Existence and Construction of Shifted Lattice Rules with an Arbitrary Number of Points and Bounded Weighted Star Discrepancy for General Decreasing Weights

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Abstract

We study the problem of constructing shifted rank-1 lattice rules for the approximation of high-dimensional integrals with a low weighted star discrepancy, for classes of functions having bounded weighted variation, where the weighted variation is defined as the weighted sum of Hardy-Krause variations over all lower dimensional projections of the integrand. Under general conditions on the weights, we prove the existence of rank-1 lattice rules such that for any $\delta > 0$, the general weighted star discrepancy is $\mathcal{O}(n^{-1+\delta})$ for any number of points $n > 1$ (not necessarily prime), any shift of the lattice, general (decreasing) weights, and uniformly in the dimension. We also show that these rules can be constructed by a component-by-component strategy. This implies in particular that a single infinite-dimensional generating vector can be used for integrals in any number of dimensions, and even for infinite-dimensional integrands when they have bounded weighted variation. These same lattices are also good with respect to the worst-case error in weighted Korobov spaces with the same types of general weights. Similar results were already available for various special cases, such as general weights and prime n, or arbitrary n and product weights, but not for the most general combination of n composite, general weights, arbitrary shift, and star discrepancy, considered here. Our results imply tractability or strong tractability of in-

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tegration for classes of integrands with finite weighted variation when the weights satisfy the conditions we give. These classes are a strict superset of those covered by earlier sufficient tractability conditions.

Key words: Numerical integration, quasi-Monte Carlo, lattice rules, star discrepancy, error bounds, tractability, worst-case error, component-by-component construction
2000 MSC: 65D30, 65D32

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1. Introduction and main story

Quasi-Monte Carlo methods approximate an integral over the d-dimensional unit cube

$$I(f) = \int_{[0,1]^d} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x},$$

by a quadrature rule of the form

$$Q_n(f) = \frac{1}{n}\sum_{k=0}^{n-1} f(\boldsymbol{t}_k), \tag{1}$$

with the set of quadrature points $P_n := \{\boldsymbol{t}_0, \boldsymbol{t}_1, \ldots, \boldsymbol{t}_{n-1}\}$. In this paper, we consider *shifted rank-1 lattice rules*, which are quadrature rules of the form (1) with

$$P_n := \left\{ \left\{ \frac{k\boldsymbol{z}}{n} + \boldsymbol{\Delta} \right\}, \quad 0 \le k \le n-1 \right\}, \tag{2}$$

where $\boldsymbol{z} \in \mathcal{Z}_n^d$ is called the generating vector, $\mathcal{Z}_n = \{z \in \{1,2,\ldots,n\} : \gcd(z,n) = 1\}$, $\boldsymbol{\Delta} \in [0,1]^d$ is an arbitrary shift, and the braces around a vector indicate that we take only the fractional part of each component. If $\boldsymbol{\Delta} = \boldsymbol{0}$ in (2), we say that the lattice rule is *unshifted*.

The *integration error* is the difference $Q_n(f) - I(f)$. We are interested in proving the existence of lattice rules for which the worst-case absolute integration error, for certain classes of functions, is bounded by a quantity that converges as $\mathcal{O}(n^{-1+\delta})$ for any $\delta > 0$ when $n \to \infty$. We want the bound to hold for any n and arbitrary shifts, and eventually also uniformly in the dimension d. We are also interested in constructing those lattice rules explicitly.

The classes of integrands considered in this paper are defined by assigning a non-negative weight $\gamma_{\mathfrak{u}} \ge 0$ to each non-empty subset of coordinates

$\mathfrak{u} \subseteq \mathcal{D} := \{1, \ldots, d\}$. Throughout the whole paper, we assume by default that only the nonempty and finite subsets of coordinates $\mathfrak{u} \subseteq \mathcal{D}$ are considered. We make this convention instead of complicating the notation by indicating this explicitly at every occurrence. When we vary d, we assume that the weight $\gamma_\mathfrak{u}$ for any fixed $\mathfrak{u}$ remains fixed (it does not depend on d). For $\mathfrak{u} \subseteq \mathcal{D}$, we define the *variation* of a function $f : [0,1]^d \to \mathbb{R}$ over the projection determined by $\mathfrak{u}$ by

$$\tilde{V}_{d,\mathfrak{u}}(f) = \int_{[0,1]^{|\mathfrak{u}|}} \left| \frac{\partial^{|\mathfrak{u}|}}{\partial \boldsymbol{x}_\mathfrak{u}} f(\boldsymbol{x}_\mathfrak{u}, \mathbf{1}) \right| \mathrm{d}\boldsymbol{x}_\mathfrak{u}, \tag{3}$$

where $\boldsymbol{x}_\mathfrak{u}$ is the $|\mathfrak{u}|$-dimensional vector that comprises the coordinates of $\boldsymbol{x}$ whose indexes belong to $\mathfrak{u}$, $(\boldsymbol{x}_\mathfrak{u}, \mathbf{1})$ is the vector in $[0,1]^d$ whose jth component is x_j if $j \in \mathfrak{u}$ and is 1 if $j \notin \mathfrak{u}$, and the $\boldsymbol{\gamma}$ subscript indicates the dependence on the collection of weights $\boldsymbol{\gamma} = \{\gamma_\mathfrak{u} : \mathfrak{u} \subseteq \mathcal{D}\}$. In this expression, we make the convention that $\infty \times 0 = 0$, so that $\gamma_\mathfrak{u} = 0$ is allowed when $\partial^{|\mathfrak{u}|} f(\boldsymbol{x}_\mathfrak{u}, \mathbf{1}) / \partial \boldsymbol{x}_\mathfrak{u} = 0$. The *weighted variation* of $f : [0,1]^d \to \mathbb{R}$ is then defined by

$$V_{d,\boldsymbol{\gamma}}(f) = \sum_{\mathfrak{u} \subseteq \mathcal{D}} \gamma_\mathfrak{u}^{-1} \tilde{V}_{d,\mathfrak{u}}(f). \tag{4}$$

