Hitting times for simple random walk

Miklós Csűrös

March 25, 1997

Consider the one dimensional random walk $\{X_t : t \in \mathbb{N}\}$ with $X_0 = 0$ and $X_t = X_t \pm 1$ with equal probability. Let $T(x) = \arg\min_{t\geq 0}\{X_t = x\}$ and $p(x,y) = \mathbb{P}\{T(x) < T(y)\}$, that is, the probability that x is reached before y. This probability is nontrivial only if exactly one of x and y is nonnegative. Because of symmetry, we have $p(x,y) = p(-x,-y)$. Since the chain is irreducible, $\forall x,y : p(x,y) + p(y,x) = 1$. Hence we have $p(x,-x) = 1/2$ and, in particular, $p(1,-1) = p(-1,1) = 1/2$.

For $x > 0$, we have the recursion

$$\begin{aligned}
p(x+1,-1) &= \mathbb{P}\{T(x+1) < T(-1)\} \\
&= \mathbb{P}\{T(x+1) < T(-1) | T(x) < T(-1)\}\, p(x,-1) + \mathbb{P}\{T(x+1) < T(-1) | T(x) > T(-1)\}\, p(-1,x) \\
&= p(1,-(x+1))p(x,-1) + 0 = p(x,-1)(1 - p(x+1,-1)),
\end{aligned}$$

and therefore,

$$p(x+1,-1) = \frac{p(x,-1)}{1+p(x,-1)}.$$

Using $p(0,-1) = 1$, and symmetry, we get

$$p(x,-1) = p(-x,1) = \frac{1}{x+1} \quad \text{and} \quad p(-1,x) = p(1,-x) = \frac{x}{x+1}.$$

One can obtain the general result

$$p(x,-y) = \frac{y}{x+y}$$

in a similarl way.

Let $(Y_t, Z_t) = (\min_{s\leq t} X_s, \max_{s\leq t} X_s)$. At time t, the visited states are exactly $\{Y_t, Y_t+1, \ldots, Z_t\}$. Observe that $Y_t \leq 0 \leq Z_t$. If $Z_t - y_t = k$, then set $A_k = Y_t$ and $B_k = Z_t$. This gives a well-defined series of growing intervals $\{[A_k, B_k] : k \in \mathbb{N}\}$. Let $W_k = 1$ if $B_{k+1} = B_k + 1$ and $W_k = -1$ if $A_{k+1} = A_k - 1$ (W_0 is undefined). Finally, let $q_k^+(a,b) = \mathbb{P}\{A_k = -a, B_k = b, W_k = 1\}$ and $q_k^-(a,b) = \mathbb{P}\{A_k = -a, B_k = b, W_k = -1\}$ for $k \geq 1$, and $q_k(a,b) = \mathbb{P}\{A_k = -a, B_k = b\}$ for $k \geq 0$. Then

$$\begin{aligned}
q_k^+(a,b) &= \mathbb{P}\{A_k = -a, B_k = b, W_k = 1 | A_{k-1} = -a, B_{k-1} = b-1, W_{k-1} = 1\}\, q_{k-1}^+(a,b-1) \\
&\quad + \mathbb{P}\{A_k = -a, B_k = b, W_k = 1 | A_{k-1} = -a, B_{k-1} = b-1, W_{k-1} = -1\}\, q_{k-1}^-(a,b-1) \\
&= p(1,-(a+b))q_{k-1}^+(a,b-1) + p(a+b,-1)q_{k-1}^-(a,b-1).
\end{aligned}$$

Similarly,

$$q_k^-(a, b) = p(-(a + b), 1)q_{k-1}^+(a, b - 1) + p(-1, a + b)q_{k-1}^-(a - 1, b).$$

The initial values are

$$\begin{array}{ll} q_1^+(0, 1) = \frac{1}{2} & q_1^-(0, 1) = 0 \\ q_1^+(1, 0) = 0 & q_1^-(1, 0) = \frac{1}{2} \end{array}$$

Using induction, we can derive

$$q_k^+(a, b) = \frac{b}{(k + 1)k} \quad \text{and} \quad q_k^-(a, b) = \frac{a}{(k + 1)k}.$$

Therefore,

$$q_k(a, b) = \frac{1}{k + 1}$$

for all $a + b = k$.