

A Dynamic Programming Procedure for Pricing American-Style Asian Options

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Pricing European-style Asian options based on the arithmetic average, under the Black and Scholes model, involves estimating an integral (a mathematical expectation) for which no easily computable analytical solution is available. Pricing their American-style counterparts, which provide early exercise opportunities, poses the additional difficulty of solving a dynamic optimization problem to determine the optimal exercise strategy. A procedure for pricing American-style Asian options of the Bermudan flavor, based on dynamic programming combined with finite-element piecewise-polynomial approximation of the value function, is developed here. A convergence proof is provided. Numerical experiments illustrate the consistency and efficiency of the procedure. Theoretical properties of the value function and of the optimal exercise strategy are also established.

(*Option Pricing; Asian Options; Path-Dependent Options; American Options; Bermudan Options; Dynamic Programming; Piecewise Polynomials*)

1. Introduction

A financial *derivative* is a contract that provides its holder with a future payment that depends on the price of one or more primitive asset(s), such as stocks or currencies. In a frictionless market, the no-arbitrage principle allows one to express the value of a derivative as the mathematical expectation of its discounted future payment, with respect to a so-called risk-neutral probability measure. *Options* are particular derivatives characterized by nonnegative payoffs. *European-style* options can be exercised at the expiration date only, whereas *American-style* ones offer early exercise opportunities to the holder.

For simple cases, such as for European call and put options written on a stock whose price is modeled as a geometric Brownian motion (GBM), as studied by

Black and Scholes (1973), analytic formulas are available for the fair price of the option. However, for more complicated derivatives which may involve multiple assets, complex payoff functions, possibilities of early exercise, stochastic time-varying model parameters, etc., analytic formulas are unavailable. These derivatives are usually priced either via Monte Carlo simulation or via numerical methods (e.g., Boyle et al. 1997, Hull 2000, Wilmott et al. 1993, and other references given there).

An important class of options for which no analytic formula is available even under the standard Black-Scholes GBM model is the class of *Asian options*, for which the payoff is a function of the arithmetic average of the price of a primitive asset over a certain time period. An Asian option can hedge the risk exposure

of a firm that sells or buys certain types of resources (raw materials, energy, foreign currency, etc.), on a regular basis over some period of time. Because the average in general is less volatile than the underlying asset price itself, these contracts are less expensive than their standard versions. Asian options are heavily traded over the counter and because of the possible lack of depth of these markets, their theoretical values often need to be computed on the fly for fair negotiations.

Asian options come in various flavors. For example, the average can be arithmetic or it can be geometric. One talks of a *plain vanilla* Asian option if the average is computed over the full trading period, and a *backward-starting* option if it is computed over a right subinterval of the trading period. This interval usually has a fixed starting point in time. The Asian option can be *fixed strike* (if the strike price is a fixed constant) or *floating strike* (if the strike is itself an average). It is called *flexible* when the payoff is a weighted average, and *equally weighted* when all the weights are equal. The prices are *discretely sampled* if the payoff is the average of a discrete set of values of the underlying asset (observed at discrete epochs), and *continuously sampled* if the payoff is the integral of the asset price over some time interval, divided by the length of that interval. The options considered in this paper are the most common: *Fixed-strike, equally weighted, discretely sampled Asian options based on arithmetic averaging*. The dynamic programming approach could also be used to price other kinds of discretely sampled Asian options.

European-style Asian (Eurasian) options can be exercised at the expiration date only, whereas *American-style ones (Amerasian)* offer earlier exercise opportunities, which may become attractive intuitively when the current asset price is below the current running average (i.e., is pulling down the average) for a call option, and when it is above the running average for a put. Here, we focus on Amerasian call options, whose values are harder to compute than the Eurasian ones because an optimization problem must be solved at the same time as computing the mathematical expectation giving the option's value. We assume that the exercise opportunities are only at the observation epochs (or a subset of them). These types of options are also called *Bermudan options*.

There is an extensive literature on the pricing of Eurasian options. In the context of the GBM model, there is a closed-form analytic solution for the value of discretely sampled Eurasian options only when they are based on the *geometric average* (Turnbull and Wakeman 1991, Zhang 1995). The idea is that under the GBM model, the asset price at any given time has the lognormal distribution, and the geometric average of lognormals is lognormal. Geman and Yor (1993) used Bessel processes and derived exact formulas for a class of continuous-time Eurasian options. For options based on the arithmetic average, solution approaches include *quasi-analytic* approximation methods based on Fourier transforms, Edgeworth and Taylor expansions, and the like (e.g., Bouaziz et al. 1994, Carverhill and Clewlow 1990, Curran 1994, Levy 1992, Ritchken et al. 1993, Turnbull and Wakeman 1991), methods based on partial differential equations (PDEs) and their numerical solution via *finite-difference* techniques (e.g., Alziary et al. 1997, Rogers and Shi 1995, Zvan et al. 1998), and *Monte Carlo simulation* coupled with variance-reduction techniques (e.g., Glasserman et al. 1999, Kemna and Vorst 1990, L'Ecuyer and Lemieux 2000, Lemieux and L'Ecuyer 1998, Lemieux and L'Ecuyer 2001).

Techniques for pricing Amerasian options are surveyed by Barraquand and Pudet (1996), Grant et al. (1997), and Zvan et al. (1998, 1999). Hull and White (1993) have adapted binomial lattices (from the binomial tree model of Cox et al. 1979) to the pricing of Amerasian options. Klassen (2001) has recently refined this approach. Broadie and Glasserman (1997a) proposed a simulation method based on nonrecombining trees in the lattice model that produces two estimators of the option value, one with positive bias and one with negative bias. By taking the union of the confidence intervals corresponding to these two estimators, one obtains a conservative confidence interval for the true value. However, the work and space requirements of their approach increases exponentially with the number of exercise opportunities. Broadie and Glasserman (1997b) then developed a simulation-based stochastic mesh method that accommodates a large number of exercise dates and high-dimensional American options. Their method may be adaptable to

Amerasian options, although this is not the route we take here. While revising this paper, we learned about related work by Wu and Fu (2000), who prove some properties of the exercise frontier for Amerasian options and propose an approach that parameterizes the exercise frontier and optimizes the parameters by a stochastic approximation algorithm combined with a simulation-based perturbation analysis gradient estimation method. Longstaff and Schwartz (2001) propose yet another simulation-based approach whose key idea is to approximate the same function as the value function in dynamic programming (DP) at each time step, but by fitting a linear combination of preselected basis functions to a data set generated by simulation using least-squares regression.

Zvan et al. (1998) have developed stable numerical PDE methods adapted from the field of computational fluid dynamics for pricing Amerasian options with continuously sampled prices. Zvan et al. (1999) have also adapted these PDE methods to options with discretely sampled asset prices, and with barriers. The numerical approach introduced in this paper is formulated in discrete time directly.

Pricing American-style options can be formulated as a Markov decision process, i.e., a stochastic DP problem, as pointed out by Barraquand and Martineau (1995) and Broadie and Glasserman (1997b), for example. The DP *value function* expresses the value of an Amerasian option as a function of the current time, current price, and current average. This value function satisfies a DP recurrence (or Bellman equation). Solving this equation yields both the option value and the optimal exercise strategy. For a general coverage of stochastic DP, we refer the reader to Bertsekas (1995).

In this paper, we write the DP equation for Amerasian options under the GBM model. Using this equation, we prove by induction certain properties of the value function and of the optimal *exercise frontier* (which delimits the region where it is optimal to exercise the option). We then detail a numerical solution approach for the DP equation, based on piecewise polynomial interpolation over rectangular finite elements. This kind of approach has been used in other application contexts, e.g., by Haurie and L'Ecuyer

(1986) and L'Ecuyer and Malenfant (1988). In fact, we reformulate the DP equation in a way that simplifies significantly the numerical integration at each step, thus improving the efficiency of the procedure. Convergence and consistency of the method as the discretization gets finer follow from the monotonicity properties of the value function. We also use an extrapolation method to accelerate the convergence. Numerical experiments indicate that the method is competitive, at least for certain situations. The DP approach could also be used for pricing other low-dimensional American-style products such as calls with dividends, puts, lookback options, and options with barriers. It would also work for other types of models than the GBM for the evolution of the underlying asset, e.g., for a CEV process (Cox 1996, Boyle and Tian 1999). The properties of the value function that we derive in §4 are easy to generalize. On the other hand, the implementation details that we develop in §5 are specific to the GBM model. These details would have to be reworked for other types of models.

The idea of this paper came after reading Grant et al. (1997). These authors also formulate the problem of pricing an Amerasian option in the dynamic programming framework, but use Monte Carlo simulation to estimate the value function at each point of some discretization of the state space, and identify a "good" exercise frontier by interpolation. Their estimate of the value function at the initial date is an estimate of the option value. These authors also propose to restrict the strategy of exercise to a class of sub-optimal rules where the exercise frontier is approximated by two linear segments, at each date of exercise opportunity. They observed in numerical examples that restricting the class of strategies in this way did not seem to diminish the value of the option significantly, but they provided no proof that this is true in general.

Here, we suggest replacing simulation at both stages by numerical integration, which is obviously less noisy, and we do not assume a priori a shape of the exercise frontier. For both the simulation approach and our approach, an approximation of the value function must be memorized, so the storage requirement is essentially the same for the two methods.

The rest of the paper is organized as follows. Section 2 presents the model and notation. In §3 we develop the DP formulation. In §4 we establish certain properties of the value function and of the optimal region of exercise. Our numerical approximation approach is detailed in §5. In §6 we report on numerical experiments. Section 7 is a conclusion.

