Surface Turbulence for Particle-Based Liquid Simulations

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Abstract

We present a method to increase the apparent resolution of particle-based liquid simulations. Our method first outputs a dense, temporally coherent, regularized point set from a coarse particle-based liquid simulation. We then apply a surface-only Lagrangian wave simulation to this high-resolution point set. We develop novel methods for seeding and simulating waves over surface points, and use them to generate high-resolution details. We avoid error-prone surface mesh processing, and robustly propagate waves without the need for explicit connectivity information. Our seeding strategy combines a robust curvature evaluation with multiple bands of seeding oscillators, injects waves with arbitrarily fine-scale structures, and properly handles obstacle boundaries. We generate detailed fluid surfaces from coarse simulations as an independent post-process that can be applied to most particle-based fluid solvers.

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Keywords: Lagrangian fluid simulation, wave turbulence

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1 Introduction

Simulating the behavior of fluids is a long standing problem that often requires visual details resolved to a very fine resolution. Simulating smoke and liquids are the two most common cases, and specialized approaches have been developed for each.

For smoke animation, Eulerian approaches are common. Here, performance scales with the underlying grid’s resolution and resolving details at fine scales quickly becomes prohibitive. Fluid up-res methods address this problem by applying fine-scale turbulence models atop coarser simulations, generating detailed results without explicitly simulating at a fine resolution [Kim et al. 2008; Narain et al. 2008; Schechter and Bridson 2008]. These methods are practical for art-direction since they guarantee that the coarse behavior will not change when fine details are added.

In contrast, detailed liquid simulations are still performed at full resolution, regardless of whether a grid- or particle-based approach is used, with high-quality particle-based simulators, such as Fluid Implicit Particle (FLIP) [Zhu and Bridson 2005; Autodesk 2014] or Smoothed Particle Hydrodynamics (SPH) [Müller et al. 2003; Next Limit Technologies 2014], having seen rapid, recent adoption.

We address the discrepancy with an “up-res” technique for particle-based liquid simulations. While an Eulerian closest point turbulence (CPT) method was recently developed for level set-based liquids [Kim et al. 2013], it can only be applied to particle-based data by converting the data to an Eulerian grid, discarding many of the simulation’s rich details. We instead add turbulent details directly to particles, solving the wave equation in a Lagrangian setting. We first convert a set of input particles from a coarse liquid simulation into a high-quality, high-density surface point set. We then perform a wave simulation that adds high-frequency features to the liquid surface, in the form of bump or displacement maps. We use standard surfacing to obtain a detailed, high-resolution liquid surface.

To our knowledge, ours is the first comprehensive up-res technique for particle-based simulations, making the following contributions:
• robust, temporally coherent, meshless surface generation, that yields smooth, simulation-ready surfaces,
• smooth constraints to ensure that our surface remains spatially and temporally faithful to the underlying particle set,
• a robust, curvature-based method for initiating surface waves,
• a novel discrete Laplace operator that is provably well-suited for meshless point representations, in addition to
• an efficient simulation strategy that synthesizes details across scales onto our high-density surface.

Our method is agnostic to the source of particle data, and can be applied to FLIP, SPH, and position-based works [Macklin et al. 2014].

2 Previous Work and Overview

Fluid simulation is a well-established area so, in addition to seminal works [Foster and Metaxas 1996; Stam 1999; Foster and Fedkiw 2001; Müller et al. 2003], we refer readers to Bridson’s book [2008] and Ihm et al.’s STAR [2014] for comprehensive surveys. Here, we focus primarily on areas most related to our work.

Fluid Up-res. Thuerey et al. [2013] surveys recent fluid up-res methods which efficiently increase the apparent spatial resolution of a coarse simulation without altering the low-frequency behavior. As noted in Section 1, these methods have been most effective in smoke simulations [Kim et al. 2008; Narain et al. 2008; Schechter and Bridson 2008; Nielsen et al. 2009; Huang et al. 2011]. Several works have also addressed the related problem of synthesizing frequency-controlled textures on moving surfaces [Yu et al. 2009].

Earlier attempts to apply up-res algorithms to liquids had limited success, both in Eulerian [Narain et al. 2008] and Lagrangian [Yuan et al. 2012; Shao et al. 2014] formulations, since they focus on increasing the resolution of the velocity field. As noted by Kim et al. [2013], the turbulence on a free surface is only loosely coupled to the fluid velocity field. Lab experiments show that surface waves tend to propagate much faster than the velocities of the underlying fluid would suggest. We thus choose to evolve a high-resolution simulation over the liquid surface to obtain more convincing results.

While CPT [Kim et al. 2013] works well for Eulerian liquids, no comprehensive Lagrangian up-res method exists. This is unfortunate, because the ad-hoc methods developed in industry [Budskberg et al. 2013] show that there is substantial interest in such techniques.

Several works have explored how to guide liquids to meet artistic goals [Shi and Yu 2005; Pan et al. 2013]. Nielsen and Bridson [2011] use a low-frequency “guide shape”, extracted from a coarse FLIP simulation, as the boundary condition for a thin, secondary, high-resolution simulation applied near the liquid boundary. Our work differs from this approach in two ways: first, we preserve the entire frequency content of the coarse simulation, including important quantities such as the silhouettes. Second, we add entirely new dynamics to the surface using a wave simulation. As such, our algorithm can complement such “guide shapes” approaches, especially since it is applied as a standalone post-process.