For any given selection of weights, let $\mathcal{F}_{d,\boldsymbol{\gamma}}$ denote the Banach space of functions f for which $V_{d,\boldsymbol{\gamma}}(f) < \infty$. These functions have integrable absolute mixed partial first derivatives for all subsets of coordinates.

For any point set P_n in $[0,1)^d$ and any choice of weights, we define the weighted star discrepancy of P_n as in [21, 22]. The local discrepancy of P_n at $\boldsymbol{x}$, for $\boldsymbol{x} = (x_1, \ldots, x_d) \in [0,1]^d$, is

$$\mathrm{discr}(\boldsymbol{x}, P_n) := \frac{|[\mathbf{0}, \boldsymbol{x}) \cap P_n|}{n} - \prod_{j=1}^{d} x_j.$$

The "classical" (unweighted) star discrepancy [16, 23] is defined as

$$D^*(P_n) := \sup_{\boldsymbol{x} \in [0,1]^d} |\mathrm{discr}(\boldsymbol{x}, P_n)|. \tag{5}$$

The *weighted star discrepancy* is then defined by

$$D^*_{d,\boldsymbol{\gamma}}(P_n) := \max_{\mathfrak{u} \subseteq \mathcal{D}} \gamma_\mathfrak{u} \sup_{\boldsymbol{x}_\mathfrak{u} \in [0,1)^{|\mathfrak{u}|}} \left| \mathrm{discr}((\boldsymbol{x}_\mathfrak{u}, \mathbf{1}), P_n) \right|. \tag{6}$$

When P_n comes from a lattice rule as in (2), we sometimes denote $D^*_{d,\boldsymbol{\gamma}}(P_n)$ by $D^*_{n,d,\boldsymbol{\gamma}}(\boldsymbol{z})$ and the latter will be the notation used for the weighted discrepancy of lattice rules.

The following weighted version of the Koksma-Hlawka inequality provides a bound on the worst-case integration error [21, p. 991]:

$$|Q_n(f) - I(f)| \;\leq\; D^*_{d,\boldsymbol{\gamma}}(P_n) \cdot V_{d,\boldsymbol{\gamma}}(f). \tag{7}$$

For any function $f \in \mathcal{F}_{d,\boldsymbol{\gamma}}$, $V_{d,\boldsymbol{\gamma}}(f)$ is a finite constant, so if we have a sequence of point sets P_n for which $D^*_{d,\boldsymbol{\gamma}}(P_n) \to 0$ at a given rate when $n \to \infty$, the error (7) converges at least at the same rate (or faster) as $D^*_{d,\boldsymbol{\gamma}}(P_n)$. This convergence bound is also uniform over the class of functions $f \in \mathcal{F}_{d,\boldsymbol{\gamma}}$, whose variation $V_{d,\boldsymbol{\gamma}}(f)$ is bounded by a given constant. Without loss of generality this constant can be taken as 1, and then the worst-case integration error, defined as

$$\mathrm{Err}_{\boldsymbol{\gamma}}(P_n) = \sup\{|Q_n(f) - I(f)| : f \in \mathcal{F}_{d,\boldsymbol{\gamma}} \text{ and } V_{d,\boldsymbol{\gamma}}(f) \leq 1\},$$

converges as $\mathcal{O}(D^*_{d,\boldsymbol{\gamma}}(P_n))$.

The classical Koksma-Hlawka inequality [16] corresponds to $\gamma_{\mathfrak{u}} = 1$ for $\mathfrak{u} = \{1, \ldots, d\}$ and $\gamma_{\mathfrak{u}} = 0$ otherwise. For that case, the existence of (unshifted) lattice rules for which $D^*_{n,d,\boldsymbol{\gamma}}(\boldsymbol{z}) = \mathcal{O}(n^{-1}(\ln n)^d)$, for any fixed dimension d, has been established long ago [16] (without an explicit construction). This convergence is not uniform in d because of the logarithmic factor, whose impact is increasingly important when d increases.

Weighted discrepancies were introduced in [6, 7, 27] then used by several authors, including (among others) [4, 9, 10, 11, 12, 21, 22, 24, 26], precisely to address this problem. The idea is to assign weights $\gamma_{\mathfrak{u}}$ to the subsets $\mathfrak{u}$ so that most large-cardinality subsets have very small weight, in a way that it becomes possible to construct point sets such that the maximum in (6) converges at a desired rate (as a function of n) uniformly in d. If a weight $\gamma_{\mathfrak{u}}$ is very small, this means that we do not care much about the uniformity of the points over the corresponding $|\mathfrak{u}|$-dimensional projection of $[0,1]^d$, presumably because we believe that f has very small variation (3) over this projection. As an extreme special case, if f satisfies $\partial^{|\mathfrak{u}|} f(\boldsymbol{x}_{\mathfrak{u}}, \mathbf{1})/\partial \boldsymbol{x}_{\mathfrak{u}} = 0$, then the variation over this projection given by (3) is zero and the weight $\gamma_{\mathfrak{u}}$ can be taken as zero (this projection has no importance). Reciprocally, if $\gamma_{\mathfrak{u}}$ is large, then the corresponding variation (3) is allowed to be large, but the sup for the corresponding $\mathfrak{u}$ in (6) is also multiplied by this large $\gamma_{\mathfrak{u}}$. This

puts a premium on reducing this particular sup, which is the classical star discrepancy of the projection of P_n on the subspace determined by the coordinates in $\mathfrak{u}$. That is, the uniformity of this projection has more importance when we minimize the weighted star discrepancy $D^*_{d,\boldsymbol{\gamma}}(P_n)$.