2. Model and Notation

2.1. Evolution of the Primitive Asset

We assume a single primitive asset whose price process $\{S(t), t \in [0, T]\}$ is a GBM, in a world that satisfies the standard assumptions of Black and Scholes (1973). Under these assumptions (see, e.g., Karatzas and Shreve 1998), there is a probability measure Q called *risk neutral* under which the primitive asset price $S(\cdot)$ satisfies the stochastic differential equation

$$dS(t) = rS(t)dt + \sigma S(t)dW(t), \quad \text{for } 0 \leq t \leq T, \quad (1)$$

where $S(0) > 0$, r is the risk-free rate, σ is the volatility parameter, T is the maturity date, and $\{W(t), t \in [0, T]\}$ is a standard Brownian motion. The solution of (1) is given by

$$S(t'') = S(t')e^{\mu(t''-t') + \sigma[W(t'') - W(t')]}, \quad \text{for } 0 \leq t' \leq t'' \leq T, \quad (2)$$

where $\mu = r - \sigma^2/2$. An important feature is that the random variable $S(t'')/S(t')$ is lognormal with parameters $\mu(t'' - t')$ and $\sigma\sqrt{t'' - t'}$, and independent of the σ -field $\mathcal{F}(t') = \sigma\{S(t), t \in [0, t']\}$, i.e., of the trajectory of $S(t)$ up to time t' . This follows from the independent increments property of the Brownian motion. In addition, from the no-arbitrage property of the Black-Scholes model, the discounted price of the primitive asset is a Q -martingale:

$$\rho(t')S(t') = E[\rho(t'')S(t'') | \mathcal{F}(t')], \quad \text{for } 0 \leq t' \leq t'' \leq T, \quad (3)$$

where $\{\rho(t) = e^{-rt}, t \in [0, T]\}$ is the discount factor process and E is (all along this paper) the expectation with respect to Q . Details about risk-neutral evaluation can be found in Karatzas and Shreve (1998).

2.2. The Bermudan-Amerasian Contract

We consider a model similar to that of Grant et al. (1997). Let $0 = t_0 \leq t_1 < t_2 < \dots < t_n = T$ be a fixed sequence of *observation dates*, where T is the *time horizon*, and let m^* be an integer satisfying $1 \leq m^* \leq n$. The *exercise opportunities* are at dates t_m , for $m^* \leq m \leq n$. If the option is exercised at time t_m , the *payoff* of the (Bermudan)-Amerasian call option is $(\bar{S}_m - K)^+ \stackrel{\text{def}}{=} \max(0, \bar{S}_m - K)$, where $\bar{S}_m = (S(t_1) + \dots + S(t_m))/m$ is the arithmetic average of the asset prices at the observation dates up to time t_m . This model is quite flexible. For $n = 1$, we get a standard European call option. For $m^* = n > 1$, we have a Eurasian option. Here, we are not really interested in these degenerate cases, but in the case where $m^* < n$. To simplify the exposition we will assume that the observation dates are equally spaced: $t_i - t_{i-1} = h$ for $i = 1, \dots, n$, for some constant h . We will also drop the "Bermudan" qualifier, with the implicit understanding that everything is really in discrete time.

3. The Dynamic Programming Formulation

3.1. Value Function and Recurrence Equations

For $m = 0, \dots, n$, denote by $v_m(s, \bar{s})$ the value of the option at the observation date t_m when $S(t_m) = s$ and $\bar{S}_m = \bar{s}$, assuming that the decisions of exercising the option or not, from time t_m onwards, are always made in an optimal way (i.e., in a way that maximizes the option value). This optimal value is a function of the *state variables* (s, \bar{s}) and of the time t_m . We take the *state space* as $[0, \infty)^2$ for convenience, although at each time step, only a subset of this space is reachable: Because $S(\cdot)$ is always positive, at time t_m one must have $\bar{s} = s > 0$ if $m = 1$ and $\bar{s} > s/m > 0$ if $m > 1$. At time t_n , $v_n(s, \bar{s}) \equiv v_n(\bar{s})$ does not depend on s , whereas at time t_0 , \bar{s} is just a dummy variable in $v_0(s, \bar{s}) \equiv v_0(s)$, which depends only on s .

At time t_m , if $S(t_m) = s$ and $\bar{S}_m = \bar{s}$, the *exercise value* of the option (for $m \geq m^*$) is

$$v_m^e(\bar{s}) = (\bar{s} - K)^+, \quad (4)$$

whereas the *holding value* (i.e., the value of the option if it is not exercised at time t_m and if we follow an optimal exercise strategy thereafter) is

$$v_m^h(s, \bar{s}) = \rho E_{m, s, \bar{s}}[v_{m+1}(S(t_{m+1}), (m\bar{s} + S(t_{m+1}))/ (m+1))] \text{ for } 0 \leq m \leq n-1, \quad (5)$$

where $E_{m, s, \bar{s}}[\cdot]$ represents the conditional expectation $E[\cdot | \mathcal{F}(t_m), S(t_m) = s, \bar{S}_m = \bar{s}]$, and $\rho = e^{-r\Delta t}$ is the discount factor over the period $[t_m, t_{m+1}]$. This holding value $v_m^h(s, \bar{s})$ is the (conditional) expected value of the option at time t_{m+1} , discounted to time t_m .

The optimal value function satisfies:

$$v_m(s, \bar{s}) = \begin{cases} v_m^h(s, \bar{s}) & \text{if } 0 \leq m \leq m^* - 1, \\ \max\{v_m^e(\bar{s}), v_m^h(s, \bar{s})\} & \text{if } m^* \leq m \leq n-1, \\ v_m^e(\bar{s}) & \text{if } m = n. \end{cases} \quad (6)$$

The optimal exercise strategy is defined as follows: In state (s, \bar{s}) at time t_m , for $m^* \leq m < n$, exercise the option if $v_m^e(\bar{s}) > v_m^h(s, \bar{s})$, and hold it otherwise. The value of the Amerasian option at the initial date t_0 , under an optimal exercise strategy, is $v_0(s) = v_0(s, \bar{s})$. The functions v_m^e , v_m^h , and v_m are defined over the entire state space $[0, \infty)^2$ for all m , via the above recurrence equations, even if we know that part of the state space is not reachable. (We do this to simplify the notation and to avoid considering all sorts of special cases.)

A natural way of solving (6) is via backward iteration: From the known function v_n and using (4)–(6), compute the function v_{n-1} , then from v_{n-1} compute v_{n-2} , and so on, down to v_0 . Here, unfortunately, the functions v_m for $m \leq n-2$ cannot be obtained in closed form (we will give a closed-form expression for v_{n-1} in a moment), so they must be approximated in some way. We propose an approximation method in §5. In the next section, we establish some properties of v_m and of the optimal strategy of exercise, which are interesting per se and are also useful for analyzing the numerical approximation techniques.

4. Characterizing the Value Function and the Optimal Exercise Strategy

4.1. The Value Function v_{n-1}

Recall that the value function v_n at the horizon $T = t_n$ has the simple form $v_n(s, \bar{s}) = (\bar{s} - K)^+$. We now derive a closed-form expression for the value function at time t_{n-1} , the last observation date before the horizon. We assume that $1 \leq m^* \leq n-1$ (otherwise, one has $v_{n-1} = v_{n-1}^h$ and the argument simplifies). From (5) we have

$$v_{n-1}^h(s, \bar{s}) = \rho E_{n-1, s, \bar{s}} \left[\left(\frac{(n-1)\bar{s} + S(t_n)}{n} - K \right)^+ \right] = \frac{\rho}{n} E_{n-1, s, \bar{s}} [(S(t_n) - \bar{K})^+],$$

where $\bar{K} = nK - (n-1)\bar{s}$.

We first consider the case where $\bar{K} \leq 0$, i.e., $\bar{s} \geq Kn/(n-1)$. The holding value can then be derived from (3) as the linear function

$$v_{n-1}^h(s, \bar{s}) = v_{n-1}^{\text{lin}}(s, \bar{s}) = \frac{s}{n} - \frac{\rho}{n} \bar{K} = \frac{s}{n} + \rho \frac{n-1}{n} \bar{s} - \rho K,$$

and the exercise value equals $\bar{s} - K > 0$. One can easily identify the optimal decision (exercise or not) for any given state (s, \bar{s}) by comparing this exercise value with the holding value v_{n-1}^{lin} . This yields an explicit expression for the value function. Consider the line defined in the (s, \bar{s}) plane by $\bar{s} - K = v_{n-1}^{\text{lin}}(s, \bar{s})$, that is,

$$s - (n - (n-1)\rho)\bar{s} + nK(1 - \rho) = 0. \quad (7)$$

The optimal strategy here is: Exercise the option if and only if (s, \bar{s}) is above the line (7). This line passes through the point $(K, K)n/(n-1)$ and has a slope of $1/(n - (n-1)\rho) < 1$, so it is optimal to exercise for certain pairs (s, \bar{s}) with $s > \bar{s}$, a possibility neglected by Grant et al. (1997). A partial intuition behind this optimal strategy is that for sufficiently large \bar{s} and for $s < \bar{s}$, the average price will most likely decrease in the future (it is pressured down by the current value), so it is best to exercise right away.

We now consider the case $\bar{K} > 0$, i.e., $\bar{s} < Kn/(n-1)$. In this case, the holding value is equivalent to the value of a European call option under the

GBM model, with strike price \bar{K} , initial price s for the primitive asset, maturity horizon $T - t_{n-1} = h$, volatility σ , and risk-free rate r . This value is given by the well-known Black-Scholes formula:

$$v^{\text{BS}}(s, \bar{s}) = \frac{1}{n} \left(\Phi(d_1)s - \rho \bar{K} \Phi(d_1 - \sigma \sqrt{h}) \right),$$

where

$$d_1 = \frac{\ln(s/\bar{K}) + (r + \sigma^2/2)h}{\sigma \sqrt{h}}$$

and Φ denotes the standard normal distribution function. If $\bar{s} \leq K$, one must clearly hold the option because the exercise value is 0. For $\bar{s} > K$, the optimal decision is obtained by comparing $v^{\text{BS}}(s, \bar{s})$ with $\bar{s} - K$, similar to what we did for the case where $\bar{K} \leq 0$. We now have a closed-form expression for v_{n-1} :

$$v_{n-1}(s, \bar{s}) = \begin{cases} \max\{\bar{s} - K, [s + (n-1)\rho\bar{s}]/n - \rho K\} & \text{if } \bar{s} \geq Kn/(n-1), \\ \max\{\bar{s} - K, v^{\text{BS}}(s, \bar{s})\} & \text{if } \bar{s} < Kn/(n-1). \end{cases}$$

We could (in principle) compute an expression for v_{n-2} by placing our expression for v_{n-1} in the DP Equations (5) and (6), although this becomes quite complicated. The functions v_n and v_{n-1} are obviously continuous, but they are not differentiable (v_n is not differentiable with respect to \bar{s} at $\bar{s} = K$). These functions are also monotone nondecreasing with respect to both s and \bar{s} . Finally, the optimal exercise region at t_{n-1} is the epigraph of some function φ_{n-1} , i.e., the region where $\bar{s} > \varphi_{n-1}(s)$, where $\varphi_{n-1}(s)$ is defined as the value of \bar{s} such that $v_{n-1}^h(s, \bar{s}) = \bar{s} - K$. In the next subsection, we show that these general properties hold as well at time t_m for $m \leq n-1$.