Surface Tracking. The importance of surface-only simulations has become increasingly clear in recent works on explicit surface tracking [Thuerey et al. 2010], and methods that use them [Yu et al. 2012]. Wojtan et al. [2011] survey these recent developments. Instead of explicitly modeling the fluid surface and carefully incorporating it into the core simulation, we propose a post-process that can be applied directly to any coarse particle simulation, remaining fully decoupled from the simulator that generated the data. Maintaining a simulation mesh is cumbersome, often requiring external geometry processing tools. Our meshless method is self-contained, simplifying implementation. While we create a mesh for rendering, meshing artifacts do not degrade the stability of the simulation.

Inspired by optimization work for liquid surfacing [Williams 2008; Bhattacharya et al. 2011], we constrain our final surface to lie in a band around the input surface. We extend ideas from Eulerian level sets and meshes to particle-based surfaces. While these works focus on generating geometry for rendering, we improve their smoothness and temporal regularity to make them suitable for simulation.

Wave Simulation. Many works consider wave simulations. From the linear wave equation [Kass and Miller 1990], and related shallow-water models [Wang et al. 2007], to bi-Laplacian variants [Yu et al. 2012] and the iWave [Tessendorf 2008]. We use the linear wave equation and rely on dispersion from stretching induced by the underlying advection. As with all these previous works, we manually set an average propagation speed for our surface waves.

On the other hand, wave particles represent surface waves [Yukel et al. 2007; Cords 2008] with a dense set of advected points. These methods have difficulty with the complex, topologically-varying surfaces that we treat, requiring many more wave particles to represent the details we achieve with our approach.

Point-Based Simulation. Our work is related to point-based techniques, but unlike previous works that deal with static point sets [Zwicker et al. 2002; Alex et al. 2003] or dynamic surfaces for rendering [Guennebaut et al. 2008], our input particles represent a volume where every particle corresponds to a quantity of liquid. Point-based rendering treats each sample as a (possibly noisy) surface point, and previous point-based simulations [Macdonald et al. 2013; Jeong and Kim 2013] operate on static point sets, and are thus not appropriate for dynamic particle sets from liquid simulations.

Overview. Figure 2 is a visual overview of our method. Given an input sequence of particles representing a liquid volume (coarse particles, dashed circles), we first construct a dense point set along the liquid interface (fine surface points, solid circles; Sections 3.2 & 3.3). We smooth and regularize these points to support point-based simulation (Section 3.4). Using per-point normals and displacement values we call wave values (green lines; Section 4.2), we perform a high-resolution wave simulation (green curve; Section 4.3) over the surface. We output the final detailed surface as a bump or height map over the high-density point set, or as a displaced point set. These points can be splatted directly or tessellated for rendering, which is easy given our regular surface point distribution.

3 Surface Construction and Maintenance

We construct a dense point set that represents the liquid’s surface, and maintain a level of smoothness and regularity necessary for point-based wave simulation. We describe our novel smooth band constraint that controls the surface’s behavior and ensures coherence between the surface points and underlying simulation. Unlike level set or mesh-based surface tracking [Osher and Fedkiw 2003; Wojtan et al. 2011], our fluid surface is represented exclusively by oriented points , each with a position and normal .

3.1 Neighborhood Relationships

To ensure that our surface points behave as a unified manifold, we draw upon work in meshless simulation (e.g., [Ihmsen et al. 2014]), taking advantage of the neighborhood relationships between point pairs. Our high-resolution surface points , and coarse input particles , will affect each other across spatial scales. Specifically:

• a coarse-scale length , obtained from the coarse particle simulator (e.g., the grid cell size in a FLIP solver), and

• robust, temporally coherent, meshless surface generation, that yields smooth, simulation-ready surfaces,
• smooth constraints to ensure that our surface remains spatially and temporally faithful to the underlying particle set,
• a robust, curvature-based method for initiating surface waves,
• a novel discrete Laplace operator that is provably well-suited for meshless point representations, in addition to
• an efficient simulation strategy that synthesizes details across scales onto our high-density surface.
A fundamental component of our approach is the generation of a smooth, temporally coherent, high-resolution output surface that does not need to explicitly track manifold connectivity. To accomplish this, we devise an implicit representation that constrains the position of the fine-scale surface points.

Our method is motivated by Williams [2008]: First, two concentric spheres of radius \( r \) and \( R \) are centered at each coarse particle, where \( R \) is the larger of the two. During surface evolution, the fine-scale surface points are constrained to remain in the volume between the union of all outer spheres and the union of all inner spheres. As depicted in Figure 3, this volume region forms “bands” around the coarse particles, delimiting the region where the advected fine scale surface is allowed to exist. This constrains surface points to regions not too far from, nor too close to, the existing coarse particles.

The \( r \) and \( R \) parameters affect the final output appearance. Small values create surfaces that more closely resemble the coarse simulation, but increase bumpiness. Large values can create surfaces that deviate significantly from the coarse simulation and exhibit over-smoothing. While they can be manually adjusted, we found that \( R = \lambda_c \) results in a reasonably smooth surface relative to the underlying simulation and \( r = R/2 \) leaves sufficient space for fine particles to evolve without closely approaching the coarse particles.