Classes of functions for which the worst-case integration error converges reasonably fast even in a large number of dimensions can be characterized and studied under the framework of tractability analysis, as follows [4, 9, 26, 27, 28]. Given a sequence of classes of functions $\mathcal{F}_{d,\boldsymbol{\gamma}}$ for $d \geq 1$, where $\boldsymbol{\gamma} = \{\gamma_{\mathfrak{u}} : \mathfrak{u} \subseteq \mathcal{D}\}$, and some numerical integration method represented here by a sequence of point sets $\{P_n,\, n \geq 1\}$, for each $\varepsilon > 0$, let $n(\varepsilon, \mathcal{F}_{d,\boldsymbol{\gamma}})$ be the minimal number n of function evaluations so that

$$\mathrm{Err}_{\boldsymbol{\gamma}}(P_n) \leq \varepsilon.$$

We say that multivariate integration is *tractable* for this sequence, with respect to the given method, if $n(\varepsilon, \mathcal{F}_{d,\boldsymbol{\gamma}})$ is bounded by a polynomial in $1/\varepsilon$ and d; that is, if $n(\varepsilon, \mathcal{F}_{d,\boldsymbol{\gamma}}) = \mathcal{O}(\varepsilon^{-p} d^q)$ for some non-negative constants p and q. It is *strongly tractable* if this holds for $q = 0$. In this paper, the integration methods considered are the rank-1 lattice rules as given by (1) and (2), while the classes of functions $\mathcal{F}_{d,\boldsymbol{\gamma}}$ are as defined earlier. Note that this tractability analysis framework assumes implicitly that each function evaluation has unit cost, because it measures the computing effort by the number of function evaluations. This is a similar setting as in [10]. This assumption is reasonable for the frequent situation where the required dimension of the points is random and unbounded, but has finite expectation, and we formally take $d \to \infty$ to handle this unboundedness [13, 14]. In other situations, d can be a truncation parameter and the required computing effort might increase (for example linearly) with d, whereas the truncation error decreases (sometimes very fast) with d. If the increase remains polynomial in d, this has no impact on tractability, but we may lose strong tractability due to an increase of q. Studying these types of situations is beyond the scope of this paper. We refer the reader to [2, 8, 12].

For the special case where the weights have the *product form* $\gamma_{\mathfrak{u}} = \prod_{j \in \mathfrak{u}} \gamma_j$ for each $\mathfrak{u} \subseteq \mathcal{D}$, where $\gamma_j \geq 0$ is a fixed constant (weight) associated with each coordinate j as introduced in [27], sufficient conditions on the weights γ_j for tractability and strong tractability by rank-1 lattice rules have been obtained under various frameworks, for example in [9, 10, 11, 12, 22]. It is proved in [22] that if $\sum_{j=1}^{\infty} \gamma_j < \infty$ (the weights γ_j are summable), then for

any $\delta > 0$, there is a constant C such that for all $n > 2$ and all $d > 0$, there exists a rank-1 rule with n points such that

$$D^*_{n,d,\boldsymbol{\gamma}}(\boldsymbol{z}) \leq C n^{-1+\delta}. \tag{8}$$

Moreover, these results are constructive in the sense that it was also proved that a vector $\boldsymbol{z} = (z_1, \ldots, z_d)$ that satisfies this bound for a given d can be obtained from *any* vector $\boldsymbol{z}' = (z_1, \ldots, z_{d-1})$ that satisfies the bound for $d-1$, and adding one coordinate to $\boldsymbol{z}'$. It means that it suffices to construct $\boldsymbol{z}$ one coordinate at a time, which reduces considerably the number of vectors $\boldsymbol{z}$ that need to be examined to find a good one. When adding a new coordinate z_d, one searches for the value of z_d that minimizes the discrepancy bound. This strategy goes by the name of *component-by-component* (CBC) construction [4, 11, 18, 19, 24, 25].

For the general (decreasing) weights $\gamma_{\mathfrak{u}}$ considered in this paper, sufficient conditions on the weights for such convergence results to hold and for the CBC construction to work were obtained in [20, 21], but only for the case where n is prime. There was also no tractability analysis.

The aim of the present article is to fill the gap by providing similar results for the more general situation of arbitrary (non-prime) n and weighted star discrepancy with general weights, which had not been covered so far. We provide sufficient conditions on the weights under which we prove that for any $\delta > 0$, there is a constant C and there are rank-1 lattice rules for which (8) holds for any $n \geq 3$, any $d > 0$, and any shift $\boldsymbol{\Delta}$. We also show that such rules can be found explicitly by a CBC construction algorithm. Our discrepancy bounds are valid uniformly over arbitrary shifts of the lattice. Bounds that hold for shifted lattices were also obtained in [20], but only for certain types of shifts of a rational form (not arbitrary shifts) and prime n, and in [9], also for prime n and product weights only. Our discrepancy bounds imply strong tractability results for the classes of integrands $\mathcal{F}_{d,\boldsymbol{\gamma}}$.

