

# Comment on “Sequential Quasi-Monte Carlo Sampling”

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Gerber and Chopin combine SMC with RQMC to accelerate convergence. They apply RQMC as in the array-RQMC method discussed below, for which convergence rate theory remains thin despite impressive empirical performance. Their proof of  $o(N^{-1/2})$  convergence rate is a remarkable contribution.

Array-RQMC simulates an array of  $N$  dependent realizations of a Markov chain, using RQMC so the distribution of the  $N$  states has low discrepancy  $D_t$  with respect to its theoretical distribution, at any step  $t$ . If  $\mathbf{x}_t = \Gamma_t(\mathbf{x}_{t-1}, \mathbf{u}_t)$ , where  $\mathbf{x}_{t-1} \sim \mathcal{U}([0, 1)^\ell)$  and  $\mathbf{u}_t \sim \mathcal{U}([0, 1)^d)$ , array-RQMC approximates the  $(\ell + d)$ -dimensional integral  $\mathbb{E}[D_t]$ , using  $N$  RQMC points. It matches each state to a point whose first  $\ell$  coordinates are near that point, and uses the next  $d$  coordinates for  $\mathbf{u}_t$ .

The *matching* step is crucial. If  $\ell = 1$ , just sort the states and points by their (increasing) first coordinate. If  $\ell > 1$ , one can map the states  $\mathbf{x}_t \in [0, 1)^\ell \rightarrow [0, 1)$ , then proceed as for  $\ell = 1$ . Gerber and Chopin do exactly this, using a Hilbert curve mapping. But the map  $[0, 1)^\ell \rightarrow [0, 1)$  is not essential. E.g., if  $\ell = 2$ , one can sort the states and the points in  $N^{-1/2}$  groups of size  $N^{-1/2}$  by their first coordinate, then sort each group by the second coordinate. This extends to  $\ell > 2$ . With this multivariate sort, L’Ecuyer et al. (2009) observed an  $\mathcal{O}(N^{-2})$  variance when pricing an Asian option with array-RQMC. I wonder what rate the Hilbert curve map can achieve for this example.

When  $\mathcal{X} \subseteq \mathbb{R}^\ell$  instead of  $[0, 1)^\ell$ , Gerber and Chopin map  $\mathbf{x}_t \rightarrow [0, 1)^\ell$  using a logistic transformation. The choice of transformation and its parameters may have a significant impact on the overall convergence. For the multivariate sort, no transformation is needed, it works directly in  $\mathbb{R}^\ell$ . The Hilbert sort could also be adapted to work directly in  $\mathbb{R}^\ell$ .

The SMC part uses importance sampling and particle resampling, a form of splitting (Kahn and Harris, 1949; L’Ecuyer et al., 2007) with fixed effort ( $N$  remains constant). Combinations of array-RQMC with several variants of splitting and with importance sampling were examined and tested by L’Ecuyer et al. (2007). Fixed-effort better fits array-RQMC than splitting variants that produce a random  $N$  at each step, and stratification improves the multinomial resampling process.

Array-RQMC was previously developed to estimate the expectation of a function of the state, not in the context of filtering for parameter estimation and for estimating posterior densities. But the same ideas apply. L’Ecuyer et al. (2008) proved an  $\mathcal{O}(n^{-3/2})$  convergence rate for the variance for  $\ell = 1$  and special types of RQMC points. Faster rates have been observed empirically in examples. Proving those rates in a general setting (under appropriate conditions) is challenging and deserves further research effort.

## References

- Kahn, H. and Harris, T. E. (1949) Estimation of particle transmission by random sampling. In *Monte Carlo Method*, vol. 12 of *Applied Mathematics Series*, 27–30. National Bureau of Standards.
- L'Ecuyer, P., Demers, V. and Tuffin, B. (2007) Rare-events, splitting, and quasi-Monte Carlo. *ACM Transactions on Modeling and Computer Simulation*, **17**, Article 9.