have six different solution techniques.

Finsterwalder's Solution

Finsterwalder (1903) as summarized by Finsterwalder and Scheufele (1937) proceeded in a manner which required only finding a root of a cubic polynomial and the roots of two quadratic polynomials rather than finding all the roots of a fourth order polynomial. Finsterwalder multiplies equation (7) by λ and adds the result to Equation (6) to produce

$$Au^2 + 2Buv + Cv^2 + 2Du + 2Ev + F = 0 \tag{10}$$

where the coefficients depend on

$$A = 1 + \lambda$$
$$B = -\cos\alpha$$
$$C = \frac{b^2 - a^2}{b^2} - \lambda\frac{c^2}{b^2}$$
$$D = -\lambda\cos\gamma$$
$$E = \left(\frac{a^2}{b^2} + \lambda\frac{c^2}{b^2}\right)\cos\beta$$
$$F = \frac{-a^2}{b^2} + \lambda\left(\frac{b^2 - c^2}{b^2}\right.$$

Finsterwalder considers this a equation in v. Solving for v,

$$v = \frac{-2(Bu+E)\pm\sqrt{4(Bu+E)^2-4C(Au^2+}}{2C}$$
$$= \frac{-(Bu+E)\pm\sqrt{(B^2-AC)u^2+2(BE-C}}{C}$$

The numerically stable way of do putation is to determine the small of the larger root.

$$v_{large} = \frac{-\mathrm{sgn}(Bu+E)}{C}\left[|Bu+\right.$$
$$\left.+\sqrt{(B^2-AC)u^2+2(BE-CD)}\right]$$
$$v_{small} = \frac{C}{Av_{large}}$$

Now Finsterwalder asks, can a be found which makes $(B^2 - AC)$ $CD)u + E^2 - CF$ be a perfect in this case v can be expressed der polynomial in terms of u. T meaning of this case is that the so corresponds to two intersecting lir order polynomial can then be sub into equation (6) or (7) either yields a quadratic equation which for u, and then using the just dete for u in the first order expression for v can be determined. Four solu duced since there are two first orde for v and when each of them is sub into equation (6) or (7) the result in u has two solutions.

The value of λ which produc square satisfies

$$(B^2 - AC)u^2 + 2(BE - CD)u +$$
$$= (up + q)^2.$$

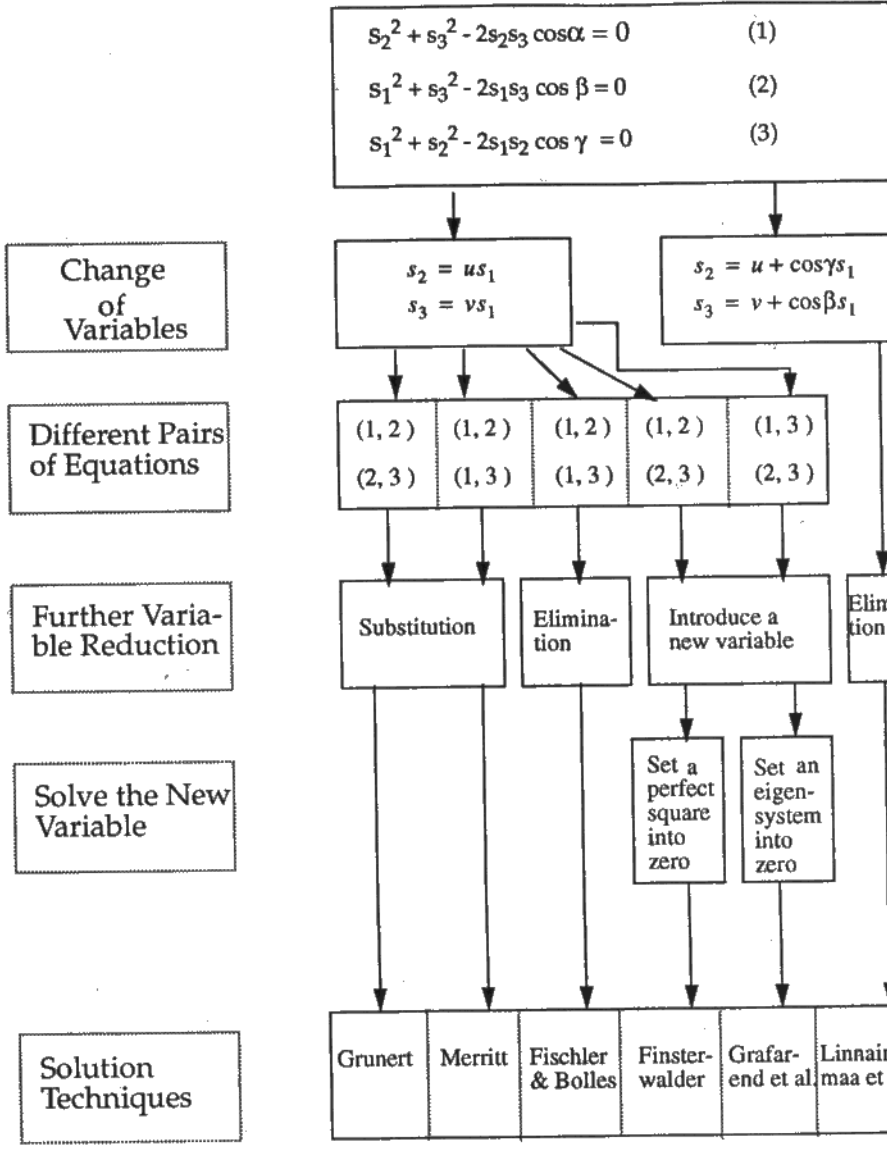

Fig. 2. Shows the differences of a algebraic derivations among six solution techniques

Hence,

$$B^2 - AC = p^2$$
$$BE - CD = pq$$
$$E^2 - CF = q^2.$$

Since $p^2q^2 = (pq)^2$,

$$(B^2 - AC)(E^2 - CF) = (BE - CD)^2$$

After expanding this out, canceling a B^2E^2 on each side and dividing all terms C there results

$$C(AF - D^2) + B(2DE - BF$$

or expressed as a determinant

$$\begin{vmatrix} A & B & D \\ B & C & E \\ D & E & F \end{vmatrix} =$$

This is a cubic equation for λ:

$$G\lambda^3 + H\lambda^2 + I\lambda + J = 0 \tag{14}$$

where

$$\begin{aligned}
G &= c^2(c^2\sin^2\beta - b^2\sin^2\gamma)\\
H &= b^2(b^2 - a^2)\sin^2\gamma + c^2(c^2 + 2a^2)\sin^2\beta\\
&\quad + 2b^2c^2(-1 + \cos\alpha\cos\beta\cos\gamma)\\
I &= b^2(b^2 - c^2)\sin^2\alpha + a^2(a^2 + 2c^2)\sin^2\beta\\
&\quad + 2a^2b^2(-1 + \cos\alpha\cos\beta\cos\gamma)\\
J &= a^2(a^2\sin^2\beta - b^2\sin^2\alpha).
\end{aligned}$$

Solve this equation for any root λ_0. This determines p and q:

$$\begin{aligned}
p &= \sqrt{B^2 - AC}\\
&= \sqrt{\cos^2\alpha - (1 + \lambda_0)\left(\frac{b^2 - a^2}{b^2} - \lambda_0\frac{c^2}{b^2}\right)}\\
q &= \operatorname{sgn}(BE - CD)\sqrt{E^2 - CF}\\
&= \operatorname{sgn}\left(-\cos\alpha\left(\frac{a^2}{b^2} + \lambda_0\frac{c^2}{b^2}\right)\cos\beta\right.\\
&\quad \left. - \left(\frac{b^2 - a^2}{b^2} - \lambda_0\frac{c^2}{b^2}\right)(-\lambda_0\cos\gamma)\right)\\
&\quad \cdot\sqrt{\left(\frac{a^2}{b^2} + \lambda_0\frac{c^2}{b^2}\right)^2\cos^2\beta - \left(\frac{b^2 - a^2}{b^2} - \lambda_0\frac{c^2}{b^2}\right)\left(\frac{-a^2}{b^2} + \lambda_0\left(\frac{b^2 - c^2}{b^2}\right)\right)}.
\end{aligned} \tag{15}$$

Then from equation (11)

$$\begin{aligned}
v &= [-(Bu + E) \pm (pu + q)]/C\\
&= [-(B \mp p)u - (E \mp q)]/C\\
&= um + n,
\end{aligned}$$

where

$$m = [-B \pm p]/C$$

and

$$n = [-(E \mp q)]/C.$$

Substituting this back into equation (7) and simplifying there results

$$\begin{aligned}
&(b^2 - mc^2)u^2 + 2(c^2(\cos\beta - n)m - b^2\cos\gamma)u\\
&\quad -c^2n^2 + 2c^2n\cos\beta + b^2 - c^2 = 0.
\end{aligned} \tag{16}$$

Hence,

$$u = \frac{-(c^2(\cos\beta - n)m - b^2\cos\gamma) \pm \sqrt{(c^2(\cos\beta - n)m - b^2\cos\gamma)^2 - (b^2 - m^2c^2)(-c^2n^2 + 2c^2n\cos\beta + b^2 - c^2)}}{(b^2 - m^2c^2)}. \tag{17}$$