Williams uses the spheres to constrain a thin-plate energy optimization to represent the surface; it is unclear how this can apply to meshless settings without costly intermediate meshing. Moreover, projecting onto the surface formed by the union of spheres is difficult due to discontinuities in the first derivatives, and an orthogonal projection creates a discontinuous surfaces unsuitable for wave simulation (Section 4.3).

To solve this problem, we instead project onto an implicit metaball-like formulation [Blinn 1982] that leads to smooth projection constraints better suited to the volumetric region between the inner- and outer-sphere unions. The efficacy of this strategy is shown in Figure 3. This novel smooth band constraint is constructed from an implicit function \( \phi(y) = g(f(y)) \) for an arbitrary point \( y \), where \( f \) is a standard metaball function and \( g \) is a rescaling function. There are many possible choices for \( f \), but we obtained good results with

\[
f(y) = \sum_{i \in \mathcal{C}} f_i(y)/\psi_i \quad \text{with} \quad f_i(y) = \exp \left( -a(y - X_i)^2 \right),
\]

where \( \psi_i \) is the metaball density of the \( i \)-th coarse particle and \( a \) is a falloff parameter discussed below. The metaball density \( \psi_i \) differs from the local density \( \rho_i^\delta \) in Equation 2, and is evaluated as

\[
\psi_i = \sum_{j \in \mathcal{C}} D(||x_i - x_j||) \quad \text{with} \quad D(z) = \exp \left( -2 \left( z/\lambda_c \right)^2 \right).
\]
Due to the kernel's fall-off. Unlike before, truncation needed to design sufficient constraints. As in Equations 1 to 3, we that vary linearly in space, permitting simpler projections.

Our constraints are smoother and define values in the interior that vary linearly in space, permitting simpler projections. As in Equations 1 to 3, we that vary linearly in space, permitting simpler projections.

\[
g(z) = \left( \sqrt{-\ln(z)/a - r} \right) / (R - r). \quad (7)
\]

We solve for a value of \(a\) that, given two coarse particle centers less than \(\mu = \frac{3}{2} \times R\) units apart, will unify the inner sphere union constraint of the two particles as if their contribution resulted from the same component of the fluid surface:

\[
a = \ln \left( \frac{2}{|1 + D(\mu)|} \right) \left( \frac{\sigma^2}{\sigma^2 - r^2} \right). \quad (8)
\]

Figure 3 (bottom middle) is the case with particle centers exactly \(\mu\) units apart. All our results use this value of \(a\), and its associated \(\mu\).

Since our band constraint function \(\phi\) is smooth, we can use its normalized gradient \(\nabla \phi\) to determine a reasonable projection direction when a particle exits the constraint region. Additionally, as the function is approximately linear with respect to the distance from the inner constraint, we can easily compute this projection as

\[
x_i = \begin{cases} 
- (R - r) \cdot \phi(x_i) \cdot g_i & \text{if } \phi(x_i) < 0 \\
- (R - r) \cdot (\phi(x_i) - 1) \cdot g_i & \text{if } \phi(x_i) > 1 \\
x_i & \text{otherwise .} 
\end{cases} \quad (9)
\]

This introduces some approximation error, so a surface point may still lie outside the constraint region after projection. However, at this stage, it keeps points sufficiently close to the constraint region.

### 3.4 Surface Smoothing and Regularization

We now have a high-density set of surface points, but their distribution must be improved (i.e., regularized) before they are suitable for surface wave simulation. This regularization proceeds in four stages: normal computation, normal regularization, tangent regularization, and point insertion and deletion.

**Normal Computation:** A smooth, artifact-free normal field is essential for the maintenance of our surface structures. We compute the normal at each fine-scale surface point using an averaged least-squares planar fit to the local gradient of \(\phi\) (see Algorithm 1).

**Regularization Along the Normal:** To improve the smoothness of our surface, we displace each surface point along its newly computed normal direction. Given a point \(i\) and one of its neighbors \(j\), we consider the plane \(\Pi_{i,j}\) spanned by vectors \(n_i\) and \((x_i - x_j)\). We find the unique circle in this plane that is equidistant to both points and orthogonal to both points' normals (see Figure 4). The projection of point \(i\) onto this circle is averaged over all its neighbors, and the point is then displaced along its averaged projected position.

This averaging is done in a neighborhood of radius \(\lambda_c\), which pushes the points towards a surface that is consistent with the computed normal field. Since the normals vary smoothly, the resulting surface is also smooth. Denoting \(n_i^\star\) the normalized projection of \(n_i\) onto \(\Pi_{i,j}\), the displacement of point \(i\) onto the circle is given by

\[
\text{proj}_{i,j} = n_i \left( \frac{(n_i + n_j) \cdot (x_i - x_j)}{2 n_i \cdot (n_i + n_j)} \right), \quad (10)
\]

so this regularization step is computed as

\[
x_i \leftarrow x_i + \sum_{j \in \mathcal{F}} W_i^\lambda (x_j) \text{proj}_{i,j} \cdot \quad (11)
\]

**Regularization Along the Tangents:** Similar to previous works, we insert repulsion forces [Turk 1991] to improve the distribution of the surface points by driving them along their tangent directions. At each surface point, we compute the weighted direction away from its neighbors, and displace it in this direction. This averaging is performed in a local neighborhood \(\lambda_t\) around the point. Explicitly, this regularization step is computed at each surface point as

\[
x_i \leftarrow x_i + 0.5 \lambda_t \sum_{j \in \mathcal{F}} W_i^\lambda (x_j) \text{T}_n(x_j - x_i), \quad (12)
\]

where \(\text{T}_n\) projects onto the tangent plane of point \(i\) and then normalizes the result, and the 0.5 factor prevents two points from moving to the same location.