Results of a similar flavor have been obtained for other types of weighted discrepancies, such as those that correspond to the weighted Korobov spaces of periodic functions studied in [4], for example. These authors provided conditions on the weights for tractability and strong tractability, and show that a CBC algorithm can construct rank-1 rules whose weighted discrepancy converges as $\mathcal{O}(n^{-\alpha+\delta})$ for any $\delta > 0$, under the assumption that n is prime, where $\alpha > 1$ depends on the function space. Their weighted discrepancy is the square worst-case error for a class of integrands specified

in Section 5, and the smoothness conditions on the integrands are stronger than for the weighted star discrepancy considered here. As an illustration, any non-periodic function that has integrable mixed first derivative belongs to the space $\mathcal{F}_{d,\boldsymbol{\gamma}}$ considered here, but not necessarily to the weighted Korobov space of [4]. The latter space has the additional feature of being a reproducing kernel Hilbert space (RKHS), whereas $\mathcal{F}_{d,\boldsymbol{\gamma}}$ is not. Other spaces of functions considered previously in the literature were also RKHS with integrable derivatives of higher order. As a corollary of our results, we extend some of the results of [4] for weighted Korobov spaces to arbitrary (non-prime) n.

Considering composite n is also important from the point of view of constructing lattice rules extensible in the number of points. Known results on the existence and construction of such extensible lattices can be found in [3, 9, 17] but these results cover only a product-weight setting or the unweighted case. We believe that extensible lattice rules with general weights could also be constructed, but we do not have a proof and this also goes beyond the scope of the present paper.

We now introduce notation to state our condition on the weights. For the remainder of this article, we select an integer $n_0 \geq 3$ and define the constant

$$c = c(n_0) := \sup_{n \geq n_0} \left(2 + 2\pi^2 n/[3\varphi(n) \log\log n]\right), \tag{9}$$

where φ is Euler's totient function. This constant is slightly larger than 2 and decreases with n_0. For example, we have $c(100) \approx 2.18$ and $c(10000) \approx 2.00127$. For any dimension d and $\delta > 0$, we define

$$C_d(\boldsymbol{\gamma}, \delta) := \sup_{n \geq n_0} n^{-\delta} \sum_{\mathfrak{u} \subseteq \mathcal{D}} \gamma_{\mathfrak{u}} \left(c \ln n\right)^{|\mathfrak{u}|}. \tag{10}$$

We always assume that $n \geq n_0$. We now state our conditions on the weights. Namely, we assume that the weight associated with a set of coordinates cannot be bigger than the weight associated with any of its (non-empty) subsets. This condition on the weights can be written as

$$\gamma_{\mathfrak{g}} \geq \gamma_{\mathfrak{u}} \quad \text{whenever} \quad \mathfrak{g} \subseteq \mathfrak{u}. \tag{11}$$

It says that the weights decrease with the cardinality of $\mathfrak{u}$; it is a standard condition [4, 21]. The expression "general decreasing weights" in our title reflects this condition.

Under these assumptions, we will prove that for all $n \geq n_0$, the average value of $D^*_{n,d,\boldsymbol{\gamma}}(\boldsymbol{z})$ over all choices of $\boldsymbol{z} \in \mathcal{Z}_n^d$ (all admissible rank-1 lattice rules with n points) is bounded by $(2/n)\sum_{\mathfrak{u}\subseteq\mathcal{D}} \gamma_{\mathfrak{u}}\,(c\ln n)^{|\mathfrak{u}|}$, which is in turn bounded by $2C_d(\boldsymbol{\gamma},\delta)n^{-1+\delta}$. This implies that there exists at least one $\boldsymbol{z}$ for which the discrepancy satisfies

$$D^*_{n,d,\boldsymbol{\gamma}}(\boldsymbol{z}) \leq \frac{2}{n}\sum_{\mathfrak{u}\subseteq\mathcal{D}} \gamma_{\mathfrak{u}}\,(c\ln n)^{|\mathfrak{u}|} \leq 2C_d(\boldsymbol{\gamma},\delta)n^{-1+\delta}. \tag{12}$$

We also show how to find such a $\boldsymbol{z}$ with reasonable effort using a CBC construction. This bound applies uniformly over arbitrary shifts $\boldsymbol{\Delta}$. A similar bound was obtained in [21] with $c = 2$, but only for unshifted lattices and under the condition that n was restricted to prime numbers.

Under the additional condition that

$$C(\boldsymbol{\gamma},\delta) := \lim_{d\to\infty} C_d(\boldsymbol{\gamma},\delta) < \infty, \tag{13}$$

the bound (12) is also uniform in d and it can then be used to prove strong tractability by lattice rules of rank-1.

Our assumptions are closely related to the question of tractability, as we shall see later. It is intuitively clear that to obtain bounds that are uniform in the dimension, we need some sort of decaying condition on the weight. For example, if all the weights were equal to 1, then from (10), we would have $C(\boldsymbol{\gamma},\delta) = \infty$.