The numerically stable way to c
compute the smaller root in term
root. Let

$$\begin{aligned}
A &= b^2 - mc^2\\
B &= c^2(\cos\beta - n)m - b^2\\
C &= -cn^2 + 2c^2n\cos\beta +
\end{aligned}$$

then

$$\begin{aligned}
u_{large} &= \frac{-\operatorname{sgn}(B)}{A}\left[|B| + \sqrt{}\right.\\
u_{small} &= \frac{C}{Au_{large}}.
\end{aligned}$$

Merritt's Solution

Merritt (1949) unaware of the
lutions also obtained a fourth
mial. Smith (1965) gives the fol
tion for Merritt's polynomial.
equation (1) by b^2, multiplies equ
and subtracts to obtain

$$\begin{aligned}
&a^2s_1^2 - b^2s_2^2 + (a^2 - b^2)s_3^2 - 2a\\
&\quad + 2b^2s_2s_3\cos\alpha = 0.
\end{aligned}$$

Similarly, after multiplying equat
and equation (3) by a^2 and sub
results

$$\begin{aligned}
&a^2s_1^2 + (a^2 - c^2)s_2^2 - c^2s_3^2 - 2a\\
&\quad + 2c^2s_2s_3\cos\alpha = 0.
\end{aligned}$$

Then using the substitution of
we obtain the following two equa

$$\begin{aligned}
&-b^2u^2 + (a^2 - b^2)v^2 - 2a^2\\
&\quad + 2b^2\cos\alpha uv + a^2 =\\
&(a^2 - c^2)u^2 - c^2v^2 - 2a^2c\\
&\quad + 2c^2\cos\alpha uv + a^2 =
\end{aligned}$$

From equation (18),

$$v^2 = \frac{2a^2\cos\beta v - 2b^2\cos\alpha uv +}{a^2 - b^2}$$

Substituting this expression for v^2 into equation (19) and simplifying to obtain

$$(a^2 - b^2 - c^2)u^2 + 2c^2 \cos\alpha uv + 2(b^2 - a^2)\cos\gamma u - 2c^2 \cos\beta v + a^2 - b^2 + c^2 = 0. \quad (21)$$

From (21),

$$v = \frac{b^2 - a^2 - c^2 + (b^2 + c^2 - a^2)u^2 + 2(a^2 - b^2)\cos\gamma u}{2c^2(u\cos\alpha - \cos\beta)}.$$

Substituting this expression for v into equation (19) produces the fourth order polynomial equation

$$B_4u^4 + B_3u^3 + B_2u^2 + B_1u + B_0 = 0 \quad (22)$$

where

$$\begin{aligned} B_4 &= (b^2 + c^2 - a^2)^2 - 4b^2c^2\cos^2\alpha \\ B_3 &= K - 2B_4\cos\gamma \\ B_2 &= B_4 + B_0 - 2K\cos\gamma \\ &\quad + 4c^4(\cos^2\alpha + \cos^2\beta + \cos^2\gamma \\ &\quad - 2\cos\alpha\cos\beta\cos\gamma - 1) \\ B_1 &= K - 2B_0\cos\gamma \\ B_0 &= (a^2 + c^2 - b^2)^2 - 4a^2c^2\cos^2\beta \end{aligned}$$

and

$$\begin{aligned} K &= 2(b^2 + c^2 - a^2)(a^2 + c^2 - b^2)\cos\gamma \\ &\quad + 4c^2(a^2 + b^2 - c^2)\cos\alpha\cos\beta. \end{aligned}$$

Merritt solves for the roots of a fourth order polynomial

$$x^4 + c_3x^3 + c_2x^2 + c_1x + c_0 = 0$$

in the following way. Add

$$\left(\frac{c_3}{2}\right)^2 x^2$$

to each side

$$\begin{aligned} &x^4 + c_3x^3 + \left(\frac{c_3}{2}\right)^2 x^2 \\ &\quad = -c_2x^2 - c_1x - c_0 + \left(\frac{c_3}{2}\right)^2 x^2 \end{aligned}$$

or

$$\begin{aligned} &x^2\left(x^2 + \frac{c_3}{2}\right)^2 \\ &\quad = \left(-c_2 + \left(\frac{c_3}{2}\right)^2\right)x^2 - c_1x - c_0. \end{aligned}$$

Now add

$$\lambda x\left(x + \frac{c_3}{2}\right) + \frac{\lambda}{2}$$

to each side. There results

$$\begin{aligned} &\left[x\left(x + \frac{c_3}{2}\right) + \frac{\lambda}{2}\right]^2 \\ &\quad = \left[-c_2 + \left(\frac{c_3}{2}\right)^2 + \lambda\right]x^2 + \\ &\quad - c_0 + \frac{\lambda^2}{4}. \end{aligned}$$

Choose λ so that the right hand square.

$$\left[-c_2 + \left(\frac{c_3}{2}\right)^2 + \lambda\right]x^2 + \left[$$
$$-c_0 + \frac{\lambda^2}{4} = (mx + n)$$

This means that

$$\begin{aligned} -c_2 + \left(\frac{c_3}{2}\right)^2 + \lambda &= \\ -c_1 + \lambda\frac{c_3}{2} &= \\ -c_0 + \frac{\lambda^2}{4} &= \end{aligned}$$

or

$$\left[-c_2 + \left(\frac{c_3}{2}\right)^2 + \lambda\right]\left[-$$
$$= \left(\frac{-c_1 + \lambda c_3/2}{2}\right)^2$$

This is a cubic which can be sol root λ_0.

Substituting the root λ_0 into th duces

$$\left[x\left(x + \frac{c_3}{2}\right) + \frac{\lambda_0}{2}\right]^2 = (n$$

from which there arises the two

$$x\left(x + \frac{c_3}{2}\right) + \frac{\lambda_0}{2} = \pm(n$$

which each can be solved for th

Fischler and Bolles' Solution

Fischler and Bolles (1981) were apparently not aware of the earlier American or earlier German solutions to the problem. From Equation (5), they obtain

$$\left(1-\frac{a^2}{b^2}\right)v^2+2\left(\frac{a^2}{b^2}\cos\beta-\cos\alpha u\right)v+u^2-\frac{a^2}{b^2}=0 \tag{23}$$

$$v^2+2(-\cos\alpha u)v+\left(1-\frac{a^2}{c^2}\right)u^2+\frac{2a^2}{c^2}\cos\gamma u-\frac{a^2}{c^2}=0. \tag{24}$$

Equation (23) is identical to equation (6) but equation (24) is different from equation (7) since it arises by manipulating a different pair of equations than was used to obtain equation (6).

Multiply (23) by

$$\left(1-\frac{a^2}{c^2}\right)u^2+\frac{2a^2}{c^2}\cos\gamma u-\frac{a^2}{c^2},$$

multiply (24) by

$$u^2-\frac{a^2}{b^2}$$

and subtract to produce

$$[(a^2-b^2-c^2)u^2+2(b^2-a^2)\cos\gamma u+(a^2-b^2+c^2)]v+2b^2\cos\alpha u^3+\left(2(c^2-a^2)\cos\beta-4b^2\cos\alpha\cos\gamma\right)u^2+[4a^2\cos\beta\cos\gamma+2(b^2-c^2)\cos\alpha]u-2a^2\cos\beta=0. \tag{25}$$

Multiply (24) by

$$\left(1-\frac{a^2}{b^2}\right)$$

and subtract from (23).

$$2c^2(\cos\alpha u-\cos\beta)v+(a^2-b^2-c^2)u^2+2(b^2-a^2)\cos\gamma u+a^2-b^2+c^2=0 \tag{26}$$

Finally, multiply (25) by

$$2c^2(\cos\alpha u-\cos\beta),$$

multiply (26) by

$$[(a^2-b^2-c^2)u^2+2(b^2-a^2)\cos\gamma u$$

and subtract to eliminate v. This fourth order polynomial equation

$$D_4u^4+D_3u^3+D_2u^2+D_1u+$$

where

$$D_4=4b^2c^2\cos^2\alpha-(a^2-b^2$$

$$D_3=-4c^2(a^2+b^2-c^2)\cos$$
$$-8b^2c^2\cos^2\alpha\cos\gamma$$
$$+4(a^2-b^2-c^2)(a^2-$$

$$D_2=4c^2(a^2-c^2)\cos^2\beta$$
$$+8c^2(a^2+b^2)\cos\alpha\cos$$
$$+4c^2(b^2-c^2)\cos^2\alpha$$
$$-2(a^2-b^2-c^2)(a^2-$$
$$-4(a^2-b^2)^2\cos^2\gamma$$

$$D_1=-8a^2c^2\cos^2\beta\cos\gamma$$
$$-4c^2(b^2-c^2)\cos\alpha\cos$$
$$-4a^2c^2\cos\alpha\cos\beta$$
$$+4(a^2-b^2)(a^2-b^2+$$

$$D_0=4a^2c^2\cos^2\beta-(a^2-b^2$$

Corresponding to each of the equation (27) for u there is an ass for v through equation (26) or eq

Grafarend, Lohse, and Schaffrim's

Grafarend, Lohse, and Schaffrim of all the previous work, except for Bolles solution, proceed in the f They begin with equations (1), (2) seek to reduce them to a homog After multiplying equation (3) by

$$\frac{-a^2}{c^2}$$

and adding the result to equati results

$$-\frac{a^2}{c^2}s_1^2+\left(1-\frac{a^2}{c^2}\right)s_2^2+s_3^2$$
$$+2\frac{a^2}{c^2}s_1s_2\cos\gamma-2\cos\alpha$$