**Insertion and Deletion:** The last regularization step adds and deletes surface points according to changes in the underlying simulation. Points are deleted when the local point density is too high. We detect this by looking for pairs of points that are closer than \(\frac{3}{4} \lambda^2\), in which case the most recently created point is deleted. Similarly, surface points are added when the local point density is

```
1 forall the surface points i do
2     n_i ← g_i;
3     compute tangent t_i^1 and bi-tangent t_i^2, orthonormal to n_i;
4     least square plane fitting in \lambda_t to the frame [t_i^1, t_i^2, n_i];
5     n_i ← normal of plane;
6 forall the surface points i do
7     n_i ← averaged normal in neighborhood \lambda_c;
8     n_i ← normalize (n_i);
```

**Algorithm 1:** Steps for computing the normals.
too low, by Poisson disk sampling [Cook 1986]. For each point, we recompute the tangential direction as it was computed in the previous tangential regularization step. This generally points in the direction of lowest density, so we place a sphere of radius $\lambda_f$ at a distance $\lambda_f$ in this direction and search for neighboring points that fall inside the sphere. If none are found, the point density is too low and we create a point at the displaced sphere’s center (Figure 4).

The three regularization steps are global operations that influence the entire surface. However, as they each consist of spatially local computations, we apply them several times to each animation frame until convergence. We used 30 iterations for the first frame and between 3 and 10 iterations per subsequent frame in all of our simulations. The number of iterations used for each scene is shown in Table 1. The accompanying video shows the uniform point density maintained by our surface operations on a deforming fluid surface.

### 3.5 Interactions with Obstacles

We have so far focused on the management of surface points in unobstructed flow, but special care must be taken when surface points approach obstacles. The regularization in Section 3.4 works best if the point distribution around each surface point is approximately uniform, which is not the case near obstacle boundaries.

Motivated by ghost and boundary [Schechter and Bridson 2012; Akinci et al. 2012] particles in SPH, we add ghost points for each surface point located close to an obstacle by reflecting the neighboring points of the current point w.r.t. the obstacle. We then simply apply the regularization steps to both “real” and ghost point sets.

### 4 Turbulence Creation and Evolution

Section 3 detailed the creation of high-density point sets that represent the underlying coarse fluid simulation surface. We can now add turbulent details atop this surface, simulating waves on its points.

#### 4.1 Curvature Evaluation

Surface wave details should appear in areas of high activity, e.g., merging or separating regions. Mean curvature is a good indicator of these regions [Kim et al. 2013], as high curvature tends to denote merging or separating regions. Mean curvature is a good indicator of these regions [Kim et al. 2013], as high curvature tends to denote merging or separating regions. Mean curvature is a good indicator of surface regions where second-order neighbors should appear in areas of high activity, e.g., turbulent details atop this surface, simulating waves on its points.

For a surface of infinite point density at distance $\lambda_i$ from the coarse particles, our measure evaluates to

$$c_{\text{min}} \approx 0.15 \lambda_i$$

for a single drop and $c_{\text{min}} \approx 0.077 \lambda_i$ for a thin sheet. These thresholds (see Appendix A for derivations) are useful for parametrizing simulations.

#### 4.2 Turbulence Creation

The simplest way to add turbulence to a surface is to add it directly to the wave heights. This can work well for grid-based methods [Kim et al. 2013] where surface curvature varies smoothly with respect to grid size, but particle-based methods often contain large, abrupt curvature variations. For instance, when a single particle falls onto a flat water surface, a discontinuous wave seeding causes an abrupt visual change in height. Curvature is always high on isolated droplets, causing uniform wave seeding over the droplet and unrealistic (non mass conserving) pulsing (see supplemental video).

A simple solution used in mesh-based simulations [Yu et al. 2012] is to turn off the wave seeding in regions of high curvature. This reduces artifacts from curvature discrepancies, but still produces abrupt changes in regions of near-maximum curvature. Moreover, it removes some of the waves caused by fine splashes characteristic of particle-based fluids, and limits the amount of new added details.

We propose a different approach, outlined in Algorithm 2 and illustrated in Figure 5. In lines 2 and 3, we throttle seeding in high

Data:
- $c_i$: surface curvature
- $c_{\text{min, max}}$: curvature thresholds
- $a_i$: oscillator amplitude
- $\Delta a$: oscillator amplitude step size
- $A$: max. oscillator amplitude
- $h_i$: internal wave height
- $v_i$: internal wave velocity
- $\Delta h_i$: max. wave frequency
- $t$: simulation time
- $i$: seed value

```
1 forall the surface points i do
2     a_i := clamp(a_i + a_i * \Delta a, 0, A);
3     s_i := clamp(s_i + a_i * \cos(t * f), 0, h_i);
4     v_i := v_i + \Delta t * h_i;
5     v_i := clamp(v_i, -W, W);
```

Algorithm 2: Pseudocode for wave evolution and seeding.