4.2. General Properties of the Value Function and of the Exercise Frontier

We now prove certain monotonicity and convexity properties of the value function at each step, and use these properties to show that the optimal strategy of exercise at each step is characterized by a function φ_m whose graph partitions the state space in two pieces: At time t_m , for $m^* \leq m < n$, if $\bar{s} \geq \varphi_m(s)$ it is optimal to exercise the option now, whereas if $\bar{s} \leq \varphi_m(s)$ it is optimal to hold it for at least another step. We derive

these properties under the GBM model, but the proofs work as well if $S(t_{m+1})/S(t_m)$ has a different distribution than the lognormal, provided that it is independent of $\mathcal{F}(t_m)$ and that the relevant expectations are finite.

PROPOSITION 1. *At each observation date t_m , for $1 \leq m < n$, the holding value $v_m^h(s, \bar{s})$ is a continuous, strictly positive, strictly increasing, and convex function of both s and \bar{s} , for $s > 0$ and $\bar{s} > 0$. The function $v_0(s)$ enjoys the same properties as a function of s , for $s > 0$. For $1 \leq m < n$, the value function $v_m(s, \bar{s})$ also has these properties except that it is only nondecreasing (instead of strictly increasing) in s .*

PROOF. The proof proceeds by backward induction on m . At each step, we define the auxiliary random variable $\tau_{m+1} = S(t_{m+1})/S(t_m)$, which has the lognormal distribution with parameters μh and $\sigma \sqrt{h}$, independently of $\mathcal{F}(t_m)$, as in (2). A key step in our proof will be to write the holding value $v_m^h(s, \bar{s})$ as a convex combination of a continuous family of well-behaved functions indexed by τ_{m+1} .

For $m = n-1$, the holding value is

$$\begin{aligned} v_{n-1}^h(s, \bar{s}) &= \rho E_{n-1, s, \bar{s}}[v_n(\bar{S}_n)] \\ &= \rho \int_0^\infty \left(\frac{(n-1)\bar{s} + s\tau}{n} - K \right)^+ f(\tau) d\tau, \end{aligned}$$

where f is the density of τ_n . The integrand is continuous and bounded by an integrable function of τ over any bounded interval for s and \bar{s} . Therefore, the holding value v_{n-1}^h is also continuous by Lebesgue's dominated convergence theorem (Billingsley 1986). The integral is strictly positive because, for instance, the lognormal distribution always gives a strictly positive measure to the event $\{(n-1)\bar{s} + s\tau_n - nK \geq n\}$, on which the integrand is ≥ 1 .

To show that $v_{n-1}^h(s, \bar{s})$ is strictly increasing in \bar{s} , let $\bar{s} > 0$ let $\delta > 0$. One has

$$\begin{aligned} &v_{n-1}^h(s, \bar{s} + \delta) - v_{n-1}^h(s, \bar{s}) \\ &= \rho \int_0^\infty \left[\left(\frac{(n-1)(\bar{s} + \delta) + s\tau}{n} - K \right)^+ \right. \\ &\quad \left. - \left(\frac{(n-1)\bar{s} + s\tau}{n} - K \right)^+ \right] f(\tau) d\tau \end{aligned}$$

$$\begin{aligned} &\geq \rho \int_{(nK-(n-1)\bar{s})/s}^{\infty} \left[\frac{(n-1)(\bar{s} + \delta) + s\tau}{n} \right. \\ &\quad \left. - \frac{(n-1)\bar{s} + s\tau}{n} \right] f(\tau) d\tau \\ &\geq \frac{n-1}{n} \delta > 0. \end{aligned}$$

The same argument can be used to prove that $v_{n-1}^h(s, \bar{s})$ is strictly increasing in s . The convexity of $v_{n-1}^h(s, \bar{s})$ follows from the fact that this function is a positively weighted average (a convex combination), over all positive values of τ , of the values of $((n-1)\bar{s} + s\tau)/n - K)^+$, which are (piecewise linear) convex functions of s and \bar{s} for each τ . It is also straightforward to verify directly the definition of convexity for v_{n-1}^h (e.g., Rockafellar 1970): For any pair (s_1, \bar{s}_1) and (s_2, \bar{s}_2) , and any $\lambda \in (0, 1)$, $v_{n-1}^h(\lambda s_1 + (1-\lambda)s_2, \lambda \bar{s}_1 + (1-\lambda)\bar{s}_2) \leq \lambda v_{n-1}^h(s_1, \bar{s}_1) + (1-\lambda)v_{n-1}^h(s_2, \bar{s}_2)$.

Because the holding function is continuous and strictly positive, the value function

$$v_{n-1}(s, \bar{s}) = \max((\bar{s} - K)^+, v_{n-1}^h(s, \bar{s}))$$

is also continuous and strictly positive. It is also convex, nondecreasing in s , and strictly increasing in \bar{s} , because it is the maximum of two functions that satisfy these properties. (The maximum can be reached at $(\bar{s} - K)^+$ only if $\bar{s} > K$, since $v_{n-1}^h(s, \bar{s}) > 0$.)

We now assume that the result holds for $m+1$, where $1 \leq m \leq n-2$, and show that this implies that it holds for m . The holding value at t_m is

$$\begin{aligned} v_m^h(s, \bar{s}) &= \rho E_{m,s,\bar{s}}[v_{m+1}(s\tau_{m+1}, (m\bar{s} + s\tau_{m+1})/(m+1))] \\ &= \rho \int_0^{\infty} v_{m+1}(s\tau, (m\bar{s} + s\tau)/(m+1)) f(\tau) d\tau, \end{aligned} \quad (8)$$

where f is the lognormal density of τ_m . The function $v_m^h(s, \bar{s})$ is also continuous and strictly positive because the integrand is continuous, strictly positive, and bounded by an integrable function of τ over every bounded interval for s and \bar{s} . The other properties can be proved via similar arguments as for the case of $m = n-1$. The proof for v_0 is also similar, as for $m > 0$. We omit the details. \square

LEMMA 2. For $s > 0$ and $0 < \bar{s}_1 < \bar{s}_2$, one has

$$\begin{aligned} v_m^h(s, \bar{s}_2) - v_m^h(s, \bar{s}_1) &< \frac{m}{m+1} (\bar{s}_2 - \bar{s}_1) \rho \\ &\text{for } 1 \leq m < n \end{aligned} \quad (9)$$

and

$$v_m(s, \bar{s}_2) - v_m(s, \bar{s}_1) \leq \bar{s}_2 - \bar{s}_1 \quad \text{for } 1 \leq m \leq n. \quad (10)$$

PROOF. The proof proceeds again by backward induction on m . We will use the property that $b^+ - a^+ \leq b - a$ when $a \leq b$. For $m = n$, we have $v_n(s, \bar{s}_2) - v_n(s, \bar{s}_1) = (\bar{s}_2 - K)^+ - (\bar{s}_1 - K)^+ \leq \bar{s}_2 - \bar{s}_1$, so (10) holds for $m = n$. We now assume that (10) holds for $m+1$, where $m < n$, and show that this implies (9) and (10) for m . From (8), we have

$$\begin{aligned} &v_m^h(s, \bar{s}_2) - v_m^h(s, \bar{s}_1) \\ &= \rho \int_0^{\infty} \left(v_{m+1} \left(s\tau, \frac{m\bar{s}_2 + s\tau}{m+1} \right) \right. \\ &\quad \left. - v_{m+1} \left(s\tau, \frac{m\bar{s}_1 + s\tau}{m+1} \right) \right) f(\tau) d\tau \\ &\leq \rho \int_0^{\infty} \left(\frac{m\bar{s}_2 + s\tau}{m+1} - \frac{m\bar{s}_1 + s\tau}{m+1} \right) f(\tau) d\tau \\ &\leq \frac{m}{m+1} (\bar{s}_2 - \bar{s}_1) \rho. \end{aligned}$$

Moreover, $v_m^e(\bar{s}_2) - v_m^e(\bar{s}_1) = (\bar{s}_2 - K)^+ - (\bar{s}_1 - K)^+ \leq \bar{s}_2 - \bar{s}_1$. Now,

$$\begin{aligned} &v_m(s, \bar{s}_2) - v_m(s, \bar{s}_1) \\ &= \max(v_m^e(\bar{s}_2), v_m^h(s, \bar{s}_2)) - \max(v_m^e(\bar{s}_1), v_m^h(s, \bar{s}_1)) \\ &\leq \max(v_m^e(\bar{s}_2) - v_m^e(\bar{s}_1), v_m^h(s, \bar{s}_2) - v_m^h(s, \bar{s}_1)) \\ &\leq \bar{s}_2 - \bar{s}_1. \end{aligned}$$

This completes the proof. \square

PROPOSITION 3. For $m = m^*, \dots, n-1$, there exists a continuous, strictly increasing, and convex function $\varphi_m : (0, \infty) \rightarrow (K, \infty)$ such that

$$v_m^h(s, \bar{s}) \begin{cases} > v_m^e(\bar{s}) & \text{for } \bar{s} < \varphi_m(s) \\ = v_m^e(\bar{s}) & \text{for } \bar{s} = \varphi_m(s) \\ < v_m^e(\bar{s}) & \text{for } \bar{s} > \varphi_m(s). \end{cases} \quad (11)$$

PROOF. Let $s > 0$ and $m^* \leq m \leq n-1$. We know from Proposition 1 and Lemma 2 that $v_m^h(s, \bar{s})$ is always strictly positive and strictly increasing in \bar{s} , with a growth rate always strictly less than $\rho m/(m+1) < 1$. On the other hand, $v_m^e(\bar{s}) = (\bar{s} - K)^+$ is 0 for $\bar{s} \leq K$ and increases at rate 1 for $\bar{s} > K$. Therefore, there is a

unique value of $\bar{s} > K$, denoted $\varphi_m(s)$, such that (11) is satisfied.