After multiplying equation (3) by

$$-\frac{b^2}{c^2}$$

and adding the result to equation (2), there results

$$\left(1-\frac{b^2}{c^2}\right)s_1^2-\frac{b^2}{c^2}s_2^2+s_3^2+2\frac{b^2}{c^2}\cos\gamma s_1 s_2 - 2\cos\beta s_1 s_3 = 0. \tag{29}$$

Next they use the same idea as Finsterwalder. They multiply equation (29) by λ and from it subtract equation (28) to produce

$$(s_1 \quad s_2 \quad s_3)A\begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = 0 \tag{30}$$

where

$$A = \begin{pmatrix} \frac{a^2-\lambda(b^2-c^2)}{c^2} & \frac{(\lambda b^2-a^2)\cos\gamma}{c^2} & -\lambda\cos\beta \\ \frac{(\lambda b^2-a^2)\cos\gamma}{c^2} & \frac{a^2-c^2-\lambda b^2}{c^2} & \cos\alpha \\ -\lambda\cos\beta & \cos\alpha & -1+\lambda \end{pmatrix}.$$

Now, as Finsterwalder did, they seek a value of λ which makes the determinant of A zero. Setting the determinant of A to zero produces a cubic for λ. For this value of λ the solution to equation (30) becomes a pair of planes intersecting at the origin.

They let $p = s_2/s_1$ and $q = s_3/s_1$ and rewrite the homogeneous equation (30) in s_1, s_2, and s_3 as a non-homogeneous equation in p and q.

$$\begin{aligned}(a^2-c^2-\lambda b^2)p^2 &+ 2c^2\cos\alpha pq + c^2(-1+\lambda)q^2 \\ &+ 2(-a^2+\lambda b^2)\cos\gamma p - 2\lambda c^2\cos\beta q \\ &+ a^2-\lambda(b^2-c^2) = 0\end{aligned} \tag{31}$$

Now since $|A| = 0$, and assuming

$$\begin{vmatrix} c^2\cos\alpha & c^2(-1+\lambda) \\ a^2-c^2-\lambda b^2 & c^2\cos\alpha \end{vmatrix} \neq 0$$

a value for (p_0, q_0) exists such that (31) can be written in the homogeneous form

$$\begin{aligned}(a^2-c^2-\lambda b^2)&(p-p_0)^2 \\ &+ 2c^2\cos\alpha(p-p_0)(q-q_0) \\ &+ c^2(-1+\lambda)(q-q_0)^2 = 0\end{aligned} \tag{32}$$

where

$$p_0 = \frac{\begin{vmatrix} \lambda c^2\cos\beta & c^2(\\ -\cos\gamma(-a^2+\lambda b^2) & c \end{vmatrix}}{\begin{vmatrix} c^2\cos\alpha & c^2(-1 \\ a^2-c^2-\lambda b^2 & c^2 c \end{vmatrix}}$$

$$q_0 = \frac{\begin{vmatrix} c^2\cos\alpha & \lambda c^2 \\ a^2-c^2-\lambda b^2 & -\cos\gamma(\end{vmatrix}}{\begin{vmatrix} c^2\cos\alpha & c^2(- \\ a^2-c^2-\lambda b^2 & c^2 \end{vmatrix}}$$

Rotating the coordinate syste that the cross term can be elim satisfy

$$\tan 2\theta = \frac{2c^2\cos\alpha}{a^2-\lambda(b^2+}$$

Define the new coordinate $(p', q$

$$\begin{pmatrix} p' \\ q' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix}$$

Then in terms of the new coo (p', q'), equation (32) becomes

$$Ap'^2 + Bq'^2 = 0$$

where

$$A = \frac{a^2-2c^2+\lambda(c^2-b^2)\pm\sqrt{(a^2-\lambda(b^2+c}}{2}$$

$$B = \frac{a^2-2c^2+\lambda(c^2-b^2)\mp\sqrt{(a^2-\lambda(b^2+c}}{2}$$

We choose the negative root sq and the positive root square term value of 2θ falls in the first and and choose the positive root sq and the negative root square t the value of 2θ falls in the sec quadrant.

Assuming $B/A < 0$, (35) resu

$$p' = \pm Kq'$$

where

$$K = \sqrt{\frac{-B}{A}}.$$

Using (34) there results

$$\begin{aligned}p[\cos\theta \pm K\sin\theta] &+ q[\sin\theta \\ &+ [-p_0(\cos\theta \pm K\sin\theta) \\ &+ q_0(-\sin\theta \pm K\cos\theta)\end{aligned}$$

Equation (36) is a function of λ. For any λ equation (36) degenerates into a pair of straight lines intercepting in p, q plane. All possible combinations of any two λs out of λ_1, λ_2, and λ_3 will give a real solution for p' and q'. Then we solve p and q. Finally, from equation (1), (2), and (3)

$$s_1 = \sqrt{\frac{c^2}{1 + p^2 - 2p\cos\gamma}} \tag{39}$$

$$s_2 = \sqrt{\frac{a^2}{1 + \left(\frac{q}{p}\right)^2 - 2\left(\frac{q}{p}\right)\cos\alpha}} \tag{40}$$

$$s_3 = \sqrt{\frac{b^2}{1 + \left(\frac{1}{q}\right)^2 - 2\left(\frac{1}{q}\right)\cos\beta}}. \tag{41}$$

However, there is a simple method proposed by Lohse (1989). Instead of translating and rotating equation (31), one can solve the quadratic equation in (31) to get a p and q relation by using different λ. Once the relation of p and q is obtained it can be substituted into (28) to solve for s_1. There are 15 possible solutions. Since we are only interested in real solutions, we only use real λ to solve (31).

Linnainmaa, Harwood, and Davis' Solution

Linnainmaa, Harwood, and Davis (1988) give another direct solution. They begin with equations (1), (2), and (3) and make a change of variables.

$$s_2 = u + \cos\gamma s_1 \tag{42}$$

$$s_3 = v + \cos\beta s_1 \tag{43}$$

equations (2) and (3) become

$$(1 - \cos^2\beta)s_1^2 + v^2 = b^2 \tag{44}$$

$$(1 - \cos^2\gamma)s_1^2 + u^2 = c^2. \tag{45}$$

Substituting (42), (43), (44), and (45) into (1) there results

$$\begin{aligned} &s_1^2(2\cos^2\gamma - 2\cos\alpha\cos\beta\cos\gamma + 2\cos^2\beta - 2) \\ &\quad - 2\cos\alpha uv + c^2 + b^2 - a^2 \\ &\quad + 2us_1(\cos\gamma - \cos\alpha\cos\beta) \\ &\quad + 2vs_1(\cos\beta - \cos\alpha\cos\gamma) = 0. \end{aligned} \tag{46}$$

Letting

$$\begin{aligned} q_1 &= 1 - \cos^2\gamma \\ q_2 &= 1 - \cos^2\beta \\ q_3 &= 2(\cos^2\gamma - \cos\alpha\cos\beta \\ &\quad + \cos^2\beta - 1) \\ q_4 &= c^2 + b^2 - a^2 \\ q_5 &= 2(\cos\alpha\cos\beta - \cos\gamma) \\ q_6 &= 2(\cos\alpha\cos\gamma - \cos\beta) \end{aligned}$$

there results

$$q_1 s_1^2 + u^2 = c^2$$

$$q_2 s_1^2 + v^2 = b^2$$

$$q_3 s_1^2 - 2\cos\alpha uv + q_4 = q_5 u s_1$$

Then they square equation (50) obtaining

$$r_1 s_1^4 + r_2 s_1^2 + r_3 = (r_4 s_1^2 + r_5$$

where

$$\begin{aligned} r_1 &= q_3^2 + 4q_1q_2\cos^2\alpha + q_1q_2^2 \\ r_2 &= 2q_3q_4 - 4(c^2q_2 + b^2q_1)c \\ &\quad - c^2q_5 - b^2q_6 \\ r_3 &= q_4^2 + 4\cos^2\alpha b^2c^2 \\ r_4 &= 4\cos\alpha q_3 + 2q_5q_6 \\ r_5 &= 4\cos\alpha q_4. \end{aligned}$$

Then to eliminate the uv term, equation (51) and simplify to obt

$$t_8 s_1^8 + t_6 s_1^6 + t_4 s_1^4 + t_2 s_1^2 + t_0$$

where

$$\begin{aligned} t_8 &= r_1^2 - r_4^2 q_1 q_2 \\ t_6 &= (b^2q_1 + c^2q_2)r_4^2 - 2r_4r_5q_1q \\ t_4 &= r_2^2 - b^2c^2r_4^2 - r_5^2q_1q_2 + 2r_4 \\ &\quad + 2r_4r_5c^2q_2 + 2r_1r_3 \\ t_2 &= (b^2q_1 + c^2q_2)r_5^2 + 2r_2r_3 - \\ t_0 &= r_3^2 - r_5^2b^2c^2. \end{aligned}$$

Equation (52) is considered as equation in s_1^2. Since s_1 must be p are at most 4 solutions to equation value for s_1 has been determined, c

and (49) can be solved for two values of u and v. Each of these can then be substituted into equation (42) and (43) to obtain the positive solutions for s_2 and s_3.