We also find that $C_d(\boldsymbol{\gamma},\delta)$ is upper bounded by

$$\tilde{C}_d(\boldsymbol{\gamma},\delta) := \sum_{\mathfrak{u}\subseteq\mathcal{D}} \gamma_{\mathfrak{u}} \left(\frac{c|\mathfrak{u}|}{e\delta}\right)^{|\mathfrak{u}|},$$

where $e = 2.71828\ldots$. This upper bound is sometimes more convenient to handle. This means that

$$\tilde{C}(\boldsymbol{\gamma},\delta) := \lim_{d\to\infty} \tilde{C}_d(\boldsymbol{\gamma},\delta) < \infty$$

implies that $C(\boldsymbol{\gamma},\delta) < \infty$. The converse is not true.

When $C(\boldsymbol{\gamma},\delta) < \infty$, we obtain in fact worst-case error bounds for infinite-dimensional integrals, that is, for $d = \infty$. Such integrals occur naturally when the number of uniform random variables required for a simulation is

random and unbounded; in that case we can take $d = \infty$, keeping in mind that the integrand will actually depend only on a finite (but unknown and unbounded) number of coordinates of $\boldsymbol{x}$, and that it is typically possible to simulate it exactly. See [2, 8, 10, 12, 13, 14] for further details and examples where $d = \infty$. Another important situation where it naturally occurs is if the random variable X of interest (the integrand) depends on the entire sample path of a Lévy process over a finite time interval $[0, T]$. Then the process can be sampled via a Lévy bridge sampling methodology [13]: sample the process first at T, then at $T/2$ conditional on its values at 0 and T, then at $T/4$ and $3T/4$ conditional on its values at 0, $T/2$, and T, and so on, ad infinitum). Here we really have $d = \infty$, although under appropriate conditions, we may have an excellent approximation of the distribution of X if we stop after a finite number of steps and interpolate the trajectory between the sampled points.

In our case, if the tractability conditions (13) are satisfied, then the results for $d = \infty$ will hold as for any finite d, because our bounds then hold uniformly in d. Moreover, we will show in detail that our results are no more restrictive that already known results on tractability. Interestingly, given a selection of weights $\gamma_{\mathfrak{u}}$ for all $\mathfrak{u}$ with $|\mathfrak{u}| < \infty$, the CBC construction method permits one to construct (in principle) an infinite-dimensional generating vector $\boldsymbol{z}$ whose first d coordinates provide a good d-dimensional generating vector for the given weights, for all $d \geq 1$. In other words, the same generating vector $\boldsymbol{z}$ can be used (truncated) in all dimensions.

The remainder of the paper is organized as follows. In Section 2, we prove the existence of lattice rules whose discrepancy does not exceed (12). In Section 3, we show the strong tractability by rank-1 rules under our conditions on the weights. In Section 4, we show that the low-discrepancy rules can be constructed by a CBC algorithm. In Section 5, we show that these rules also provide good worst-case error bounds for weighted Korobov spaces of functions, thus extending results of [4] to arbitrary (non-prime) n.

2. Existence of good rank-1 lattice rules for any n

A common approach for showing the existence of a lattice rule whose discrepancy is smaller than a given target, in a given class of rules, is to show that the average discrepancy over all rules in that class is smaller than the target. We use this type of averaging argument here, over the set of generating vectors $\boldsymbol{z} \in \mathcal{Z}_n^d$, for any given n. We start with some technical definitions

and notations. The dependence on d is often implicit in the notation.

For each non-empty set $\mathfrak{u} \subseteq \mathcal{D}$, let

$$E_{n,\mathfrak{u}} := \{\boldsymbol{h} \in (\mathbb{Z} \setminus \{0\})^{|\mathfrak{u}|} : -n/2 < h_j \le n/2 \text{ for all } j \in \mathfrak{u}\},$$

and for any $\boldsymbol{z} \in \mathcal{Z}_n^d$, let

$$R_n(\boldsymbol{z}, \mathfrak{u}) := \sum_{\substack{\boldsymbol{h} \in E_{n,\mathfrak{u}} \\ \boldsymbol{h}\cdot\boldsymbol{z}_\mathfrak{u} \equiv 0 \,(\bmod\, n)}} \prod_{j \in \mathfrak{u}} |h_j|^{-1} = \frac{1}{n} \sum_{k=0}^{n-1} \prod_{j \in \mathfrak{u}} \left(\sum_{-n/2 < h \le n/2}{}' \frac{e^{2\pi \mathrm{i} h k z_j / n}}{|h|} \right), \quad (14)$$

where $\sum'$ denotes the sum over the nonzero integer vectors, $\boldsymbol{z}_\mathfrak{u}$ contains the coordinates of $\boldsymbol{z}$ whose indexes belong to $\mathfrak{u}$, and the second equality follows from [23, Theorem 2.8] applied to the function

$$g_\mathfrak{u}(\boldsymbol{x}) = \prod_{j \in \mathfrak{u}} \left(\sum_{-n/2 < h \le n/2}{}' \frac{e^{2\pi \mathrm{i} h x_j}}{|h|} \right).$$

Note that this quantity $R_n(\boldsymbol{z}, \mathfrak{u})$ differs from the well known quality criterion R from [16, 23], in the sense that it is defined for each projection $\mathfrak{u} \subseteq \mathcal{D}$ (see also [21]) and because here we do not allow vectors $\boldsymbol{h}$ having components equal to 0. Finally, we denote

$$e_{n,d}(\boldsymbol{z}) := \sum_{\mathfrak{u} \subseteq \mathcal{D}} \gamma_\mathfrak{u} R_n(\boldsymbol{z}, \mathfrak{u}). \quad (15)$$

The following result is a slight extension of [21, Lemma 1], in the sense that the bound on the weighted star discrepancy holds uniformly over arbitrary shifts of the lattice.