To show that $\varphi_m(s)$ is strictly increasing in s , let $0 < s_1 < s_2$. If we suppose that $\varphi_m(s_1) \geq \varphi_m(s_2)$, we obtain the contradiction

$$\begin{aligned} 0 &< v_m^h(s_2, \varphi_m(s_2)) - v_m^h(s_1, \varphi_m(s_2)) \\ &= v_m^h(s_2, \varphi_m(s_2)) - v_m^h(s_1, \varphi_m(s_1)) + v_m^h(s_1, \varphi_m(s_1)) \\ &\quad - v_m^h(s_1, \varphi_m(s_2)) \\ &= \varphi_m(s_2) - K - [\varphi_m(s_1) - K] + v_m^h(s_1, \varphi_m(s_1)) \\ &\quad - v_m^h(s_1, \varphi_m(s_2)) \\ &= v_m^h(s_1, \varphi_m(s_1)) - v_m^h(s_1, \varphi_m(s_2)) \\ &\quad - [\varphi_m(s_1) - \varphi_m(s_2)] \leq 0, \end{aligned}$$

where the first inequality is by Proposition 1 and the last inequality is by (9) in Lemma 2. Therefore, $\varphi_m(s_1) < \varphi_m(s_2)$, i.e., $\varphi_m(s)$ is strictly increasing in s .

Now consider two states (s_1, \bar{s}_1) and (s_2, \bar{s}_2) where it is optimal to exercise the option at time t_m , i.e., for which $\bar{s}_1 \geq \varphi_m(s_1)$ and $\bar{s}_2 \geq \varphi_m(s_2)$. Let $(s_\lambda, \bar{s}_\lambda) = \lambda(s_1, \bar{s}_1) + (1 - \lambda)(s_2, \bar{s}_2)$ for $0 \leq \lambda \leq 1$. Because v_m^h is convex by Proposition 1,

$$\begin{aligned} v_m^h(s_\lambda, \bar{s}_\lambda) &\leq \lambda v_m^h(s_1, \bar{s}_1) + (1 - \lambda)v_m^h(s_2, \bar{s}_2) \\ &\leq \lambda(\bar{s}_1 - K) + (1 - \lambda)(\bar{s}_2 - K) \\ &= \bar{s}_\lambda - K = v_m^e(\bar{s}_\lambda). \end{aligned}$$

Therefore, in state $(s_\lambda, \bar{s}_\lambda)$ it is also optimal to exercise the option, i.e., $\bar{s}_\lambda \geq \varphi_m(s_\lambda)$. We have just proved that for any two points lying above the function φ_m , the straight line joining these two points is also above the function φ_m . This implies that φ_m is a convex function. The convexity then implies the continuity. \square

For $m = m^*, \dots, n - 1$, we define the (optimal) *exercise frontier* at time t_m as the graph of φ_m , i.e., the locus of points (s, \bar{s}) such that $v_m^h(s, \bar{s}) = v_m^e(\bar{s})$. The function $\varphi_m(s)$ is the *optimal exercise function* and its epigraph is the (optimal) exercise region. It is optimal to exercise the option at time t_m if $\bar{s} \geq \varphi_m(s)$, and hold it until the next exercise date t_{m+1} if $\bar{s} \leq \varphi_m(s)$. The optimal exercise function is illustrated in §6 for a numerical example.

5. Numerical Solution of the DP Equations

We now elaborate on the numerical approach that we have implemented for approximating the solution of the DP equations and the optimal exercise function. The general idea is to partition the positive quadrant of the plane (s, \bar{s}) by a rectangular grid and to approximate the value function at each observation date by a simple polynomial function on each rectangle of the grid (i.e., piecewise polynomial). However, instead of fitting the approximation to v_m directly, we will make the change of variable $\bar{s}' = (m\bar{s} - s)/(m - 1)$ at time t_m and redefine the value function in terms of (s, \bar{s}') by setting $w_m(s, \bar{s}') = v_m(s, \bar{s})$ before fitting a piecewise polynomial approximation to this new function w_m . This change of variable greatly simplifies the integration when the approximation is incorporated into Equation (5): It makes the integral unidimensional and allows us to compute it formally (explicitly) instead of numerically. The polynomial functions we tried are linear in s , and linear, quadratic, or cubic in \bar{s}' . Other types of approximations could also be used for w_m (see, e.g., de Boor 1978).

5.1. A Piecewise Polynomial Approximation

At time t_m , let

$$\bar{s}' = \begin{cases} (m\bar{s} - s)/(m - 1) & \text{if } m > 1, \\ 0 & \text{if } m \leq 1, \end{cases} \quad (12)$$

which is the value of \bar{S}_{m-1} if $S(t_m) = s$ and $\bar{S}_m = \bar{s}$ and define

$$w_m(s, \bar{s}') = v_m(s, \bar{s}), \quad (13)$$

where $\bar{s} = ((m - 1)\bar{s}' + s)/m$ if $m \geq 1$ and $\bar{s} = 0$ if $m = 0$. The function w_m has the same properties as stated for v_m in Proposition 1, except that w_1 does not depend on \bar{s}' . The recurrence (5)–(6) can be rewritten in terms of w_m as

$$\begin{aligned} w_m^e(s, \bar{s}') &= v_m^e(s, \bar{s}) = (\bar{s} - K)^+, \quad (14) \\ w_m^h(s, \bar{s}') &= v_m^h(s, \bar{s}) = \rho E_{m, s, \bar{s}}[w_{m+1}(s\tau_{m+1}, \bar{s})] \\ &\quad \text{for } 0 \leq m \leq n - 1, \quad (15) \end{aligned}$$

$$w_m(s, \bar{s}') = \begin{cases} w_m^h(s, \bar{s}') & \text{if } 0 \leq m \leq m^* - 1, \\ \max \{w_m^e(s, \bar{s}'), \\ w_m^h(s, \bar{s}')\} & \text{if } m^* \leq m \leq n - 1, \\ w_m^e(s, \bar{s}') & \text{if } m = n. \end{cases} \quad (16)$$

The idea now is to approximate each value function w_m by a continuous and piecewise polynomial function of (s, \bar{s}') , of degree 1 (linear) in s and degree d in \bar{s}' . We have experimented with $d = 1, 2$, and 3 , i.e., (piecewise) bilinear, linear-quadratic, and linear-cubic functions.

We first define a grid by selecting $0 = a_0 < a_1 < \dots < a_p < a_{p+1} = \infty$ and $0 = b_0 < b_1 < \dots < b_q < b_{q+1} = \infty$. The grid points $G = \{(a_i, b_j) : 0 \leq i \leq p \text{ and } 0 \leq j \leq q\}$, partition of the positive quadrant $[0, \infty) \times [0, \infty)$ into the $(p+1)(q+1)$ rectangles $R_{ij} = \{(s, \bar{s}') : a_i \leq s < a_{i+1} \text{ and } b_j \leq \bar{s}' < b_{j+1}\}$, for $i = 0, \dots, p$ and $j = 0, \dots, q$. In our implementations, we always took $a_1 = b_1$ and $a_p = b_q$, and, to simplify, we assume that these conditions hold in the following discussion (however, our convergence proof does not make this assumption). Under these conditions, if $(s, \bar{s}') \in G$, then \bar{s} in (15) is necessarily in the interval $[b_1, b_q]$ for $m > 0$, which means that we need an approximation of w_{m+1} only over the box $B = [0, \infty) \times [b_1, b_q]$.

We regroup the rectangles covering the box B as follows. Suppose for now that $q-1$ is a multiple of d . For $j \in J^{(d)} \stackrel{\text{def}}{=} \{1, d+1, 2d+1, \dots, q-d\}$, let

$$R_{i,j}^{(d)} = \begin{cases} R_{0,j} \cup \dots \cup R_{0,j+d-1} \\ \cup R_{1,j} \cup \dots \cup R_{1,j+d-1} & \text{for } i = 1, \\ R_{i,j} \cup \dots \cup R_{i,j+d-1} & \text{for } 1 < i < p-1, \\ R_{p-1,j} \cup \dots \cup R_{p-1,j+d-1} \\ \cup R_{p,j} \cup \dots \cup R_{p,j+d-1} & \text{for } i = p-1. \end{cases}$$

We consider approximating functions that are linear in s and polynomial of degree d in \bar{s}' over each of these rectangles $R_{i,j}^{(d)}$, and continuous across these rectangles.

To construct these piecewise polynomials, at each step m , for $m = n-2, \dots, 2$, we first compute an approximation of w_m at each point of G . This is done via the DP equations (4)–(6), using an available approximation \hat{w}_{m+1} of the function w_{m+1} (in a manner

to be detailed in a moment). Given the approximation $\tilde{w}_m(a_i, b_j)$ of $w_m(a_i, b_j)$ at each point $(a_i, b_j) \in G$, we interpolate \tilde{w}_m with a piecewise polynomial of degree d along the line $s = a_i$ for each i , and then interpolate linearly in s between the lines $s = a_i$ and $s = a_{i+1}$, for $i = 1, \dots, p-1$. At time t_{n-1} , we use the exact closed-form expression for the value function (since we know it) instead of an approximation. At times t_1 and t_0 , $w_m(s, \bar{s}')$ depends only on s , so only the piecewise linear interpolation in s is needed.