Comparisons of the Algebraic Derivations

The main difference between the Grunert solution and the Merritt solution is that they use different pairs of equations. As a result, the coefficients in their fourth order polynomials are different. However, if we replace b with c, c with b, β with γ, and γ with β in equation (9), then we can obtain equation (22). Therefore, from the algebraic point of view, their solutions are identical. But Merritt converts the fourth order polynomial into two quadratics instead of solving it directly. The difference between Fischler and Bolles' and Grunert's solution is that the former just multiplies some terms to two pairs of equations and then subtracts each other without expressing one variable in terms of the other.

Grunert and Merritt use the substitution to reduce the two variables into one variable. The advantage of the substitution approach is that it is pretty trivial. But there exists a singular region when the denominator is zero in equations (8) and (21). This is discussed more fully in the next subsection. Fischler and Bolles and Lin nainmaa et al. use direct elimination to reduce the variables. Though the approaches are not trivial, it does not generate any singular point during the derivation.

Linnainmaa et al. use $s_2 = u + \cos\gamma s_1$ and $s_3 = v + \cos\beta s_1$ as the change of variables. Naturally, this leads to another different derivation to the problem. Although we consider equation (52) as a fourth order equation in s_1^2, the complexity of the coefficients is much higher than that of Grunert's fourth order equation.

Finsterwalder and Grafarend et al. introduce the same variable, but they use different approaches to solve λ. Finsterwalder solves equation (10) for v and seeks a λ to make the term inside the square root be a perfect square. Grafarend et al. actually rewrite the quadratic equations into matrix form $(s_1\ s_2\ s_3)P(s_1\ s_2\ s_3)^t = 0$ and $(s_1\ s_2\ s_3)Q(s_1\ s_2\ s_3)^t = 0$, then try to solve

which is another form of equa
this point these two approaches a
equivalent.

Singularity of Solutions

It is well-known that there exist
structures for the three point
on which the resection is unsta
1966) or indeterminate (Smith
unstable geometric structure,
in the position of the center
will result in a large change in
three vertices. For the indeterm
structure, the position of three
be solved. Besides the singul
geometric structures, there also
gularities caused by the algebra
solutions. In the following par
give detailed explanations and

The danger cylinder is a typi
unstable geometric structure a
geometric structure where the c
tivity is located on a circular
through the three vertices of a t
ing its axis normal to the plane
An illustration of the danger c
in Figure 3.a. The reason for t
be explained as follows. Instea
the position of three vertices,
let the coordinates of the cente
be unknown, (x, y, z), as in the
lem. Since the problem is ma
three unknown variables $s_1, s_2,$
actually no difference between
or the center of perspectivity.
of s_1, s_2, and s_3 are functions o
equations (1), (2), and (3) int
$f_2(x, y, z) = 0$, and $f_3(x, y, z) =$
derivatives. We then have

$$\frac{1}{s_1 s_2 s_3} AB \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} =$$

where

$$A = \begin{pmatrix} x - x_1 & y - y_1 & z \\ x - x_2 & y - y_2 & z \\ x - x_3 & y - y_3 & z \end{pmatrix}$$

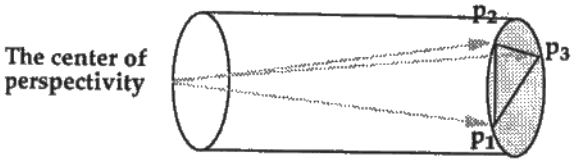

Fig. 3a. An illustration of the danger cylinder.

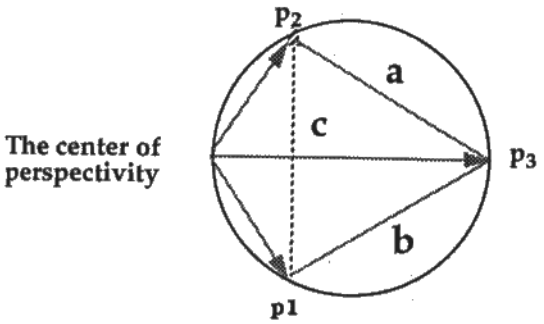

Fig. 3b. An illustration of the concylic case.

$$B = \begin{pmatrix} 0 & s_2 - s_3\cos\alpha & s_3 - s_2\cos\alpha \\ s_1 - s_3\cos\beta & 0 & s_3 - s_1\cos\beta \\ s_1 - s_2\cos\gamma & s_2 - s_1\cos\gamma & 0 \end{pmatrix},$$

and $p_i = \begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix}$, $i = 1,2,3$ as defined before are the position of three vertices of the triangle in the camera coordinate frame.

To make the system stable the dx, dy, dz must have no solutions other than zeros; that is, matrices A and B must be non-singular. The determinant of matrix A is proportional to the volume of a tetrahedron formed by three vertices and the center of perspectivity. As long as these four points are not coplanar the matrix A is nonsingular. When the matrix B is singular; that is, where the determinant of B is zero, we can expand the determinant of B and express s_1, s_2, s_3, $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ in terms of x, y, z. Then we can obtain an equation that represents the equation of a circular cylinder circumscribing three vertices of the triangle with its axis normal to the plane of the triangle. For example, let the center of perspectivity be located at the origin, $p_1 = \begin{pmatrix} 0 \\ 0 \\ 10 \end{pmatrix}$, $p_2 = \begin{pmatrix} 2 \\ 2 \\ 10 \end{pmatrix}$, and $p_3 = \begin{pmatrix} -2 \\ 2 \\ 10 \end{pmatrix}$. Then the second row B will be a zero vector; that is, singular.

When the center of perspectivi tices of a triangle are concylic as ure 3.b, the resection problem is Note that the problem cannot b the five coefficients of equation (to zeros; that is, the four point ar example, let three side lengths three angles $\alpha = \gamma = 60°$ and $\beta =$ coefficients of polynomials of six be equal to zeros.

The singularity in the algebraic occur in the Grunert, Finsterwa Grafarend et al. solutions whe inator term in the formula equ example, let three side lengths three angles $\alpha = \gamma = \beta = 60°$, lateral triangle parallel to the ima the triangle center at z axis, then must equal one and thus v or u As a result the denominator (co in the Grunert solution and (u co the Merritt solution equal zero. solutions have a singularity.

Determination of the Absolute Orie

Once the position of three vertice gle is determined, the transforma which governs where the 3D came system is with respect to the 3D nate system can be calculated.

The problem can be stated as f three points in the 3D camera coor and their corresponding three poi world coordinate system, we want a rotation matrix R and translat which satisfies

$$p_i = Rp_i' + T \qquad i = 1,2$$

where $p_i = \begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix}$ $i = 1,2,3$ are the

3D camera coordinate system, p_i'

$1,2,3$ are the points in the 3D wor

Table I. The summary of characteristic of six solutions.

Authors	Features	Algebraic singularit
Grunert 1841	Direct solution, solve a fourth order polynomial	Yes
Finsterwalder 1903	Form a cubic polynomial and find the roots of two quadratics	Yes
Merritt 1949	Direct solution, solve a fourth order polynomial	Yes
Fischler and Bolles 1981	Another approach to form a fourth order polynomial	No
Linnainmaa et al. 1988	Generate an eighth order polynomial	No
Grafarend et al. 1989	Form a cubic polynomial and find intersection of two quadratics	Yes

system, R is a 3 by 3 orthonormal matrix, i.e., $RR^t = I$, and $T = \begin{pmatrix} t_x \\ t_y \\ t_z \end{pmatrix}$. The problem can be solved by a linear (Schut 1960), an iterative (Wolf 1974; Slama 1980), or noniterative closed-form solution (Horn 1988). We give a simple linear solution in Appendix I.

4 The Experiments

To characterize the numerical sensitivity of the six different 3 point resection solutions we perform experiments. The experiments study the effects of rounding errors and numerical instability of each of these six different solutions in both single and double precision mode. In addition, we examine the relation of the equation manipulation order. This is accomplished by changing the order in which the three corresponding point pairs are given to the resection procedure.

Since singularities and unstable structures exist in the three point perspective pose estimation problem, we wanted to know how often it can happen in the testing data. To isolate these singularities and unstable structures, we ran 100000 experiments on the Grunert solution, because it has both algebraic and geometric singularities. Then we screened out the singular cases by picking those trials whose error is larger than a certain value.

4.1 Test Data Generation

The coordinates of the vertices of the 3D trian-

gle are randomly generated by
dom number generator. The ra
and z coordinates are within [
25], and $[f + a, b]$ respectively.
plane is located in front of ca
tance of focal length, f, the z
be larger than the focal length
$b \geq f + a$. The a and b are used
test the solution under differen
Projecting the 3D spatial coor
image frame we obtain the pe
coordinates u and v.