While not identical to mean curvature, this measure is a very reliable criterion for wave generation. Moreover, thresholding any such criterion in a meaningful way across discretizations is very important. We derive practical thresholds $\{c_{\text{min}}, c_{\text{max}}\}$, evaluating Equation 13 at two extremal scenarios: single drops and thin sheets. For a surface of infinite point density at distance $\lambda_i$ from the coarse particles, our measure evaluates to $c_{\text{max}} = 0.15 \lambda_i$ for a single drop and $c_{\text{min}} \approx 0.077 \lambda_i$ for a thin sheet. These thresholds (see Appendix A for derivations) are useful for parametrizing simulations.

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Motivated by ghost and boundary [Schechter and Bridson 2012; Akinci et al. 2012] particles in SPH, we add ghost points for each surface point located close to an obstacle by reflecting the neighboring points of the current point w.r.t. the obstacle. We then simply apply the regularization steps to both “real” and ghost point sets.

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#### 4.1 Curvature Evaluation

Surface wave details should appear in areas of high activity, e.g., merging or separating regions. Mean curvature is a good indicator of these regions [Kim et al. 2013], as high curvature tends to denote under-resolved areas in the base simulation. Second-order neighborhood fitting is commonly used to compute curvature on point surfaces: this works well for smooth surfaces [Wang et al. 2013], but we deal with turbulent surfaces where quadratic fits can generate extreme and noisy curvatures, leading to instabilities over time.

We instead propose a new, robust alternative for evaluating curvature on point clouds. The curvature $c_i$ at point $i$ is defined as the signed distance of its neighbors to its tangent plane, easily obtained from the normal computed at each point (Section 3.4), which gives:

$$c_i = \sum_{j \in F} W_i^\lambda_j (\mathbf{n}_i \cdot (\mathbf{x}_i - \mathbf{x}_j)).$$ (13)

- $c_i$: surface curvature
- $c_{\text{min, max}}$: curvature thresholds
- $a_i$: oscillator amplitude
- $\Delta a$: oscillator amplitude step size
- $A$: max. oscillator amplitude
- $h_i$: internal wave height
- $v_i$: internal wave velocity
- $\Delta h_i$: max. wave frequency
- $t$: simulation time
- $i$: seed value

1 forall the surface points i do
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3     s_i := clamp(s_i + a_i * \cos(t * f), 0, h_i);
4     v_i := v_i + \Delta t * h_i;
5     v_i := clamp(v_i, -W, W);

Algorithm 2: Pseudocode for wave evolution and seeding.
curvature regions based on the curvature thresholds of Section 4.1. We seed with time-varying cosine oscillators (line 6), but these oscillations are never directly visualized. Instead, we apply the wave equation to them and only new waves that have propagated out from the cosine oscillations are visualized. This avoids double accumulation of wave values in regions of high curvature: once from their own oscillator and once from the waves generated by neighboring oscillators. This also causes waves to appear at the boundary of the seeding regions: for an isolated droplet, the seeding region has no boundary and, as no wave is seeded, the pulsing artifact is removed.

Our seeding is easy to implement: in addition to the wave value \( h_i \) used to solve the wave equation, we store a displayed wave value \( d_i \). We store cosine oscillators separately in seed values \( s_i \). At each step, we first add seed values to the displayed wave values to obtain the wave values. We perform wave simulation on \( h_i \) and subtract seed values from \( h_i \) to recover \( d_i \). The \( d_i \) thus only contain new features propagated out from the seed values, in addition to features that have evolved from previous timesteps. We never visualize \( h_i \).

Finally, inspired by smoke up-res methods [Kim et al. 2008], our approach seeds features across multiple frequency octaves. To enable further control, the base seeding frequency \( f_i \) of the cosine oscillators, and the number of additional octaves \( f_o \), are exposed as user parameters. The accompanying video shows a comparison of our seeding method versus more direct approaches.

### 4.3 Turbulence Evolution

We evolve waves on the surface using the standard wave equation,

\[
\partial_t^2 h = c \Delta h, \tag{14}
\]

where \( c \) is the wave speed and \( \Delta \) is the Laplace-Beltrami operator. We solve this equation explicitly with a symplectic Euler scheme that also requires the wave velocity \( v_i \) to be stored at each surface point. The linear wave equation, due to its dispersive nature, only approximates capillary surface wave behavior. The wave speed \( c \) must thus be selected empirically, for instance by estimating the desired traveled distance of waves between two given frames. Despite this, the linear wave model is widely used [Thuerey et al. 2010; Chentanez and Müller 2010] and yields convincing results. Similar to these works, we add a small amount of diffusion to approximate wave dissipation, but this is not required to maintain stability.

Some previous works solve differential equations involving the Laplace-Beltrami operator on point surfaces [Liang et al. 2012; Liang and Zhao 2013], requiring the evaluation of surface curvature as well as second derivatives of the embedded function. In our case, the curvature of the surface is small compared to the length \( \lambda_f \) at which the wave equation operates, so we instead approximate the Laplace-Beltrami operator in Equation 14 with a simpler, flat Laplace operator. We avoid the metric tensor computations of Laplace-Beltrami, which are both costly and difficult to robustly evaluate on point surfaces. Figure 6 compares our flat Laplace operator to Laplace-Beltrami, showing results that are indistinguishable even after 1000 simulation steps. Here, the approximation was 6 times faster to compute and gave a maximum height difference of only 1.4%. Appendix B further validates our approximation.