Proposition 1. *Under condition (11), the weighted star discrepancy of an arbitrarily shifted lattice rule* (2) *satisfies*

$$D^*_{n,d,\boldsymbol{\gamma}}(\boldsymbol{z}) \le \frac{1}{n} \max_{\mathfrak{u} \subseteq \mathcal{D}} |\mathfrak{u}| \gamma_\mathfrak{u} + \frac{e_{n,d}(\boldsymbol{z})}{2}. \quad (16)$$

Proof. It is known from [21] that

$$D^*_{n,d,\boldsymbol{\gamma}}(\boldsymbol{z}) \le \max_{\mathfrak{u} \subseteq \mathcal{D}} \gamma_\mathfrak{u} \left(1 - (1 - 1/n)^{|\mathfrak{u}|} + \frac{1}{2} \sum_{\substack{\boldsymbol{h} \in E^*_{n,\mathfrak{u}} \\ \boldsymbol{h}\cdot\boldsymbol{z}_\mathfrak{u} \equiv 0 \,(\bmod\, n)}}{}' \prod_{j \in \mathfrak{u}} \frac{1}{\max(1, |h_j|)} \right),$$

where

$$E^*_{n,\mathfrak{u}} := \{\boldsymbol{h} \in \mathbb{Z}^{|\mathfrak{u}|} : -n/2 < h_j \le n/2 \text{ for all } j \in \mathfrak{u}\}.$$

By [9, Lemma 6], this bound holds for arbitrarily shifted lattice rules. The inequality (16) then follows from [21, Lemma 1]. □

On the right side of (16), only the second term depends on $\boldsymbol{z}$. That is, the discrepancy bound depends on $\boldsymbol{z}$ only via the quantity (15). We now turn our attention to the analysis of this quantity.

The next theorem gives an upper bound on the average value of $e_{n,d}(\boldsymbol{z})$ over all admissible generating vectors $\boldsymbol{z} \in \mathcal{Z}_n^d$, where we recall that $\mathcal{Z}_n = \{z \in \{1, 2, \ldots, n\} : \gcd(z, n) = 1\}$. There are $(\varphi(n))^d$ such generating vectors.

Theorem 2. *For any integer $n \ge n_0$ and any dimension d, we have*

$$M_{n,d,\boldsymbol{\gamma}} := \frac{1}{(\varphi(n))^d} \sum_{\boldsymbol{z} \in \mathcal{Z}_n^d} e_{n,d}(\boldsymbol{z}) \le \frac{2}{n} \sum_{\mathfrak{u} \subseteq \mathcal{D}} \gamma_{\mathfrak{u}} \left(c \ln n\right)^{|\mathfrak{u}|},$$

where $c \ge 2$ is defined in (9).

Proof. From the definition of the mean and from (14) and (15), we have

$$M_{n,d,\boldsymbol{\gamma}} = \frac{1}{(\varphi(n))^d} \sum_{\boldsymbol{z} \in \mathcal{Z}_n^d} \sum_{\mathfrak{u} \subseteq \mathcal{D}} \gamma_{\mathfrak{u}} \left(\frac{1}{n} \sum_{k=0}^{n-1} \prod_{j \in \mathfrak{u}} \left({\sum_{-n/2 < h \le n/2}}' \frac{e^{2\pi \mathrm{i} h k z_j / n}}{|h|} \right) \right).$$

By separating out the $k = 0$ term, we can write

$$M_{n,d,\boldsymbol{\gamma}} = \frac{1}{n} \sum_{\mathfrak{u} \subseteq \mathcal{D}} \gamma_{\mathfrak{u}} S_n^{|\mathfrak{u}|} + L_{n,d,\boldsymbol{\gamma}}, \tag{17}$$

where

$$S_n := {\sum_{-n/2 < h \le n/2}}' \frac{1}{|h|},$$

and

$$L_{n,d,\boldsymbol{\gamma}} = \frac{1}{(\varphi(n))^d} \sum_{\boldsymbol{z} \in \mathcal{Z}_n^d} \sum_{\mathfrak{u} \subseteq \mathcal{D}} \gamma_{\mathfrak{u}} \left(\frac{1}{n} \sum_{k=1}^{n-1} \prod_{j \in \mathfrak{u}} \left({\sum_{-n/2 < h \le n/2}}' \frac{e^{2\pi \mathrm{i} h k z_j / n}}{|h|} \right) \right)$$

$$= \frac{1}{n}\sum_{\mathfrak{u}\subseteq\mathcal{D}}\gamma_{\mathfrak{u}}\left(\sum_{k=1}^{n-1}\prod_{j\in\mathfrak{u}}\left(\frac{1}{\varphi(n)}\sum_{z_j\in\mathcal{Z}_n}\sum_{-n/2<h\le n/2}{}'\frac{e^{2\pi \mathrm{i}hkz_j/n}}{|h|}\right)\right).$$