4.1.1 Permutation of Test Data.
merical stability of each resecti
permute the order of the three
angle and the order of the persp
of the 3D triangle vertices. Ass
order of vertices is 123 for verte
and vertex three, respectively, th
permutations are 312, 231, 132
The permutation of triangle ver
muting in a consistent way the
lengths, the 3D vertices and th
2D perspective projection verti

4.2 The Design of Experiments

In this section we will summa
ters in the experiments discusse
The experimental procedure of
be presented too. The param
ods involved in accuracy and
permutation are denoted by

N_1 – the number of trials = 10000
N_2 – the number of trials = 100000
P – different number of precisions = 2
d_1 – the first set of depths along z axis
d_2 – the second set of depths along z axis
$S_w = \sum_{i=0}^{4} |S_i|$ – the worst sensitivity value for all coefficients.
$S_{wn} = \sum_{i=0}^{4} |S_{a_i}^x|$ – the worst normalized sensitivity value for all coefficients.
$\epsilon_{ware} = \sum_{i=0}^{4} \epsilon_{ware_i}$ – the worst absolute error for all coefficients
$\epsilon_{wrre} = \sum_{i=0}^{4} \epsilon_{wrre_i}$ – the worst relative error for all coefficients
$\epsilon_{sware} = \sum_{i=0}^{4} |S_i \times \epsilon_{ware_i}|$ – the worst polynomial zero drift due to the absolute error
$\epsilon_{swrre} = \sum_{i=0}^{4} |S_{a_i}^x \times \epsilon_{wrre_i}|$ – the worst polynomial zero drift due to the relative error

where $S_i = \frac{dx}{da_i}\Big|_{x=z_j} = \frac{-\frac{\partial P}{\partial a_i}}{\frac{\partial P}{\partial x}}\Big|_{x=z_j}$, $P = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$, and $S_{a_i}^x = \frac{a_i}{x}\frac{\partial x}{\partial a_i}$. The ϵ_{wrre_i} and ϵ_{ware_i} are the total relative and absolute rounding errors propagated from the first to the last mathmetical operations of each coefficient of the polynomial P.

4.2.1 The Design Procedures. The experimental procedures and the characteristics to be studied are itemized as follows:

Step 0. Do the following steps N times.
Step 1. Generate the coordinates of vertices of the 3D triangle.
$-25 < x_i < 25$ where $i = 1, 2, 3$
$-25 < y_i < 25$
For z_i coordinate there are several sets to be tested.
1. $d_1 = \{(a,b) \mid (a,b) \in \{(0,5), (4,20)\}, f = 1\}$
2. $d_2 = \{(a,b) \mid (a,b) \in \{(0,5), (4,20), (24,75)\}, f = 1\}$

Step 2. For single and double precision do the resection calculation.
Step 3. Permutation of the vertices. Let the original vertex order be 123 (vertex one, vertex two and vertex three, respectively), then we permute the order as 312, 231, 132, 321, and 2
Step 4. For each of the resection t termine the location of t if the calculation can suc

Step 4.1. For any calculation v ceeded record the ab error (ADE) associa permutation. The r distance error (MAD as follows:

$$\epsilon_\mu = \sum_{i=1}^{n}$$

where n is the numb ments and

$$\epsilon_i = \sum_{i=1}^{n} (\epsilon_{i1} + \epsilon_{i2} + \epsilon_{i3})$$
$$\epsilon_{i1} = \sqrt{((x_{ci1} - x_{i1})^2 + (y_{ci1} -$$
$$\epsilon_{i2} = \sqrt{((x_{ci2} - x_{i2})^2 + (y_{ci2} -$$
$$\epsilon_{i3} = \sqrt{((x_{ci3} - x_{i3})^2 + (y_{ci3} -$$

and $(x_{ci}, y_{ci}, z_{ci})^t$ is t point coordinates and is the correct generat dinates. The error st tion is expressed as fo

$$sd = \sqrt{\frac{\sum_{i=1}^{n} (\epsilon}{(n -}}$$

Step 5. This procedure is only ap nert's solution

Step 5.1. Calculate the sensitivit each coefficient and t for all coefficients bas cussion in A.2.3.
Step 5.2. Calculate the worst ab ative rounding errors ficient based on the A.2.4. The number digits is the same as representation of mac plication and division. and subtraction the po nificant digits in each c be checked.

Step 5.3. Calculate the polynomial zero drift.

Step 5.4. Record the values of the sensitivity S_w, the normal sensitivity S_{wn}, the worst absolute rounding error ϵ_{ware}, the worst relative rounding error ϵ_{wrre}, the worst polynomial zero drift due to absolute rounding error, and the worst polynomial zero drift due to relative rounding error ϵ_{swrre} for each permutation.

Step 5.5. Based on the smallest value of ϵ_{ware}, ϵ_{wrre}, S_w, S_{wn}, ϵ_{sware}, or ϵ_{swrre} picks the corresponding error generated by the corresponding permutation and accumulate the number of its rank in the six permutation. Rank each permutation in terms of the error associated with the permutation. The rank one is associated with the smallest error and the rank six is associated with the largest error.

Step 6. Check for singular cases.
Redo the whole procedure again by changing N_1 to N_2 and d_1 to d_2 and use Grunert's solution only. If the largest absolute distance error is greater than 10^{-7} redo steps 5 and record the corresponding values for the large error cases.

5 Results and Discussion

In this section we discuss the results of the experiments. The software is coded in the C language and the experimer both a Sun 3/280 workstation a computer. Unless stated otherw in the following paragraphs are the Sun 3/280. Table II shows random permutation of six diff From Table II we find that Finst tion (solution two) gives the be Merritt's solution gives the Grunert's solution (solution one lution (solution four) and Grafa six) are about the same order a ond best accuracy. The reasons results can be explained in terms polynomial and the complexity Linnainmaa's solution (solution an eighth order polynomial. Th have to solve the eighth order complexity of the coefficients o solution is still higher than Finsterwalder's solution only ne third order polynomial. The hig nomial and higher complexity c to be less numerically stable. ritt's solution also converts the fo nomial problem into a third or problem, but it gives a worse because the conversion process most numerically stable. An ex directly solves Merritt's fourth o was conducted. A Laguerre's m to find the zeros of a polynomi are similar to that of Grunert's

The histogram of the absolute

Table II. Results of random permutation of six solutions in double precision and single precision.

Algorithms	Precision	Mean absolute distance error	Standard deviation
Sol. 1 (Grunert)	D.P.	0.19e−08	0.16e−06
	S.P.	0.31e−01	0.88e−00
Sol. 2 (Finsterwalder)	D.P.	0.22e−10	0.90e−09
	S.P.	0.89e−02	0.51e−01
Sol. 3 (Merritt)	D.P.	0.11e−05	0.64e−04
	S.P.	0.28e−01	4.15e−00
Sol. 4 (Fischler)	D.P.	0.62e−08	0.59e−06
	S.P.	0.14e−01	0.34e−00
Sol. 5 (Linnainmaa)	D.P.	0.74e−07	0.61e−05
	S.P.	0.32e−01	0.82e−00
Sol. 6 (Grafarend)	D.P.	0.46e−08	0.43e−06
	S.P.	0.20e−01	0.75e−01

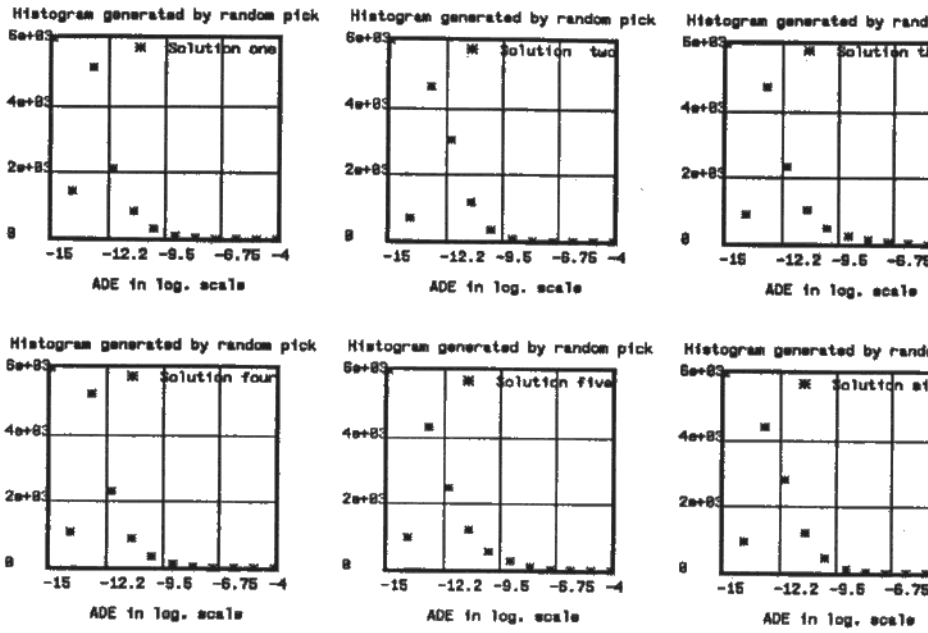

Fig. 4. Shows histograms of the absolute distance error of random permutations in log. scale for six solut

Table III. The best and the worst mean absolute distance error in single precision.

Algorithms	The best MADE	Standard deviation	The worst MADE	Standard deviation
Sol. 1 (Grunert)	0.10e−03	0.25e−02	0.81e−01	1.45e−00
Sol. 2 (Finsterwalder)	0.74e−04	0.12e−02	0.59e−01	1.71e−00
Sol. 3 (Merritt)	0.17e−02	0.54e−01	1.29e−00	8.53e−00
Sol. 4 (Fischler)	0.87e−04	0.47e−03	0.40e−01	0.47e−00
Sol. 5 (Linnainmaa)	0.16e−02	0.14e−00	0.11e−00	2.16e−00
Sol. 6 (Grafarend)	0.77e−04	0.14e−02	0.94e−01	2.75e−00

(ADE) of 10000 trials are shown in Figure 4. From the histogram of the ADE we can see all the solutions can give an accuracy to the order of 10^{-13} in double precision. The populations of high accuracy results of solution one and solution four are larger than that of solution two. But the population of less accuracy for solution one and solution four also is a little bit more than that of solution two.