A common method [Liang and Zhao 2013] for computing the flat Laplace operator on a point surface is to use the derivatives of a local quadratic least squares approximation of the function. Although this works with densely sampled surfaces, we generate waves at scales that can be of the same order as the distance between points. Only a few points can then be used for the quadratic approximation, which is imprecise and leads to instabilities over time. Our supplemental video and Figure 7 illustrate such instabilities.

We instead compute the Laplace operator by using the tangent plane, which was previously obtained during surface normal computation (section 3.4). By projecting nearby points onto the tangent plane, the displacement defined by the wave values \( h_i \) becomes a function defined on the tangent plane. In this coordinate system, an affine approximation \( P \) of the function is first computed using standard least-square minimization. Subtracting the values of \( P \) on each neighboring point eliminates the zeroth- and first-order derivatives of the function, so the Laplacian can be evaluated directly as a weighted sum of discrete directional second-order derivatives:

\[
\Delta h_i = \sum_{j \in F} W_i^{2\lambda_f} (\mathbf{x}_j) \frac{4 ((h_j - P(\mathbf{x}_i)) - (h_i - P(\mathbf{x}_j)))}{\| \mathbf{x}_i - \mathbf{x}_j \|^2}. \tag{15}
\]

We have found that a neighborhood radius size of \( 2\lambda_f \) works well. Appendix B shows how our operator approximates the Laplacian. To our knowledge, we are the first to introduce this discrete operator for computing the Laplacian on a meshless set of points. Our supplemental video and Figure 7 show that it matches computations using the usual quadratic least squares approximation. Furthermore, since it uses an affine least squares fit instead of a quadratic fit, our operator remains stable even for waves at the highest frequency representable by the surface points. As such, it is particularly well suited to our meshless point representations, and we observed it was 2 times faster to evaluate than the quadratic least squares Laplacian.

We inject ghost points when surface points approach obstacles (Section 3.5) and also use these ghost points during wave computations, where we simply copy the wave value on a ghost point from its original “real” surface point. This gives Neumann boundary conditions on the obstacles and yields the expected wave reflections.

### 5 Results and Discussion

**Up-resed Simulations.** We apply our method to coarse simulations generated using Houdini 13’s FLIP solver. We apply our method to a large 12.5 million particles input simulation (Figures 1 and 12). Even at this high resolution, we enhance the visual quality by generating waves on a 500K surface point representation.
<table>
<thead>
<tr>
<th>Scene</th>
<th># particles</th>
<th># surface points</th>
<th>Total (sec/frame)</th>
<th>Advection</th>
<th>Regularization</th>
<th>Curvature</th>
<th>Laplacian</th>
<th>Wave</th>
<th>Disk I/O</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dam Break</td>
<td>1250k</td>
<td>500k</td>
<td>85.4</td>
<td>8.6%</td>
<td>53.1% (10)</td>
<td>5.4%</td>
<td>2.9%</td>
<td>0.3%</td>
<td>2.0%</td>
</tr>
<tr>
<td>River</td>
<td>400k</td>
<td>280k</td>
<td>42.2</td>
<td>14.7%</td>
<td>48.7% (3)</td>
<td>9.9%</td>
<td>10.8%</td>
<td>0.8%</td>
<td>15.1%</td>
</tr>
<tr>
<td>Double Drop</td>
<td>1400k</td>
<td>350k</td>
<td>25.9</td>
<td>8.1%</td>
<td>57.0% (5)</td>
<td>5.4%</td>
<td>7.9%</td>
<td>1.0%</td>
<td>4.9%</td>
</tr>
<tr>
<td>Stir</td>
<td>390k</td>
<td>17k</td>
<td>1.5</td>
<td>15.0%</td>
<td>49.9% (5)</td>
<td>6.7%</td>
<td>4.1%</td>
<td>1.1%</td>
<td>4.6%</td>
</tr>
<tr>
<td></td>
<td>390k</td>
<td>66k</td>
<td>5.4</td>
<td>10.8%</td>
<td>58.1% (5)</td>
<td>7.5%</td>
<td>6.0%</td>
<td>0.8%</td>
<td>4.8%</td>
</tr>
<tr>
<td></td>
<td>390k</td>
<td>145k</td>
<td>15.2</td>
<td>8.7%</td>
<td>61.0% (5)</td>
<td>7.0%</td>
<td>9.8%</td>
<td>0.6%</td>
<td>3.9%</td>
</tr>
</tbody>
</table>

Table 1: Timings for the various steps of our algorithm. All performance statistics were computed on an Intel i7 quad core running at 3.4 GHz with 32GB of RAM. In the regularization column, the parenthesis indicate the number of regularization steps used per frame.

Our complete post-process requires 85.4s/frame, compared to the 241s/frame used for the input simulation. This illustrates our scalability beyond resolutions achievable with regular fluid solvers.

Figure 9 features a complex moving obstacle [Lait 2011], and demonstrates various levels of up-resing: from a 390K input coarse particle simulation, we generate 17K, 66K and 145K surface points. Each successive point set is able to resolve higher frequency details corresponding to successively higher visual fidelity. In contrast to previous work [Kim et al. 2013], our method does not require any additional information apart from the points used to represent the final surface to simulate a similar amount of wave detail.

Figure 11 shows a complex, turbulent riverbed. Even at 400K particles, the coarse simulation cannot capture the intense turbulence that characterizes a river’s flow. Our up-resed output conveys this imagery, adding surface waves both where the water collides with rocks, as well as in stationary eddies behind these same obstacles.