Denoting

$$T_n(k) := \frac{1}{\varphi(n)}\sum_{z\in\mathcal{Z}_n}\sum_{-n/2<h\le n/2}{}'\frac{e^{2\pi \mathrm{i}hkz/n}}{|h|}, \tag{18}$$

we have

$$L_{n,d,\boldsymbol{\gamma}} = \frac{1}{n}\sum_{\mathfrak{u}\subseteq\mathcal{D}}\gamma_{\mathfrak{u}}\sum_{k=1}^{n-1}\left(T_n(k)\right)^{|\mathfrak{u}|}. \tag{19}$$

To bound (19), we will first make use of [22, Lemma 1], which yields

$$\sum_{k=1}^{n-1}|T_n(k)| \le c\ln n, \tag{20}$$

where c is defined in (9). Using Jensen's inequality as in [5, Theorem 19, p. 28], from which

$$\left(\sum a_i^s\right)^{1/s} \le \left(\sum a_i^t\right)^{1/t} \tag{21}$$

for arbitrary non-negative numbers a_i and $0<t<s$, we obtain

$$\sum_{k=1}^{n-1}\left(T_n(k)\right)^{|\mathfrak{u}|} \le \sum_{k=1}^{n-1}|T_n(k)|^{|\mathfrak{u}|} \le \left(\sum_{k=1}^{n-1}|T_n(k)|\right)^{|\mathfrak{u}|} \le (c\ln n)^{|\mathfrak{u}|}.$$

Replacing this in (19), we get

$$L_{n,d,\boldsymbol{\gamma}} \le \frac{1}{n}\sum_{\mathfrak{u}\subseteq\mathcal{D}}\gamma_{\mathfrak{u}}(c\ln n)^{|\mathfrak{u}|},$$

which by substituting in (17) leads to

$$M_{n,d,\boldsymbol{\gamma}} \le \frac{1}{n}\sum_{\mathfrak{u}\subseteq\mathcal{D}}\gamma_{\mathfrak{u}}\left(S_n^{|\mathfrak{u}|}+(c\ln n)^{|\mathfrak{u}|}\right) \le \frac{1}{n}\sum_{\mathfrak{u}\subseteq\mathcal{D}}\gamma_{\mathfrak{u}}\left((2\ln n)^{|\mathfrak{u}|}+(c\ln n)^{|\mathfrak{u}|}\right),$$

where in the last step we used the fact that $S_n \le 2\ln n$ (see [15, 22]). The conclusion then follows by using that $c \ge 2$. □

Of course, there always exists at least one vector $\boldsymbol{z}$ that does as well (or better) than the average. This gives the following.

Corollary 3. *For any integer $n \geq n_0$ and dimension d, there exists a generating vector $\boldsymbol{z}$ such that*

$$e_{n,d}(\boldsymbol{z}) \leq M_{n,d,\boldsymbol{\gamma}} \leq \frac{2}{n} \sum_{\mathfrak{u} \subseteq \mathcal{D}} \gamma_{\mathfrak{u}} \left(c \ln n\right)^{|\mathfrak{u}|}.$$

Next, we state our main result.

Theorem 4. *For any $n \geq n_0$, there is at least one generating vector $\boldsymbol{z}$ (depending on n) such that*

$$\frac{e_{n,d}(\boldsymbol{z})}{2} \leq C_d(\boldsymbol{\gamma}, \delta) n^{-1+\delta} \tag{22}$$

and

$$D^*_{n,d,\boldsymbol{\gamma}}(\boldsymbol{z}) \leq \frac{2}{n} \sum_{\mathfrak{u} \subseteq \mathcal{D}} \gamma_{\mathfrak{u}} \left(c \ln n\right)^{|\mathfrak{u}|} \leq 2 C_d(\boldsymbol{\gamma}, \delta) n^{-1+\delta}, \tag{23}$$

where $C_d(\boldsymbol{\gamma}, \delta)$ is defined in (10). If the weights satisfy (13), this implies that the discrepancy converges as $\mathcal{O}(n^{-1+\delta})$ uniformly in d.

Proof. For any fixed d, from Corollary 3, it follows that there exists a generating vector $\boldsymbol{z}$ such that

$$\frac{e_{n,d}(\boldsymbol{z})}{2} \leq \frac{1}{n} \sum_{\mathfrak{u} \subseteq \mathcal{D}} \gamma_{\mathfrak{u}} \left(c \ln n\right)^{|\mathfrak{u}|}.$$

Combining this with (10), we obtain (22). To prove (23), it suffices to bound $\max_{\mathfrak{u} \subseteq \mathcal{D}} |\mathfrak{u}| \gamma_{\mathfrak{u}}$ by $\gamma_{\mathfrak{u}} (c \ln n)^{|\mathfrak{u}|} \leq C_d(\boldsymbol{\gamma}, \delta) n^{\delta}$ and the result will follow from (16). Using the fact that $|\mathfrak{u}| \leq S_n^{|\mathfrak{u}|}$ for any $n \geq 3$ (see [21]; this is the technical reason why we took $n_0 \geq 3$) and that $S_n \leq 2 \ln n < c \ln n$, we obtain $|\mathfrak{u}| \gamma_{\mathfrak{u}} \leq \gamma_{\mathfrak{u}} (c \ln n)^{|\mathfrak{u}|} \leq n^{\delta} C_d(\boldsymbol{\gamma}, \delta)$. □

The following upper bound on $C_d(\boldsymbol{\gamma}, \delta)$, defined in the introduction, is sometimes easier to compute or estimate than $C_d(\boldsymbol{\gamma}, \delta)$ itself.