As we can expect, the double precision calculation gives a much better results than the single precision calculation. For single precision most of the solutions give the accuracy of the ADE to the order of 10^{-5}. Generally speaking, the results of double precision are about 10^7 times better than the results of single the single precision mode the ro routine fails in several cases an up the MADE. Therefore, if po precision calculation is recomme 3-points perspective projection ca

The best MADE and the wor six permutations for the double the single precision are shown in Table IV. The best results are ab better than the worst results. Finst lution, Grunert's solution and Fisc give the same best accuracy.

Because Grunert's solution ha best accuracy and is easier to anal

Table IV. The best and the worst mean absolute distance error in double precision.

Algorithms	The best MADE	Standard deviation	The worst MADE	Standard deviation
Sol. 1 (Grunert)	0.41e−12	0.90e−11	0.60e−08	0.26e−06
Sol. 2 (Finsterwalder)	0.34e−12	0.73e−11	0.20e−09	0.51e−08
Sol. 3 (Merritt)	0.26e−10	0.15e−08	0.18e−04	0.13e−02
Sol. 4 (Fischler)	0.69e−12	0.19e−10	0.13e−07	0.69e−06
Sol. 5 (Linnainmaa)	0.35e−11	0.24e−09	0.36e−06	0.23e−04
Sol. 6 (Grafarend)	0.44e−12	0.16e−10	0.88e−08	0.48e−06

to demonstrate how analysis methods can discriminate the worst and the best from the six permutations. The analysis methods can be applied to the other solution techniques as well. In the following paragraphs we discuss the results of analysis.

For each trial there are six permutation by which the data can be presented to the resection technique. In the controlled experiments where the correct answer are known, the six resection results can be ordered from least error (best pick) to the highest error (worst pick) using the square error distance between the correct 3D position of the triangle vertices and the calculated 3D position of the triangle vertices. The fraction of times each selection technique selects the data permutation giving the best (least) error to the worst (most) error for two different depths are plotted in Figures 5 and 6. The histogram of the absolute distance error of the six selection methods is shown in Figure 7. Figures 5 and 6 show that the drift of zeros is not affected by the absolute error (i.e. WS) or the relative error (i.e. WRRE). The worst sensitivity (i.e. WS) and the worst relative error (i.e. WRRE) do not permit an accurate choice to be made for the picking order. The worst normalized sensitivity produces the best results and can effectively stabilize the calculation of the coefficients of the polynomial.

The absolute drift of polynomial zeros is changed by both the absolute error of coefficients and the sensitivity of the polynomial zero with respect to the coefficients. Thus, the ϵ_{sware} methods can suppress the probability of picking the worst result from the six permutations. Both the relative error of coefficients and the

Table V. The comparison of the mean error of randomly order, the best and mean absolute distance error picked S_w, S_{wn}, ϵ_{swrre} and ϵ_{sware} for two diff

Picking methods	Mean absolute
Depth	$1 < z < 5$
Random order	0.19e−08
The best	0.41e−12
The worst	0.60e−08
ϵ_{ware}	0.99e−11
ϵ_{wrre}	0.40e−08
Sw	0.15e−08
Swn	0.89e−12
ϵ_{swrre}	0.90e−12
ϵ_{sware}	0.93e−12

worst normalized sensitivity of zero with respect to the coeffi relative drift of the zeros. H method also gives a pretty good comparisons of the MADE of the best and the worst and the by the ϵ_{ware}, ϵ_{wrre}, S_w, S_{wn}, ϵ_{swr} two different depths are shown

The goal is to achieve the bes accuracy of the best permutat ten thousand times better tha obtained by the worst case and t tained by choosing a random pe S_{wn}, ϵ_{sware}, and ϵ_{swrre} methods mately a half of the accuracy best permutation. Any of these can be used to choose a permutat gives reasonably good accuracy. worst normalized sensitivity on sensitivity calculation. So it is to quickly pick the right permuta the histograms of probability of ϵ_{swrre} do not have very high va

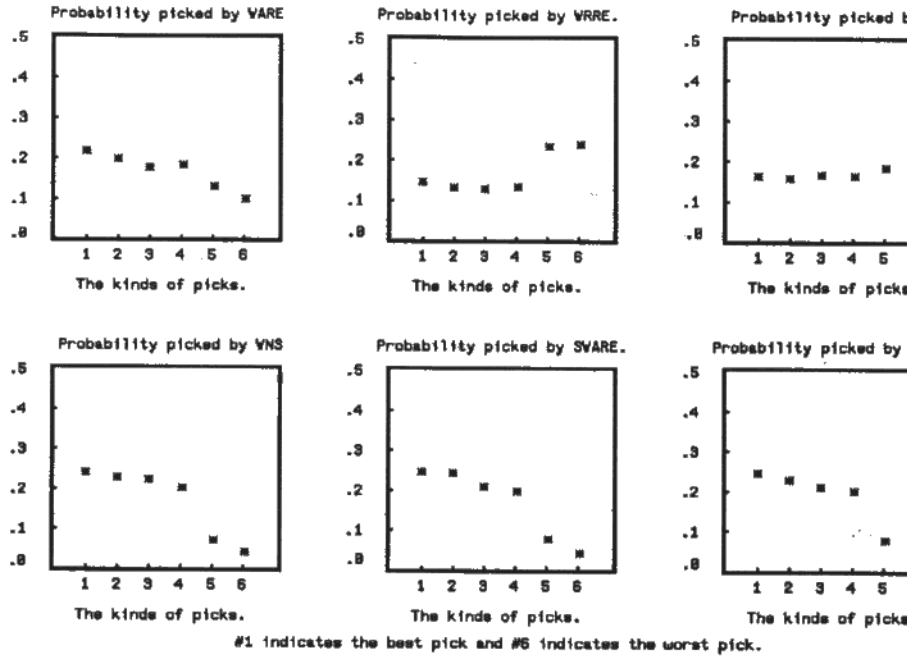

Fig. 5. Shows, for each of the six selection techniques, the fraction of times the technique selected the giving the best (least) error to the worst (most) error for all 10000 experimental cases for which the depth $1 < z < 5$.

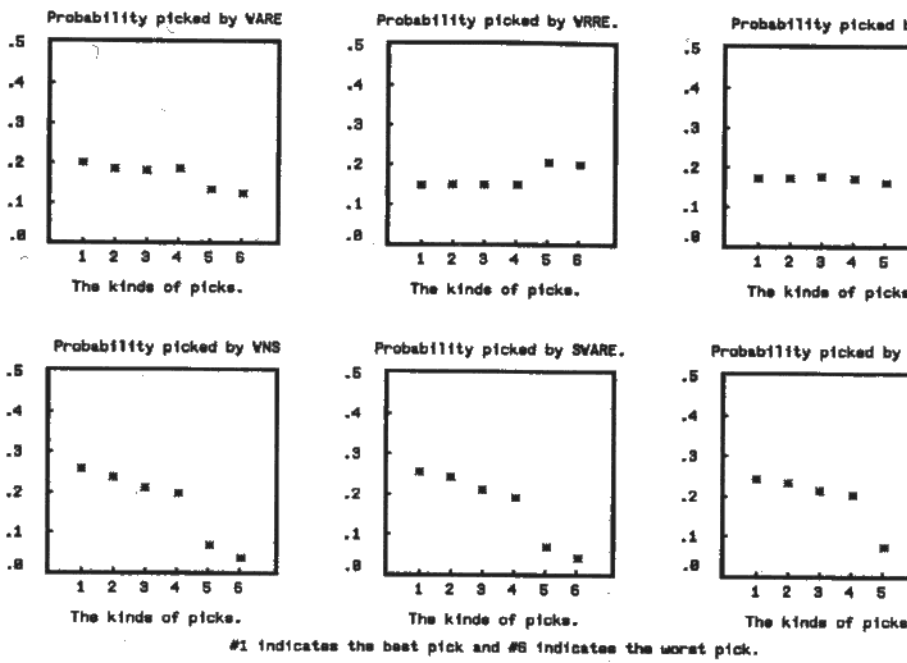

Fig. 6. Shows, for each of the six selection techniques, the fraction of times the technique selected the giving the best (least) error to the worst (most) error for all 10000 experimental cases for which the depth $5 < z < 20$.