Comparison with Full Resolution FLIP. Figure 10 compares the results of a high-resolution 4 million particles FLIP simulation with our method applied to a low resolution simulation comprising just 2500 particles. The 4 million particles are able to resolve certain fine structures, such as the splash, however, the final surface only contains rough, low frequency waves. In comparison, our method crisply resolves waves with many more frequencies. The 4 million particles scene requires 142s/frame while ours uses only 4.74s/frame, corresponding to a 30× speedup. Obtaining comparable waves with only a FLIP simulation would require even more particles, increasing our speedup for an equal-quality wave motion.

Implementation and Performance. Our method relies heavily on surface point lookups in small neighborhoods, so it is crucial to use an acceleration structure to store the surface point data. We used a hashing structure [Teschner et al. 2003] to improve the efficiency of these operations, yielding a speedup in the range of 20× for 27K surface points to 200× for 290K surface points compared to brute-force lookups. Most of the computations are performed on individual surface points, so trivial parallelizations using OpenMP further accelerated these operations by another 4 to 8×. Full computation breakdowns for our four scenes are provided in Table 1.

Limitations. Since we are designing an upres technique, we deliberately leave the coarse dynamics of the simulation untouched. Consequently, we do not reduce the size of the smallest underlying simulation structures. While this does not affect the majority of a simulation, fine isolated structures (i.e., droplets) are limited by the size of the input simulation. As mentioned in Section 4.2, seeding waves on isolated particles leads to visible mass loss and undesirable behavior, so we leave them untouched. However, our method remains compatible with techniques that directly address this problem, such as Ihmsen et al.’s approach [2012] (see Figure 8).

The timings in Table 1 show the majority of our computation is spent in surface regularization and neighborhood queries. This is as expected since the λ scale used for normal evaluation and regularization (Section 3.4), as well as for curvature computations (Section 4.1), does not decrease as the number of surface points increases. The number of neighbors thus grows quadratically with the total number of surface points. However, there are redundancies in these queries, since they all relate to the same underlying coarse simulation regardless of the final point count. A hierarchical method designed to instead select a constant-sized subset of representative neighbors for use in these queries would reduce the complexity of the regularization steps to that of all other steps. Designing such a selection method is a natural direction for future work.

6 Conclusion

We have presented a fully Lagrangian method for enhancing a particle-based liquid simulation using surface waves. This was
made possible using a combination of several novel techniques, including a robust method for point surface creation and maintenance and a stable discrete Laplace operator, both of which can apply more broadly in settings involving surface operations on animated point sets. We also proposed a novel wave injection strategy based on bands of oscillators, and we have demonstrated that our method can efficiently process coarse input simulations (of both low- and high-resolutions) into highly detailed and turbulent liquid surfaces.

In the future, we plan to explore more complex and expressive waves models, art-directable editing controls for the fine-scale details, closer coupling to secondary particle systems for drops and foam effects, and the application of our method to other types of secondary surface simulations for Lagrangian data.

**Appendix A: Threshold Computations**

To evaluate our curvature measure on a surface of infinite point density, we consider the integral form of (13) as $|S| \to \infty$. Here, surface points all have equal density $\rho$, so we can replace $W$ with $K$. We omit the neighborhood size $\lambda_c$ from the kernel notation for brevity. We work in a local frame with the surface point of interest at $(0, \lambda_c, 0)$ with normal $(0, 1, 0)$ and $xz$-tangent plane, yielding

\[
p(\theta, \phi) = (\lambda_c \sin \theta \cos \phi, \lambda_c \cos \theta, \lambda_c \sin \theta \sin \phi)
\]

\[
c_i = \int_{S} K((0, \lambda_c, 0) - \mathbf{x}, \mathbf{e}_y) (\lambda_c - y) dS \int_{S} K((0, \lambda_c, 0) - \mathbf{x}) dS.
\]

For the case of a single drop, we consider the surface $\{x^2 + y^2 + z^2 = \lambda_c^2\}$. Solving the integral analytically in spherical coordinates

\[
K((0, \lambda_c, 0) - \mathbf{x}, 0) = (1 - \cos(\theta)) K((0, \lambda_c, 0) - \mathbf{x}) d\theta \quad K((0, \lambda_c, 0) - \mathbf{x}) d\phi
\]

\[
c_i = \frac{3\lambda_c}{20}
\]

For the case of a thin sheet, we consider the surface $\{(y \geq 0 \cap \{x^2 + y^2 = \lambda_c^2\}) \cup (y \leq 0 \cap \{z = \lambda_c\})\}$. Since $K$ is non-zero only in the cylindrical part of the thin sheet, we can use cylindrical coordinates $p(x, \theta) = (x, \lambda_c \cos \theta, \lambda_c \sin \theta)$:

\[
c_i = \int_{x = -\lambda_c}^{\lambda_c} \int_{\theta = -\pi/3}^{\pi/3} \lambda_c (1 - \cos(\theta)) K((0, \lambda_c, 0) - p(x, \theta)) \lambda_c dx d\theta
\]

\[
\int_{x = -\lambda_c}^{\lambda_c} \int_{\theta = -\pi/3}^{\pi/3} K((0, \lambda_c, 0) - p(x, \theta)) \lambda_c dx d\theta
\]

which evaluates numerically to $0.0771413 \lambda_c$ (to double precision).