More specifically, for $2 \leq m \leq n-2$, over the line segment defined by $s = a_i$ and $\bar{s}' \in [b_j, b_{j+d}]$ for $j \in J^{(d)}$, we approximate w_m by the Newton interpolation polynomial

$$\hat{w}_m(a_i, \bar{s}') = P_{i,j}^m(\bar{s}') \stackrel{\text{def}}{=} \alpha_{i,j,0}^m + \alpha_{i,j,1}^m(\bar{s}' - b_j) + \dots + \alpha_{i,j,d}^m(\bar{s}' - b_j) \cdots (\bar{s}' - b_{j+d-1}). \quad (17)$$

The coefficients $\alpha_{i,j,\ell}^m$ are obtained by interpolating the known values of \tilde{w}_m at the points $(a_i, b_j), \dots, (a_i, b_{j+d})$, i.e., by solving the system of linear equations defined by

$$\tilde{w}_m(a_i, \bar{s}') = P_{i,j}^m(\bar{s}') \quad \text{for } \bar{s}' = b_j, \dots, b_{j+d}. \quad (18)$$

This is done for $i = 1, \dots, p$, and $j \in J^{(d)}$. Then, over each rectangle $R_{i,j}^{(d)}$ for $i < p$, the approximation $\hat{w}_m(s, \bar{s}')$ of $w_m(s, \bar{s}')$ is defined by the linear interpolation

$$\hat{w}_m(s, \bar{s}') = \frac{a_{i+1} - s}{a_{i+1} - a_i} \hat{w}_m(a_i, \bar{s}') + \frac{s - a_i}{a_{i+1} - a_i} \hat{w}_m(a_{i+1}, \bar{s}'). \quad (19)$$

In the case where $q-1$ is not a multiple of d , if j^* is the largest $j < q$ such that d divides $j-1$, we use a polynomial approximation of degree $q-j^* (< d)$ in \bar{s}' over the rectangles $R_{i,j^*}^{(d)}$, redefined by $R_{i,j^*}^{(d)} = R_{i,j^*} \cup \dots \cup R_{i,q-1}$.

5.2. Explicit Integration for Function Evaluation

We now examine how to compute the approximation $\tilde{w}_m(a_i, b_j)$ given the available approximation \hat{w}_{m+1} of w_{m+1} . Observe that w_m^h in (15) is expressed as an expectation with respect to a *single* random variable,

τ_{m+1} , and we have chosen our change of variable $(s, \bar{s}) \rightarrow (s, \bar{s}')$ precisely to obtain this property. Moreover, the fact that our approximation of w_{m+1} is piecewise linear in s makes the integral very easy to compute *explicitly* when this approximation is put into (15). More specifically, the holding value $w_m^h(s, \bar{s}')$ is approximated by

$$\begin{aligned} \tilde{w}_m^h(s, \bar{s}') &= \rho E_{m, s, \bar{s}}[\hat{w}_{m+1}(s\tau_{m+1}, \bar{s})] \\ &= \rho E_{m, s, \bar{s}} \left[\sum_{i=1}^{p-1} \left(\frac{a_{i+1} - s\tau_{m+1}}{a_{i+1} - a_i} P_{i, \xi}^{m+1}(\bar{s}) \right. \right. \\ &\quad \left. \left. + \frac{s\tau_{m+1} - a_i}{a_{i+1} - a_i} P_{i+1, \xi}^{m+1}(\bar{s}) \right) I_i \right], \end{aligned} \quad (20)$$

where ξ is the largest $j \in J^{(d)}$ such that $\bar{s} \geq b_j$ and I_i is the indicator of the event $\{(s\tau_{m+1}, \bar{s}) \in R_{i, \xi}\}$.

The function \tilde{w}_m^h is to be evaluated only over the points of G . For $k = 0, \dots, p$ and $l = 0, \dots, q$, if $(s, \bar{s}') = (a_k, b_l)$, we have $\bar{s} = c_{k,l} \stackrel{\text{def}}{=} ((m-1)b_l + a_k)/m$. By evaluating (20) at $(s, \bar{s}') = (a_k, b_l)$ and rearranging the terms, we obtain

$$\tilde{w}_m^h(a_k, b_l) = \rho \sum_{i=1}^p D_{k,i} P_{i, \xi}^{m+1}(c_{k,l}), \quad (21)$$

where ξ is the element of $J^{(d)}$ for which $c_{k,l} \in [b_\xi, b_{\xi+d})$,

$$D_{k,i} = \begin{cases} E \left(\frac{a_2 - a_k \tau}{a_2 - a_1} I_1 \right) & \text{for } i = 1, \\ E \left(\frac{a_{i+1} - a_k \tau}{a_{i+1} - a_i} I_i + \frac{a_k \tau - a_{i-1}}{a_i - a_{i-1}} I_{i-1} \right) & \text{for } 1 < i < p, \\ E \left(\frac{a_k \tau - a_{p-1}}{a_p - a_{p-1}} I_{p-1} \right) & \text{for } i = p, \end{cases}$$

τ is a lognormal random variable with parameters μh and $\sigma\sqrt{h}$, and I_i is the indicator of the event $\{a_k \tau < a_2\}$ if $i = 1$, $\{a_{p-1} \leq a_k \tau\}$ if $i = p-1$, $\{a_i \leq a_k \tau < a_{i+1}\}$ if $1 < i < p-1$. Defining $A_{k,i} = E[I_i]$ and $B_{k,i} = E[a_k \tau I_i]$, we can rewrite

$$\begin{aligned} D_{k,i} &= \begin{cases} \frac{a_2 A_{k,1} - B_{k,1}}{a_2 - a_1} & \text{for } i = 1, \\ \frac{a_{i+1} A_{k,i} - B_{k,i}}{a_{i+1} - a_i} + \frac{B_{k,i-1} - a_{i-1} A_{k,i-1}}{a_i - a_{i-1}} & \text{for } 1 < i < p, \\ \frac{B_{k,p-1} - a_{p-1} A_{k,p-1}}{a_p - a_{p-1}} & \text{for } i = p. \end{cases} \end{aligned}$$

Knowing that $E[I\{a_k \tau < a_i\}] = \Phi(x_{k,i})$ and $E[\tau I\{a_k \tau < a_i\}] = \Phi(x_{k,i} - \sigma\sqrt{h}) e^{rh}$ where Φ is the standard normal distribution and $x_{k,i} = [\ln(a_i/a_k) - \mu h]/(\sigma\sqrt{h})$, we find that

$$A_{k,i} = \begin{cases} \Phi(x_{k,2}) & \text{for } i = 1, \\ \Phi(x_{k,i+1}) - \Phi(x_{k,i}) & \text{for } 1 < i < p-1, \\ 1 - \Phi(x_{k,p-1}) & \text{for } i = p-1, \end{cases}$$

and

$$B_{k,i} = \begin{cases} a_k \Phi(x_{k,2} - \sigma\sqrt{h}) e^{rh} & \text{for } i = 1, \\ a_k [\Phi(x_{k,i+1} - \sigma\sqrt{h}) - \Phi(x_{k,i} - \sigma\sqrt{h})] e^{rh} & \text{for } 1 < i < p-1, \\ a_k [1 - \Phi(x_{k,p-1} - \sigma\sqrt{h})] e^{rh} & \text{for } i = p-1. \end{cases}$$

The constants $A_{k,i}$, $B_{k,i}$, and $D_{k,i}$ are precomputed before doing the first iteration, and the $D_{k,i}$ are then used to evaluate (21) at each step m .

This yields the approximate value function

$$\tilde{w}_m(a_k, b_l) = \max(\tilde{w}_m^h(a_k, b_l), (c_{k,l} - K)^+). \quad (22)$$

These values at the grid points are then interpolated to obtain the function \hat{w}_m as explained earlier. Integration and interpolation stages are repeated successively until $m = 0$, where an approximation of w_0 and of the option value $v_0(S(0))$ is finally obtained. An important advantage of choosing the same grid G for all m is that the coefficients $D_{k,i}$ can be precomputed once for all. It would also be possible to use an adaptive grid, where the grid points change with the observation dates. This could be motivated by the fact that the probability distribution of the state vector changes with time. In that case, the mathematical expectations $D_{k,i}$ would depend on m and would have to be recomputed at each observation date.

With the approximation \hat{w}_m in hand, at any given time step m the (approximate) optimal decision in any state (s, \bar{s}) is easily found by comparing $\hat{w}_m(s, \bar{s}')$ with $v_m^e(\bar{s}) = (\bar{s} - K)^+$. If one is willing to memorize the approximations \hat{w}_m , nothing else is needed. In fact, it suffices to memorize the approximations \hat{w}_m over a set of rectangles that covers the optimal exercise frontier for each m . If one does not want to store these approximations, one may compute and memorize an approximation of the function φ_m for each m . This involves

additional work, but storing these one-dimensional functions requires less memory than storing the two-dimensional functions \hat{w}_m . To approximate φ_m , one can first approximate $\varphi_m(s)$ on a grid of values of s by the value \bar{s} such that $(\bar{s} - K)^+ = \hat{w}_m(s, \bar{s})$. This value of \bar{s} , denoted $\hat{\varphi}_m(s)$, can be found by a bisection algorithm or by more refined methods. One can then fit a spline to the points $(s, \hat{\varphi}_m(s))$ by least squares or something of that flavor, and retain it as an approximation of the optimal exercise frontier at each exercise date.

As it turns out, this procedure evaluates, with no extra cost, the option value and the optimal decision at all observation dates and in all states. This could be used, for instance, to estimate the sensitivity of the option value with respect to the initial price. Eurasian options can be evaluated via this procedure as well, because they are a special case.

5.3. Computational Speed-up and Complexity Analysis

Whenever $\tilde{w}_m^h(a_k, b_l) < w_m^e(a_k, b_l)$ at some point (a_k, b_l) for $m \geq m^*$, i.e., if it is optimal to exercise at that point, we know that we are above the optimal exercise function, so for all $j \geq l$, $\tilde{w}_m^h(a_k, b_j) < w_m^e(a_k, b_j)$ and there is no need to compute $\tilde{w}_m^h(a_k, b_j)$. Our implementation exploits this property to save computations for the pricing of Amerasian options. If $m < m^*$, we cannot avoid the computation of $\tilde{w}_m^h(a_k, b_j)$ because one is not allowed to exercise the option. For this reason, Amerasian options are somewhat faster to evaluate than Eurasian options (see the numerical results in §6). This is one example of how the theoretical results proved in §4 can be exploited to improve computational efficiency. These results can be exploited not only by DP but by other methods as well.