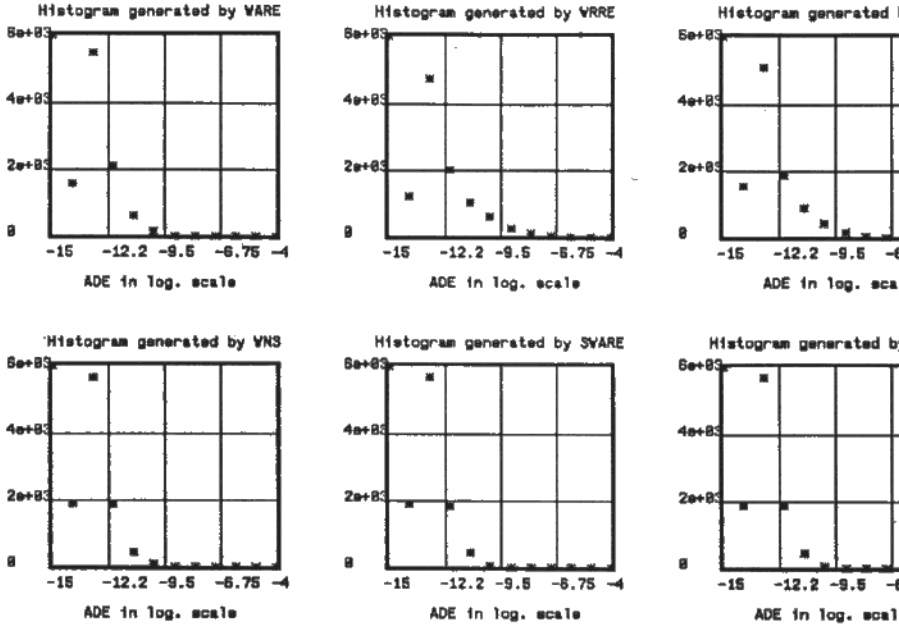

Fig. 7. Shows histograms of the absolute distance error of six selection methods in log. sc

best pick, they still have a very accurate absolute distance error compared to the best absolute distance error. This reveals that in many cases the accuracy of six permutations are too close to be discriminated.

In order to study the frequency with which singularities and instabilities may happen we pick the large error cases whose absolute distance error is greater than 10^{-7}, run more trials and add different depths for Grunert's technique. Around each singularity we find a region within the parameter space leading to large absolute distance errors in the Grunert solution, diverging with decreasing distance to the point of singularity. Because the real singularities may seldomly happen in the numerical calculation, most cases we only have to deal with very large errors in the vicinity of singular points in the parameter space. Because the set of the vicinities of all singularities in the parameter space does not have the full symmetry of permutation group, we always can find a better parametrization of our experiment. Our task is to define an objective function on the parameter space, which allows us to select a parametrization from the six possible parametrizations, which est absolute distance error to the The results are shown in Table

Table VI and Table VII wh obtained from the Vax 8500 runn ating system contain the statistics distance error of the different sel for the three different depth case sample of all 100000 experimen and based on the subsample of l in Table VII. The sample size f about 69 for the first depths, al second depth and about 495 for Table VII shows that the singul really happen in these experime mean ADE is about 10^{-2}. Howe ity of singular points the error compared to that of Table VI. Table VII also show that the sel work fine in these cases.

When the experiments of Tabl VII are run in the Sun 3/280, ilar to these obtained from the the magnitude differences in n racy of results between two syst

Table VI. The same as Table V. But it runs 100000 trials and with three different depths.

Picking	Depth [1...5] MADE	Std. dev.	Depth [5...20] MADE	Std. dev.	Depth [25...75] MADE	Std dev.
Random	2.22e−07	6.58e−05	4.49e−09	6.11e−07	1.44e−07	1.72e−05
Best	6.69e−12	1.79e−09	2.01e−12	2.41e−10	4.18e−11	3.88e−09
Worst	1.69e−06	4.39e−04	7.14e−07	1.90e−04	1.61e−04	5.50e−02
ϵ_{ware}	1.06e−08	3.31e−06	4.49e−10	6.24e−08	3.70e−07	1.14e−04
ϵ_{wrre}	1.68e−06	4.39e−04	6.31e−07	1.88e−04	1.83e−06	3.81e−04
Sw	5.99e−09	1.33e−06	5.98e−07	1.88e−04	1.34e−08	2.13e−06
Swn	9.18e−12	1.88e−09	3.76e−12	3.55e−10	2.43e−10	3.89e−08
ϵ_{sware}	7.64e−12	1.80e−09	3.66e−12	4.17e−10	1.21e−10	1.18e−08
ϵ_{swrre}	7.57e−12	1.80e−09	4.16e−12	4.53e−10	1.21e−10	1.17e−08

Table VII. The same as Table VI. But it only considers large error cases.

Picking	Depth [1...5] MADE	Std. dev.	Depth [5...20] MADE	Std. dev.	Depth [25...75] MADE	Std dev.
Random	1.35e−05	6.35e−05	4.03e−06	2.57e−05	1.22e−04	2.43e−03
Best	7.23e−08	5.16e−07	2.43e−09	1.64e−08	1.37e−08	2.38e−07
Worst	1.18e−04	5.78e−04	2.59e−03	2.52e−02	5.43e−02	1.20e−00
ϵ_{ware}	6.76e−07	4.30e−06	2.94e−07	1.21e−06	9.56e−06	2.08e−04
ϵ_{wrre}	6.02e−05	3.18e−04	2.58e−03	2.52e−02	1.29e−04	2.43e−03
Sw	6.21e−06	4.70e−05	1.14e−07	5.25e−07	1.09e−04	2.42e−03
Swn	5.42e−07	4.23e−06	8.02e−09	4.49e−08	2.18e−08	2.97e−07
ϵ_{sware}	5.20e−07	4.23e−06	7.93e−09	4.43e−08	1.78e−08	2.87e−07
ϵ_{swrre}	5.20e−07	4.23e−06	8.01e−09	4.49e−09	1.73e−08	2.87e−07

an order of one except in worst cases with depth[5...20] and depth[25...75] and in Sw case with depth[5...20] whose magnitude differences are an order of two and three, respectively.

6 Conclusions

We have reviewed the six solutions of the three point perspective pose estimation problem from a unified perspective. We gave the comparisons of the algebraic derivations among the six solutions and observed the situations in which there may be numerical instability and indeterminate solutions. We ran hundreds of thousands of experiments to analyze the numerical stability of the solutions. The results show that the Finsterwalder solution gives the best accuracy, about 10^{-10} in double precision and about 10^{-2} in single precision. We have shown that the use of different pairs of equations and change of variables can produce different numerical behaviors. We have described an analysis method to almost always produce a numerically stab for the Grunert solution. The an described here can pick the solut about 0.9×10^{-12}, which is very best accuracy 0.41×10^{-12} that ca by picking the best permutation about thousand times better tha which is achieved by picking the mutation.

Acknowledgment

The authors wish to thank the their helpful suggestions.

Appendix I: A Simple Linear So Absolute Orientation

Let us restate the problem. Give in the 3D camera coordinate syst

corresponding three points in the 3D world coordinate system, we want to determine a rotation matrix R and translation vector T which satisfies

$$p_i = Rp_i' + T \qquad i = 1,2,3 \qquad \text{(a.1)}$$

where $p_i = \begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix}$ $i = 1,2,3$ are the points in the 3D camera coordinate system, $p_i' = \begin{pmatrix} x_i' \\ y_i' \\ z_i' \end{pmatrix}$ $i = 1,2,3$ are the points in the 3D world coordinate system, R is a 3 by 3 orthonormal matrix, i.e., $RR^t = I$, and $T = \begin{pmatrix} t_x \\ t_y \\ t_z \end{pmatrix}$.

In order to solve the problem linearly we express the rotation matrix as follows

$$R = \begin{pmatrix} r_{11} r_{12} r_{13} \\ r_{21} r_{22} r_{23} \\ r_{31} r_{32} r_{33} \end{pmatrix}$$

Then, equation (a.1) is an underconstraint system of 9 equations in 12 unknowns. However, as stated in Ganapathy (1984) those unknowns in the rotation matrix are not independent. There exist some constraints as follows

$$r_{11}^2 + r_{12}^2 + r_{13}^2 = r_{21}^2 + r_{22}^2 + r_{23}^2 = r_{31}^2 + r_{32}^2 + r_{33}^2 = 1$$

$$\begin{aligned} r_{13} &= r_{21} r_{32} - r_{22} r_{33} \\ r_{23} &= r_{12} r_{31} - r_{11} r_{32} \\ r_{33} &= r_{11} r_{22} - r_{12} r_{21} \end{aligned} \qquad \text{(a.2)}$$

Since three vertices of the triangle are coplanar, with the constraints above we can assume $z_i = 0$, $i = 1,2,3$. Thus, equation (a.1) can be written as

$$\begin{aligned} x_i &= r_{11} x_i' + r_{12} y_i' + t_x \\ y_i &= r_{21} x_i' + r_{22} y_i' + t_y \qquad i = 1,2,3 \\ z_i &= r_{31} x_i' + r_{32} y_i' + t_z \end{aligned}$$

In terms of matrix form we have

$$AX = B$$

where

$$A = \begin{pmatrix} x_1' & y_1' & 0 & 0 & 0 & 0 \\ 0 & 0 & x_1' & y_1' & 0 & 0 \\ 0 & 0 & 0 & 0 & x_1' & y_1' \\ x_2' & y_2' & 0 & 0 & 0 & 0 \\ 0 & 0 & x_2' & y_2' & 0 & 0 \\ 0 & 0 & 0 & 0 & x_2' & y_2' \\ x_3' & y_3' & 0 & 0 & 0 & 0 \\ 0 & 0 & x_3' & y_3' & 0 & 0 \\ 0 & 0 & 0 & 0 & x_3' & y_3' \end{pmatrix}$$

$$X = [r_{11}\ r_{12}\ r_{21}\ r_{22}\ r_{31}\ r_{32}\ t_x\ t_y$$

$$B = [x_1\ y_1\ z_1\ x_2\ y_2\ z_2\ x_3\ y_3\ z_3]^t$$

The matrix A will not be singu the three points are not collin has a unique solution. After t solved, equation (a.2) can be us r_{23} and r_{33}.