**Appendix B: Laplace-Beltrami Approximation**

We justify our Laplace approximation and detail computations used for Figures 6 and 7.

**Flat Laplace Computation**

First, we show that (15) indeed approximates the Laplace operator at surface point $i$. We orient the coordinate system to have origin at point $i$ and tangent $xy$-plane. We express $W_{i}^{2\lambda_f}$ centered at Figure 9: We upres a 380K FLIP simulation (left) with 17K (middle, top), 66K (middle, bottom), and 145K (right) surface points.

Figure 11: We upres an input 400K particle FLIP simulation (middle bottom half; zoom-ins bottom) with 280K surface points (middle top half; zoom-ins top). Our surface waves interact realistically with the turbulent flow over the rocks and the resulting stationary eddies.
the origin as $W$, and override $\mathcal{F}$ as the neighboring points in those coordinates. We can rewrite the discrete operator as

$$\sum_{x \in \mathcal{F}} 4W(x)((h(x) - P(x)) - (h(0) - P(0)))/||x||^2$$

$$= \sum_{x \in \mathcal{F}} 4W(x)((h(x) - P(x)) + \nabla P(x) \cdot x - h(0) + P(0))/||x||^2$$

$$= \sum_{x \in \mathcal{F}} 4W(x)(h(x) - \nabla h(0) + O(\lambda_f^2) \cdot x - h(0))/||x||^2$$

$$= \sum_{x \in \mathcal{F}} 4W(x)(h(x) - \nabla h(0) \cdot x - h(0))/||x||^2 + O(\lambda_f^2)$$

where the error term in (16) results from the superconvergence demonstrated in Appendix A in Liang’s work [Liang 2013], and is possible due to the approximate point distribution symmetry maintained by our method (Section 3.4). As the point density approaches infinity, the operator converges to

$$\int_{D} 4K(x)(h(x) - \nabla h(0) \cdot x - h(0))/||x||^2 \, dx + O(\lambda_f^2)$$

$$= \int_{D} 2K(x)h''_x(0) \, dx + O(\lambda_f^2).$$

The error term disappears once $\lambda_f$ approaches zero as the point density increases. We split (19), effect a clockwise rotation of 90°, and use the symmetry of $K$ to obtain

$$= \int_{D} K(x)h''_x(0) \, dx + \int_{D} K(x)h''_y(0) \, dx$$

$$\approx \int_{D} K(x)h''_x(0) \, dx + \int_{D} K(x)h''_y(0) \, dx$$

$$= \int_{D} K(x)h''_y(0) \, dx + \int_{D} K(x)h''_x(0) \, dx$$

$$= \int_{D} K(x, y) \Delta h(0) \, dx$$

where the equality between (22) and the left most equation in (23) leverages the invariance of the Laplacian under rigid deformation.

**Flat Surface Approximation**

We can now justify our claim in Section 4.3 that locally approximating the curved surface with a flat surface yields negligible errors in the evaluation of differential quantities. We parameterize a quadratic surface $q$ about a given surface point as

$$q(x, y) = q_00 + q_{10}x + q_{01}y + q_{20}x^2 + q_{11}xy + q_{02}y^2.$$  (24)

Following [Liang and Zhao 2013], we compute a least square quadratic approximation centered at the surface point for both the surface ($f$) and the wave function ($h$) to approximate the Laplace-Beltrami operator at this point. The full expression for the operator can be derived as

$$2 \left( 1 + f_{00} + 2f_{10} + f_{01} + f_{11} + f_{10}f_{01} ight) h_{02}$$

$$+ 2 \left( 1 + f_{10} + f_{01} + f_{11} + f_{10}f_{01} ight) h_{20}$$

$$+ 2f_{11}h_{10} + 3f_{01}h_{20} + 3f_{00}h_{11} + 3f_{10}h_{02} + 2f_{00}h_{01} + 2f_{10}h_{11} + 3f_{00}h_{01} + 2f_{10}h_{11} + 3f_{00}h_{01}$$

$$+ 3f_{00}h_{01} + 2f_{10}h_{11} + 3f_{00}h_{01}$$

$$+ 2f_{11}h_{10} + 3f_{01}h_{20} + 3f_{00}h_{11} + 3f_{10}h_{02} + 2f_{00}h_{01} + 2f_{10}h_{11} + 3f_{00}h_{01}.$$  (25)

Notice that if we align our local coordinate system with the surface, i.e., $f_{10} = f_{01} = 0$, (25) simplifies to $2h_{20} + 2h_{02}$, which is the flat Laplace operator. While our normal computation in Algorithm 1 attempts to get as close as possible to this perfect alignment, numerical errors will be present. Still, Figure 6 shows that $f_{10}$ and $f_{01}$ are small enough to permit our approximation with a flat Laplacian. Moreover, by only keeping the first-order terms of (25), i.e., ignoring terms that depend at least quadratically on $f_{10}$ and $f_{01}$, we arrive at the first-order approximation

$$2h_{20} + 2h_{02} + 3f_{00}h_{01} + 2f_{10}h_{11} + 3f_{10}h_{02} + 2f_{00}h_{01}.$$  (26)

We apply this expression in Figure 6 when computing the Laplace-Beltrami operator, where $(2h_{20} + 2h_{02})$ is evaluated using (15) and the wave function derivatives $h_{10}$ and $h_{01}$ are evaluated using the affine approximation $P$ already computed in (15).

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**References**