Computing time can also be reduced by exploiting the fact that whenever $\tilde{w}_m(a_k, b_l)$ is small enough to be negligible (say, less than some fixed threshold ϵ_1), there is no need to compute $\tilde{w}_m(a_k, b_j)$ for $j < l$; it can be replaced by 0. We took $\epsilon_1 = 10^{-6}$ for the numerical examples of §6. Time can also be saved by taking advantage of the fact that (typically) several terms in the sum (21) are negligible, so there is no need to sum explicitly *all* the terms.

The time complexity of the algorithm for computing the value function is $O(p^2)$ to precompute the $D_{k,i} s$, plus $O(np^2q)$ to compute the sum (21) for each of the pq grid points at each of the n time steps, assuming that we compute *all* the terms of each sum, plus $O(npq)$ to solve the systems of linear equations giving the coefficients in (18). The overall time complexity is thus $O(np^2q)$.

For comparison, the time complexity of the PDE-based algorithm of Zvan et al. (1999) is $O(n'pq)$, where n' is the number of time steps in their time discretization. If the distance h between the successive observation dates is very small, n' can be of the same order as n , but if h is large (the observation dates are sparse), one must take $n' \gg n$ to reduce the time-discretization error. With DP, there is no need to discretize the time. This analysis suggests that PDE methods should be favored when h is small and that DP should become increasingly attractive as h increases.

5.4. Convergence Proof

Proving the convergence of the DP algorithm as the grid size becomes finer and finer is not straightforward because the state space is unbounded and the value function increases unboundedly when s or \bar{s} goes to infinity. However, if $c = \min(a_p, b_q) \rightarrow \infty$, then by Lemma 3 of Conze and Viswanathan (1991) and standard large deviations approximation for the normal distribution, we have

$$\begin{aligned} & Q \left[\max_{0 \leq t \leq T} S(t)/S(0) > c \right] \\ &= Q \left[\max_{0 \leq t \leq T} \ln[S(t)/S(0)] > \ln c \right] \\ &= 1 - \Phi \left(\frac{\ln c - \mu t}{\sigma \sqrt{t}} \right) + \exp \left(\frac{2\mu \ln c}{\sigma^2} \right) \Phi \left(\frac{-\ln c - \mu t}{\sigma \sqrt{t}} \right) \\ &= O \left(\frac{1}{\ln c} \exp \left[-\frac{\ln^2 c}{2\sigma^2 t} + O(\ln c) \right] \right) \\ &= O \left(\frac{1}{\ln c} c^{-\ln c / (2\sigma^2 t)} \right). \end{aligned} \tag{23}$$

Thus, the probability that the trajectory of $\{(S(t), \bar{S}(t)), 0 \leq t \leq T\}$ ever exits the box $B = (0, a_p] \times (0, b_q]$ decreases to 0 at a rate faster than $O(1/p(c))$ for any polynomial p . On the other hand, one can show that

the error on the value function can only increase linearly when s or \bar{s} goes to infinity. As a consequence, the effect of the approximation error outside the box B becomes negligible if the box is large enough. This is the basic idea of the proof of the next proposition. Besides proving convergence, the proposition also says that with the bilinear interpolation ($d = 1$), if we neglect the effect of the approximation error outside the box B , our DP procedure gives an *upper bound* on the option value.

Define $\delta_a = \sup_{1 \leq i \leq p} (a_i - a_{i-1})$ and $\delta_b = \sup_{1 \leq j \leq q} (b_j - b_{j-1})$.

PROPOSITION 4. *Assume that each approximation \hat{w}_m is nondecreasing. If $p \rightarrow \infty, q \rightarrow \infty, a_p \rightarrow \infty, b_q \rightarrow \infty, \delta_a \rightarrow 0$, and $\delta_b \rightarrow 0$, then for any constant $c > 0$,*

$$\sup_{0 \leq m < n} \sup_{(s, \bar{s}) \in (0, c]^2} |\hat{w}_m(s, \bar{s}) - w_m(s, \bar{s})| \rightarrow 0.$$

Moreover, if $d = 1, \hat{w}_m(s, \bar{s}) \geq w_m(s, \bar{s})$ for all m and all $(s, \bar{s}) \in (0, c]^2$, when a_p and b_q are large enough.

PROOF. In Lemma 2, we showed that the derivative of $v_m(s, \bar{s})$ and $v_m^h(s, \bar{s})$ with respect to \bar{s} never exceeds 1. We now show that their derivatives with respect to s are also bounded by the constant C_m , defined recursively by $C_n = 0$ and

$$C_m = \left(C_{m+1} + \frac{m}{(m+1)^2} \right) \rho E[\tau_{m+1}] = C_{m+1} + \frac{m}{(m+1)^2}.$$

By backward induction on m , it can be shown that for $s_1 \leq s_2$,

$$v_m(s_2, \bar{s}) - v_m(s_1, \bar{s}) \leq (s_2 - s_1)C_m. \quad (24)$$

This is clearly true for $m = n$, because $v_n(s, \bar{s})$ does not depend on s . If we assume that (24) holds for $m + 1$, then, using Lemma 2,

$$\begin{aligned} & v_m^h(s_2, \bar{s}) - v_m^h(s_1, \bar{s}) \\ &= \rho \int_0^\infty \left(v_{m+1} \left(s_2 \tau, \frac{m\bar{s} + s_2 \tau}{m+1} \right) \right. \\ & \quad \left. - v_{m+1} \left(s_1 \tau, \frac{m\bar{s} + s_1 \tau}{m+1} \right) \right) f(\tau) d\tau \\ &\leq \rho \int_0^\infty \left((s_2 - s_1) \tau C_{m+1} + \frac{m}{m+1} \frac{(s_2 - s_1) \tau}{m+1} \right) f(\tau) d\tau \end{aligned}$$

$$\begin{aligned} &= \rho(s_2 - s_1) \left(C_{m+1} + \frac{m}{(m+1)^2} \right) E[\tau_{m+1}] \\ &= (s_2 - s_1) C_m. \end{aligned}$$

Note that the slope of v_m with respect to s never exceeds that of v_m^h . This implies (24) and completes the induction. Moreover, the derivative of w_m does not exceed that of v_m , and similarly for w_m^h , so Lemma 2 and (24) hold for these functions as well.

Using these bounds on the slope, and the fact that w_m is increasing with respect to each of its arguments, we obtain that

$$\sup_{1 \leq k < p, 1 \leq \ell < q} [w_m(a_{k+1}, b_{\ell+1}) - w_m(a_k, b_\ell)] \leq C_m \delta_a + \delta_b. \quad (25)$$

Let us define $\epsilon_n = 0$ and, for $m = n - 1, n - 2, \dots, 0$,

$$\epsilon_m = 2\rho\epsilon_{m+1} + C_m \delta_a + \delta_b.$$

We also define an increasing sequence of square boxes $B_m = [0, c_m]^2$ as follows. Choose $c_0 = c$ arbitrarily, and for $m = 0, \dots, n - 1$, choose $c_{m+1} \geq c$ large enough so that for all $(s, \bar{s}) \notin B_m$,

$$\begin{aligned} & E_{m, s, \bar{s}} [|\hat{w}_{m+1}(s\tau_{m+1}, \bar{s}) - w_{m+1}(s\tau_{m+1}, \bar{s})| \\ & \quad \times I((s\tau_{m+1}, \bar{s}) \notin B_{m+1})] \leq \rho\epsilon_{m+1}, \end{aligned} \quad (26)$$

where I is the indicator function. Such a c_{m+1} exists because both \hat{w}_{m+1} and w_{m+1} are bounded by a linear function and the probability of exiting the box B_{m+1} decreases faster than the inverse of any (positive) linear function of c_{m+1} when $c_{m+1} \rightarrow \infty$, thanks to (23).

We now show, by backward induction on m , that $|\hat{w}_m - w_m|$ is bounded by ϵ_m over the box B_m . This is clearly true for $m = n$, because $\hat{w}_n = w_n$. Now, if we assume that $|\hat{w}_{m+1}(s, \bar{s}) - w_{m+1}(s, \bar{s})| \leq \epsilon_{m+1}$ for $(s, \bar{s}) \in B_{m+1}$, then we have, for $(s, \bar{s}) \in B_m$,

$$\begin{aligned} & \tilde{w}_m(s, \bar{s}) - w_m(s, \bar{s}) \\ &\leq \rho \int_0^\infty |\hat{w}_{m+1}(s\tau, \bar{s}) - w_{m+1}(s\tau, \bar{s})| f(\tau) d\tau \\ &\leq \rho\epsilon_{m+1} Q[(s\tau_{m+1}, \bar{s}) \in B_{m+1}] \\ & \quad + E_{m, s, \bar{s}} [|\hat{w}_{m+1}(s\tau_{m+1}, \bar{s}) - w_{m+1}(s\tau_{m+1}, \bar{s})| \\ & \quad \quad \times I((s\tau_{m+1}, \bar{s}) \notin B_{m+1})] \\ &\leq 2\rho\epsilon_{m+1}. \end{aligned}$$

Then, because \hat{w}_m is a nondecreasing interpolation of \tilde{w}_m in the box B_m and using (25), we have that

$$|\hat{w}_m(s, \vec{s}) - w_m(s, \vec{s})| \leq 2\rho\epsilon_{m+1} + C_m\delta_a + \delta_b = \epsilon_m.$$

Under the assumptions of the proposition, $\epsilon_m \rightarrow 0$ and this proves the first part.

In the case of the piecewise-linear interpolation ($d = 1$), it is also easily seen that the interpolation always *overestimates* w_m , because w_m is increasing and convex. That is, by backward induction on m one can show that $\hat{w}_m \geq \tilde{w}_m \geq w_m$. \square

5.5. Grid Choice, Refinement, and Convergence Acceleration

In the experiments reported here, the grid G was chosen as follows: We took $p = q$, $b_i = a_i$ for all i , $a_1 = S(0) \exp(\mu t_{n-1} - 5\sigma\sqrt{t_{n-1}})$, $a_{p-1} = S(0) \exp(\mu t_{n-1} + 5\sigma\sqrt{t_{n-1}})$, $a_p = S(0) \exp(\mu t_{n-1} + 6\sigma\sqrt{t_{n-1}})$, and for $2 \leq i \leq p-2$, a_i was the quantile of order $(i-1)/(p-2)$ of the lognormal distribution with parameters μt_{n-1} and $\sigma\sqrt{t_{n-1}}$. This choice is heuristic and certainly not optimal. Ideally, the density of grid points should be increased in the regions that are visited with high probability and where the value function tends to be less linear.