Appendix II: The Numerical Solutions

A.1 The Problem Definition

In general, all the solutions giv can be used to solve the three p resection problem. However, the numerical calculations are different solution techniques. F each solution technique the nu will be different when the order manipulation or variables is di ample, if we let $s_1 = us_2$ and of $s_2 = us_1$ and $s_3 = vs_1$, then of equation (9) will be changed can be reflected by replacing α α with β and β with α. As a res the numerical accuracy of the f

The order of the equation ma bined with choosing different p for substitution can produce six ical behaviors for each solutio these effects we preorder the projection and the correspondi the six different possible permu

In this appendix we describ methods that can be used to de merical stability of the solutions

three corresponding point pairs to the re section procedure.

A.2 The Analysis—A Rounding Error Consideration

There are several sensitivity measures which can be used. They include the numerical relative and absolute errors, and the drift of polynomial zeros. We are mainly concerned about how the manipulation order affects the rounding error propagation and the computed roots of the polynomial. Since both the absolute rounding error and the relative rounding error may affect the final accuracy, we consider both factors. The sensitivity analysis focuses on the roots of the polynomial formed by the three-point perspective solutions. In contrast, the polynomial zero drift considers both the errors and the sensitivity of polynomial zero. However, all factors can affect the numerical results. Each of these measures will be used to predict sensitivity in terms of the mean absolute error.

A.2.1 The Effect of Significant Digits. In this analysis all computations are conducted in both single precision and double precision for the six techniques. The quantity measured is the mean absolute distance error for each precision.

A.2.2 The Histogram of the Mean Absolute Distance Error. The histogram analysis will give the distribution of the absolute distance error. A technique may give a large number of highly accurate results, but produce a few large errors due to degenerate cases; others may give accurate results to all trials without any degenerate cases. The analysis of the histogram of the errors will help us to discriminate between which techniques are uniformly good from those which are only good sometimes.

A.2.3 The Sensitivity Analysis of Polynomial Zeros. The global accuracy is affected by the side lengths, the angles at the center of perspectivity with respect to side lengths, and the permutation order in which the input data is given. These

effects will appear in the coeffici
puted polynomial and affect th
zeros of the polynomial. For
polynomial a small change in th
efficient will dramatically change
one or more zeros. This change
agate to the solution produced
perspective resection technique.
of the zeros of a polynomial
a change in the coefficients is
assuming the zero location is a
coefficients (Vlach and Singhal
j-th zero z_j of the polynomial $P($
$= a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x +$

$$P(a_0, a_1, \ldots, a_n, x(a_0, a_1, \ldots, a$$

Differentiating with respect to a

$$\frac{\partial P}{\partial a_i} + \frac{\partial P}{\partial x}\frac{dx}{da_i}\bigg|_{x=z_j} =$$

Rearranging the equation gives

$$S_i = \frac{dx}{da_i}\bigg|_{x=z_j} = \frac{-\frac{\partial P}{\partial a_i}}{\frac{\partial P}{\partial x}}\bigg|$$

where $a_0, a_1, \ldots, a_n$ are the coef
polynomial, z_j is the j-th zero of

Consider the total sensitivity,
coefficients on a particular zero.

$$S = \sum_{i=0}^{n} S_i$$

To avoid the cancellation among
negative terms, we take the abs
each term and consider the wo
express the worst sensitivity S_w by

$$S_w = \sum_{i=0}^{n} |S_i|$$

A large sensitivity of the zero with
coefficients may lead to a large err
result. Laguerre's method is use
zeros of polynomial. It has adva
extracting the zeros with small ab
to better preserve accuracy in the
the polynomial and can converge

zero from a real initial estimate. The accuracy for the iterative stop criterion is the rounding error of the machine.

A.2.4 The Numerical Stability. Discussion in most numerical books show how calculations involving finite-digit arithmetic can lead to significant errors in some circumstances. For example, the division of a finite-digit result by a small number, i.e., multiplying by a relative large number, is numerically unstable. Another example is the subtraction of large and nearly equal numbers which can produce an unacceptable rounding error. In order to study how large a rounding absolute error can be produced by the mathematical operation, we will calculate the worst absolute and relative error for each kind of arithmetic operation. Let fl be the floating point mathematical operator. Hence, the rounding error produced by fl on two numbers which themselves have rounding error or truncation error (Wilkinson 1963) can be modeled as follows:

$$fl(\hat{x}_1 + \hat{x}_2) = (x_1(1+\epsilon_{x1}) + x_2(1+\epsilon_{x2}))(1+\epsilon_r)$$
$$\cong (x_1+x_2)\left(1+\epsilon_r+\frac{x_1}{x_1+x_2}\epsilon_{x1} + \frac{x_2}{x_1+x_2}\epsilon_{x2}\right)$$
$$fl(\hat{x}_1 - \hat{x}_2) = (x_1(1+\epsilon_{x1}) - x_2(1+\epsilon_{x2}))(1+\epsilon_r)$$
$$\cong (x_1-x_2)\left(1+\epsilon_r+\frac{x_1}{x_1-x_2}\epsilon_{x1} - \frac{x_2}{x_1-x_2}\epsilon_{x2}\right)$$
$$fl(\hat{x}_1 \times \hat{x}_2) = x_1x_2(1+\epsilon_{x1})(1+\epsilon_{x2})(1+\epsilon_r)$$
$$\cong x_1x_2(1+\epsilon_r+\epsilon_{x1}+\epsilon_{x2})$$
$$fl\left(\frac{\hat{x}_1}{\hat{x}_2}\right) = \frac{x_1(1+\epsilon_{x1})}{x_2(1+\epsilon x2)}(1+\epsilon_r)$$
$$\cong \left(\frac{x_1}{x_2}\right)(1+\epsilon_r+\epsilon_{x1}-\epsilon_{x2})$$
$$-0.5\times 10^{1-d} \le \epsilon_r \le 0.5\times 10^{1-d}$$

where d is the number of significant digits of $fl(\hat{x}_1 + \hat{x}_2)$; ϵ_r is relative error introduced by each operation; the relative errors of x_1 and x_2 are ϵ_{x1} and ϵ_{x2} respectively, and these are propagated from the previous operations. The higher order terms are very smal omitted.

DEFINITION. *A sequence* $\langle OP_1, O$ *of binary mathematical operators* addition, subtraction, multiplicati *applied to a series of numbers* $(x_1$ *at a time is given as follows:*

$$OP_{i=1}^{n-1}(x_i, x_{i+1}) f(\epsilon_{x_i}, \epsilon_{x_i}$$
$$= \hat{x}(1+\epsilon_{total})$$

where f *is a function of* $\epsilon_{x_i}, \epsilon_{x_{i+1}}$ *result of the operation assuming* computation and ϵ_{total} *is the to propagated from the first operatio eration. Hence,* $\hat{x}(1+\epsilon_{total})$ *is calculation using finite precision. the relative error of* x_i*;* $\epsilon_{x_{i+1}}$ *is the* x_{i+1}.

We consider the worst case for i.e., $\epsilon_r = 0.5 \times 10^{1-d}$. Thus, th rounding error(ϵ_{wrre_i}) is express

$$\epsilon_{wrre_i} = \epsilon_{total}$$

and the worst absolute roundin given

$$\epsilon_{ware_i} = \hat{x} \times \epsilon_{total}$$

The ϵ_{ware_i} and ϵ_{wrre_i} will be accur of the coefficient. As in the se section we expect a large rela error lead to a large final error

A.2.5 Polynomial Zero Drift. tivity helps us to understand ho of the polynomial coefficients a The worst relative and absolute quantitative measurement of e of a polynomial zero from its c pends on both sensitivity and In this paragraph we will give polynomial zero drift. Define sensitivity $S_{a_i}^x$ of zero with res cient by

$$S_{a_i}^x = \frac{a}{x}\frac{\partial x}{\partial a_i}$$

and the function x

$$x|_{x=z_j} = x(a_0, a_1, \ldots$$

Then, the worst normalized sensitivity (S_{wn}) is given by

$$S_{wn} = \sum_{i=0}^{n} |S_{a_i}^{x}|$$

The polynomial zero drift can be expressed as follows:

$$dx|_{x=z_j} = \sum_{i=0}^{n} \frac{\partial x}{\partial a_i} da_i$$

Divide both sides of the above equation by x and in terms of normalized sensitivity we obtain

$$\left.\frac{dx}{x}\right|_{x=z_j} = \sum_{i=0}^{n} S_{a_i}^{x} \frac{da_i}{a_i}$$

Consider the worst absolute drift case due to the absolute rounding error we have

$$\epsilon_{sware} = \sum_{i=0}^{n} |S_i \times \epsilon_{ware_i}|$$

and the worst relative drift case due to the relative rounding error we have

$$\epsilon_{swrre} = \sum_{i=0}^{n} |S_{a_i}^{x} \times \epsilon_{wrre_i}|$$

As discussed above the final error is expected in proportion to the value of the worst drift ϵ_{swrre} and ϵ_{sware}.

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