To assess the discretization error, we applied the algorithm repeatedly for different grid sizes, doubling the value of p ($=q$) each time, until there was no significant change in the option value estimate. We also made a sensitivity analysis with respect to ϵ_1 : For $\epsilon_1 < 10^{-6}$ (e.g., 10^{-7} or 10^{-8}) the results for the option value were the same as those reported here. Also, replacing the numbers 5 and 6 by larger numbers in the definitions of a_1 , a_{p-1} , and a_p did not change the results.

The sequence of DP approximations of the option value for successive values of p , where each value of p is twice the previous one, converges to the true option value when $p \rightarrow \infty$. The straightforward way of estimating the option value is then to take the approximation obtained with the finest grid (largest p). But one can do better by using extrapolation methods designed for accelerating the convergence of sequences and that transform a given sequence into another sequence that converges more rapidly under very broad conditions. One of these

methods (the most powerful, some say) is the ϵ -algorithm introduced by Wynn in 1956 (see Wynn 1966, Brezinski 1978, Sidi 1996), which works as follows. Let $s_1^{(1)}, \dots, s_\ell^{(1)}$ be the original (finite) sequence of length ℓ , where ℓ is *odd*, and let $s_k^{(0)} = 0$ for all k . For $j = 2, \dots, \ell$ and $k = 0, \dots, \ell - j + 1$, define

$$s_k^{(j)} = s_{k+1}^{(j-2)} + \left(s_{k+1}^{(j-1)} - s_k^{(j-1)} \right)^{-1}.$$

The ℓ th and last sequence has a single term, $s_0^{(\ell)}$, which is the final approximation of the convergence point of the original sequence. This algorithm performs surprisingly well in various contexts. The reader can consult the references for the justification and theory. In our context, we take it as a heuristic and always use $\ell = 3$: The original sequence is formed by taking the DP approximation for three successive grid sizes, say $(p/4) \times (p/4)$, $(p/2) \times (p/2)$, and $p \times p$. The CPU time required for computing $s_0^{(3)}$ is approximately the sum of CPU times required by DP for the three grids, and is only marginally more than the CPU time for the $p \times p$ grid alone. Of course, such extrapolation methods can be used to accelerate convergence of any method that uses successively refined discretizations (e.g., PDE methods, binomial trees, etc.), not just DP.

6. Numerical Experiments and Examples

EXAMPLE 1. For our first example, we take $S(0) = 100$, $K = 100$, $T = 1/4$ (years), $\sigma = 0.15$, $r = 0.05$, $h = 1/52$, $m^* = 1$, and $n = 13$. This is a 13-week contract, with an exercise opportunity at each observation epoch, which is every week. We also consider three slight modifications of this example: We first increase the volatility σ from 0.15 to 0.25, we then increase T from 1/4 to 1/2 (26 weeks) while keeping $n = 13$, and finally we increase K from 100 to 105 to obtain an out-of-the-money option. In each case, we evaluate the Eurasian and Amerasian options with five grid sizes, as indicated in Tables 1 and 2. We made the computations with the linear-linear ($d = 1$), linear-quadratic ($d = 2$), and linear-cubic ($d = 3$) approximations. In Tables 1 and 2, for each parameter set and each grid size, the DP approximation of $v_0(S(0))$ obtained with

Table 1 Approximations of the Eurasian Call Option Prices

(K, T, σ)	$p = 150$	$p = 300$	$p = 600$	$p = 1,200$	$p = 2,400$	Simulation
(100, 0.25, 0.15)	2.16581	2.16510	2.16493	2.16489	2.16487	[2.16483, 2.16492]
	2.16547	2.16502	2.16491	2.16488	2.16487	
(100, 0.25, 0.25)	3.36556	3.36439	3.36411	3.36402	3.36402	[3.3639, 3.3642]
	3.36501	3.36426	3.36408	3.36403	3.36402	
(100, 0.50, 0.25)	4.92929	4.92765	4.92726	4.92716	4.92713	[4.9269, 4.9274]
	4.92851	4.92746	4.92721	4.92715	4.92713	
(105, 0.50, 0.25)	2.80811	2.80647	2.80607	2.80597	2.80595	[2.8058, 5.8062]
	2.80734	2.80628	2.80602	2.80596	2.80594	
CPU (h:min:sec)	0:00:01	0:00:10	0:01:16	0:09:47	1:18:25	00:00:13
	0:00:01	0:00:10	0:01:15	0:09:50	1:23:13	

$d = 1$ is given above that obtained with $d = 2$. We see that the two values are essentially the same with the finest grid, and that $d = 2$ does slightly better than $d = 1$ for the coarse grids. We also tried with $d = 3$, but the convergence was not really faster and the CPU times were much higher.

The timings reported here are for a 750Mhz PC running the Linux operating system. The programs are written in FORTRAN and were compiled with the GNU g77 compiler. The CPU times are given for the first set of parameters values and are approximately the same for the other sets, in each the two tables. With $d = 2$, there are more coefficients to determine in each rectangular box than with $d = 1$, but there is only half the number of boxes for the same grid size.

The approximation of $v_0(S(0))$ converges rapidly as the grid size is refined. With the coarsest grid of 150×150 , the error is already less than one quarter of a cent, and the result is obtained in less than 1 second. The values provided by the ϵ -algorithm (not shown

in the table) converge even faster: With $p = 600$, these values never differ by more than 0.00001 from the values given in the table for $p = 2,400$, which can be considered as practically exact.

We have also estimated the value of the Eurasian option by Monte Carlo simulation, using the geometric average as a control variate, as suggested by Kemna and Vorst (1990). The column labeled "simulation" gives the 95% confidence interval thus obtained, with one million replications. It agrees with the DP approximations.

When we compare the value of the Amerasian option with its Eurasian counterpart, we see that the privilege of early exercise increases the value of the option, as expected. The contract becomes more expensive when the volatility or the maturity date is increased, because this gives more chance of achieving a large average. It becomes cheaper when the strike price is increased.

To quantify the impact of increasing the number of early exercise opportunities (and observation dates), we performed additional experiments with the same parameter sets as in Table 2, but with different values of n ranging from 1 to 52. For each of the four parameter sets in Table 3, the top and bottom lines give the value of the Amerasian and Eurasian call options, respectively, computed via DP with a 600×600 grid. The linear-linear and linear-quadratic approximations give exactly the same values for the first four decimal digits shown here.

We see that increasing n decreases the option value. The explanation is that increasing the number of observation dates increases the stability of the average

Table 2 Approximations of the Amerasian Call Option Prices

(K, T, σ)	$p = 150$	$p = 300$	$p = 600$	$p = 1,200$	$p = 2,400$
(100, 0.25, 0.15)	2.32209	2.32114	2.32091	2.32085	2.32084
	2.32156	2.32101	2.32088	2.32085	2.32084
(100, 0.25, 0.25)	3.65221	3.65058	3.65019	3.65009	3.65006
	3.65129	3.65037	3.65013	3.65008	3.65006
(100, 0.50, 0.25)	5.33506	5.33273	5.33217	5.33203	5.33200
	5.33375	5.33242	5.33209	5.33201	5.33199
(105, 0.50, 0.25)	2.96851	2.96633	2.96580	2.96567	2.96564
	2.96732	2.96605	2.96573	2.96565	2.96563
CPU (h:min:sec)	0:00:01	0:00:06	0:00:48	0:06:04	0:48:37
	0:00:01	0:00:06	0:00:48	0:06:16	0:51:15

Table 3 Amerasian (Top) and Eurasian (Bottom) Option Values As a Function of n

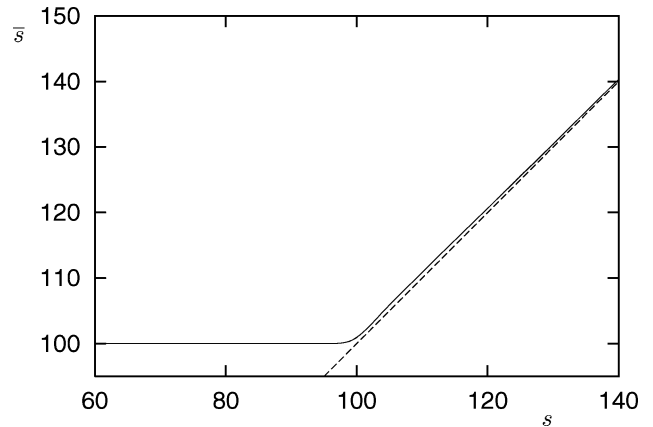
(K, T, σ)	n					
	1	2	4	13	26	52
(100, 0.25, 0.15)	3.635	2.842	2.512	2.321	2.289	2.276
	3.635	2.842	2.443	2.165	2.103	2.072
(100, 0.25, 0.25)	5.598	4.395	3.920	3.650	3.608	3.592
	5.598	4.395	3.788	3.364	3.270	3.222
(100, 0.50, 0.25)	8.260	6.463	5.745	5.332	5.266	5.239
	8.260	6.462	5.558	4.927	4.787	4.716
(105, 0.50, 0.25)	5.988	4.245	3.475	2.966	2.858	2.804
	5.988	4.245	3.389	2.806	2.678	2.614

prices, and this offsets the advantage of having more exercise opportunities. Note that $n = 1$ corresponds to a standard European call. For $n = 2$, it is optimal to exercise at time t_1 only if $S(t_1) = \bar{s}_1 \geq 2K$ (see §4.1), which is an extremely rare event with our choice of parameters. This is why the Amerasian and Eurasian options have practically the same value when $n = 2$.

Figures 1 and 2 show the optimal exercise frontier at times t_{n-2} and t_2 , respectively, for this example, for the Amerasian option with parameters $(K, T, \sigma, n) = (100, 0.5, 0.25, 52)$. These figures illustrate the fact that the farther away from the time horizon we are, the higher the exercise frontier is: It makes sense to wait even if the current price is somewhat below the current average, because things have time to change. The function w_n (not shown here) depends almost exclusively on \bar{s} (very little on s) and is almost piecewise linear when we are near the time horizon, but the dependence on s and the nonlinearity increase when we move away (backwards) from the time horizon.

EXAMPLE 2. Our second example is the one considered by Grant et al. (1997). The time increment is fixed at $h = 1/365$ (1 day), the first observation date is at $t_1 = 91/365$ (91 days), and the first exercise opportunity is at $t_m^* = 105/365$ (105 days). The other parameters are $S(0) = 100$, $K = 100$, $T = 120/365$, $\sigma = 0.20$, and $r = 0.09$. Table 4 gives our approximation of $v_0(S(0))$ for the Amerasian option with different grid sizes, as in Table 2, with $d = 2$ (the linear-quadratic approximation). In the second line of each entry (the numbers in parentheses) we give the extrapolation value obtained with the ϵ -algorithm with $\ell = 3$, based on the DP

Figure 1 The Optimal Exercise Frontier at Time t_{n-2} for Example 1 (Solid Line)

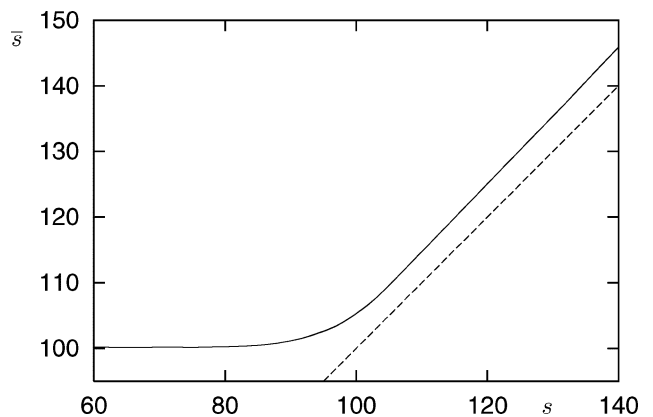


Note. The dotted line is the diagonal $\bar{s} = s$.

approximations with $p/4$, $p/2$, and p . These extrapolation values stabilize faster than the DP approximations themselves.

The column labeled GVW gives the 95% confidence interval reported by Grant et al. (1997) for the value of the option with the strategy obtained by their procedure. The difference from our values can be explained in part by the fact that their procedure systematically underestimates the values of Amerasian call options, because the exercise strategy found by their simulation procedure is suboptimal, so when they use this strategy in a simulation, their estimator of the optimal price has a negative bias. Further negative bias is

Figure 2 The Optimal Exercise Frontier at Time t_2 for Example 1 (Solid Line)



Note. The dotted line is the diagonal $\bar{s} = s$.

Table 4 Approximation of the Option Value for the GVW Example

(K, T, σ)	$p = 150$	$p = 300$	$p = 600$	$p = 1,200$	$p = 2,400$	GVW
(100, 0.2)	5.80859	5.80274 (5.799)	5.80059 (5.799)	5.79976 (5.799)	5.79943 (5.799)	5.80 ± 0.02
(105, 0.2)	3.35848	3.35284 (3.349)	3.35075 (3.350)	3.34994 (3.349)	3.34962 (3.349)	3.35 ± 0.02
(100, 0.3)	7.97369	7.96367 (7.957)	7.95988 (7.958)	7.95838 (7.957)	7.95776 (7.957)	7.92 ± 0.02
(105, 0.3)	5.57682	5.56705 (5.560)	5.56329 (5.561)	5.56185 (5.561)	5.56124 (5.561)	5.53 ± 0.02
CPU (h:min:s)	0:00:01	0:00:08	0:01:09	0:08:53	1:08:40	

introduced when they assume that the exercise frontier at each stage is determined by two straight lines. With our DP procedure and the ϵ -algorithm, approximately 10 seconds of CPU time suffice to obtain the option value to within one-third of a cent.

EXAMPLE 3. We tried our method with the example given in Tables 3 and 4 of Zvan et al. (1999), denoted here by ZFV. The parameters are $n = 250$, $S(0) = K = 100$, $T = 0.25$, $r = 0.1$, and $\sigma = 0.2$ and 0.4 . Such a large value of n (i.e., frequent observation dates) should favor the ZFV method, in view of the analysis of §5.3. There are two important differences between the model used by ZFV and ours: (1) ZFV define $\bar{S}_m = (S(0) + \dots + S(t_m))/(m + 1)$, i.e., they include $S(0)$ in the average and we do not; (2) ZFV try to approximate the value of a continuously sampled Amerasian option that can be exercised at any time, whereas we are really interested in the value of the Bermudan option. To make the results somewhat more comparable, we adopt their definition of \bar{S}_m for the present example. But because of the second difference, we are still not pricing exactly the same Amerasian contract, and this will show up in the experimental results (Table 6).

The results are in Tables 5 and 6 for the Eurasian and Amerasian cases, respectively, with the linear-quadratic interpolation ($d = 2$). The CPU times given are for $\sigma = 0.2$; those for $\sigma = 0.4$ are similar. For each entry, the number in parentheses is the extrapolation value provided by the ϵ -algorithm for the corresponding value of p . In Table 6, DP converges to slightly different values than ZFV, and this can be explained by the difference (2) between the two models, as discussed earlier. To confirm this, we tried DP on the same model but with larger values of n , and the values were much closer to those of ZFV.

The CPU times used by ZFV and DP are not really comparable because we used a 750 Mhz computer, whereas ZFV used a 200 Mhz one. The ZFV method converges faster than DP as a function of p ; it gives a better approximation for coarse grid sizes. The quality of the approximation depends on the choice of grid points and on the type of approximation used over the boxes, and our DP implementation could probably be improved in this respect. According to the theoretical analysis in §5.3, the advantage of ZFV over DP should decrease if the observation dates become sparser and increase if they are more frequent.

Table 5 Estimated Prices of Eurasian Call Options for the ZFV Example

p	DP with Linear-Quadratic Interpolation			ZFV			
	$\sigma = 0.2$	$\sigma = 0.4$		CPU	$\sigma = 0.2$	$\sigma = 0.4$	CPU
145	2.988	5.278		0:00:07	2.929	5.160	0:02:42
289	2.942	5.187	(5.159)	0:00:51	2.929	5.161	0:11:50
600	2.932	5.167	(5.161)	0:07:32			
1,200	2.930	5.163	(5.162)	0:58:14			
2,400	2.930	5.162	(5.162)	7:23:37			

Table 6 Estimated Prices of Amerasian Call Options for the ZFV Example

ρ	DP with Linear-Quadratic Interpolation			ZFV		
	$\sigma = 0.2$	$\sigma = 0.4$	CPU	$\sigma = 0.2$	$\sigma = 0.4$	CPU
145	3.267	5.919	0:00:04	3.213	5.828	0:02:48
289	3.217 (3.201)	5.830 (5.799)	0:00:30	3.213	5.825	0:11:54
600	3.206 (3.203)	5.807 (5.801)	0:04:05			
1,200	3.204 (3.203)	5.803 (5.801)	0:34:24			
2,400	3.203 (3.203)	5.802 (5.801)	4:21:10			

Table 7 Approximation of $v_0(100)$ for the Hull and White Example (Example 4)

ρ	DP				Binomial Tree		
	Eurasian	CPU	Amerasian	CPU	h	Eurasian	Amerasian
150	4.542	0:00:03	4.957	0:00:01	0.100	4.687	5.377
300	4.524	0:00:24	4.937	0:00:14	0.050	4.614	5.145
600	4.520	0:03:21	4.932	0:01:57	0.010	4.539	4.962
1,200	4.519	0:26:46	4.931	0:15:04	0.005	4.529	4.942
2,400	4.519	3:41:03	4.930	2:02:54	0.003	4.525	4.936

EXAMPLE 4. Here we consider the same example as in Table 5 of Hull and White (1993). We have $n = 80$, $S(0) = 50$, $K = 50$, $T = 1$, $\sigma = 0.3$, $r = 0.1$, and $S(0)$ is included in the average as for the previous example (i.e., $\bar{S}_m = (S(0) + \dots + S(t_m))/(m + 1)$). Hull and White (1993) estimated the option value for this example using a binomial tree. Table 7 compares their results with ours. The h in the table refers to the parameter h used by their method. Its value must be quite small to get accurate results. The DP method converges nicely and needs less than 30 seconds of CPU time to approximate the value within one penny for both the Eurasian and Amerasian calls.

7. Conclusion

We studied in this paper a pricing method for the (Bermudan-)Amerasian option on a single asset, under the GBM model, via dynamic programming coupled with a piecewise-polynomial approximation of the value function after an appropriate change of variable. We proved continuity, monotonicity, and convexity properties of the value function and of the optimal exercise function (which delimits the optimal region of exercise). These properties characterize the

optimal exercise strategy for the option. Our computational method appears competitive with other available methods.

One of our examples illustrates that increasing the number of exercise opportunities tends to decrease the value of the option when the average is taken over the dates where there is an exercise opportunity: The increase in the stability of the average price offsets the value for having more exercise opportunities.

The DP approach does not rely on the form of the exercise region and could be used for pricing other types of discretely sampled American-style options for which the relevant information process can be modeled as a Markov process over a low-dimensional state space (for the case considered in this paper, the Markov process is two-dimensional). The GBM could be replaced by a more general CEV process (Cox 1996), or by other models for the underlying asset. For the CEV process, the quantities $D_{k,i}$ in (21) can also be computed in closed form. In general, however, the implementation would have to be reworked. A key ingredient is the ability to approximate the value function at each time step. Here we have used piecewise polynomials, with the pieces determined by a rectangular grid that remains the same at all steps. Adapting the grid to the shape of the value function at each step (with the same number of pieces) could provide a better approximation but would bring additional overhead. Perhaps a good compromise would be to readjust the grid every η steps (say), for some integer η , and readjust it only in the areas just below the optimal exercise frontier, where the value function is significantly nonlinear.

It can be useful to study, for each case of practical interest, how to exploit the structure of the problem to characterize the value function and the optimal

exercise strategy, and to improve the efficiency of the numerical method, as we have done here. When the dimension of the state space is large, e.g., if the payoff depends on several underlying assets, approximating the value function becomes generally much more difficult (we hit the “curse of dimensionality”) and pricing the option then remains a challenging problem.

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