Proposition 5. *Under condition (11), one has*

$$C_d(\boldsymbol{\gamma}, \delta) \leq \tilde{C}_d(\boldsymbol{\gamma}, \delta) := \sum_{\mathfrak{u} \subseteq \mathcal{D}} \gamma_{\mathfrak{u}} \left(\frac{c|\mathfrak{u}|}{e\delta} \right)^{|\mathfrak{u}|}$$

so that $\tilde{C}(\boldsymbol{\gamma}, \delta) < \infty$ *implies* $C(\boldsymbol{\gamma}, \delta) < \infty$. *However, the converse is not true. We also have*

$$\tilde{C}_d(\boldsymbol{\gamma}, \delta) \leq \sum_{\mathfrak{u} \subseteq \mathcal{D}} \gamma_{\mathfrak{u}} (c/\delta)^{|\mathfrak{u}|} |\mathfrak{u}|!. \tag{24}$$

Proof. Consider the function $f(x) = x^{-\delta}(c \ln x)^s$, where $x \geq 1$ and s is a positive integer. By solving the equation $f'(x) = 0$, we find that f reaches its maximum at $x_0 = e^{s/\delta}$. It follows that

$$f(x) \leq f(e^{s/\delta}) = \left(\frac{cs}{e\delta} \right)^s.$$

Therefore,

$$n^{-\delta} (c \ln n)^{|\mathfrak{u}|} \leq \left(\frac{c|\mathfrak{u}|}{e\delta} \right)^{|\mathfrak{u}|},$$

for any $n \geq n_0$, from which we find that

$$n^{-\delta} \sum_{\mathfrak{u} \subseteq \mathcal{D}} \gamma_{\mathfrak{u}} (c \ln n)^{|\mathfrak{u}|} \leq \sum_{\mathfrak{u} \subseteq \mathcal{D}} \gamma_{\mathfrak{u}} \left(\frac{c|\mathfrak{u}|}{e\delta} \right)^{|\mathfrak{u}|} = \tilde{C}_d(\boldsymbol{\gamma}, \delta).$$

By (10), we obtain

$$C_d(\boldsymbol{\gamma}, \delta) \leq \tilde{C}_d(\boldsymbol{\gamma}, \delta).$$

We now prove (24). By Stirling's formula, for any integer $s \geq 1$, we can write $s^s = s! e^s e^{-\lambda_s} (2\pi s)^{-1/2}$, where $1/(12s+1) < \lambda_s < 1/(12s)$. Therefore,

$$\left(\frac{cs}{e\delta} \right)^s = (c/\delta)^s s! e^{-\lambda_s} (2\pi s)^{-1/2} < (c/\delta)^s s!.$$

Using this with $s = |\mathfrak{u}|$, we get

$$\left(\frac{c|\mathfrak{u}|}{e\delta} \right)^{|\mathfrak{u}|} < (c/\delta)^{|\mathfrak{u}|} |\mathfrak{u}|!$$

which yields (24).

□

3. Tractability analysis

Theorem 4 provides the basis of the following sufficient tractability conditions on the weights.

Theorem 6. *For any $\delta > 0$, if there is a constant q for which*

$$\sup_{d \geq 1} d^{-q} C_d(\boldsymbol{\gamma}, \delta) < \infty,$$

then we have $n(\varepsilon, \mathcal{F}_{d,\boldsymbol{\gamma}}) = \mathcal{O}(\varepsilon^{-p} d^q)$ for $p = 1/(1-\delta)$ and the integration problem is tractable. If this holds for $q = 0$, that is if $C(\boldsymbol{\gamma}, \delta) < \infty$, then the problem is strongly tractable.

Proof. We recall that $n = n(\varepsilon, \mathcal{F}_{d,\boldsymbol{\gamma}})$ is the minimum number of function evaluations required to have $\mathrm{Err}_{\boldsymbol{\gamma}}(P_n) \leq \varepsilon$. From (7) and (23), we have that

$$|Q_n(f) - I(f)| \leq D^*_{n,d,\boldsymbol{\gamma}}(\boldsymbol{z}) \cdot V_{d,\boldsymbol{\gamma}}(f) \leq 2\, C(\boldsymbol{\gamma}, \delta) n^{-1+\delta} V_{d,\boldsymbol{\gamma}}(f).$$

If $d^{-q} C_d(\boldsymbol{\gamma}, \delta) < \infty$ and $V_{d,\boldsymbol{\gamma}}(f) \leq 1$, then $|Q_n(f) - I(f)| \leq \varepsilon$ whenever $n^{1-\delta} \geq 2\, C_d(\boldsymbol{\gamma}, \delta)/\varepsilon$, that is, whenever

$$n \geq n(\varepsilon, \mathcal{F}_{d,\boldsymbol{\gamma}}) := \lceil (2\, C_d(\boldsymbol{\gamma}, \delta)/\varepsilon)^{1/(1-\delta)} \rceil.$$

This last expression is $\mathcal{O}(\varepsilon^{-p} d^q)$, so tractability holds. If this holds for $q = 0$, then replacing $C_d(\boldsymbol{\gamma}, \delta)$ by $C(\boldsymbol{\gamma}, \delta)$ in this expression gives a bound that does not depend on d, and hence we have strong tractability. □

We now turn our attention to special cases of weight selection for which tractability has been studied previously (see [4, 9, 12, 21, 22, 27, 28], for example). The tractability conditions found earlier for these situations are special cases of our conditions, in the sense that if they hold, then our conditions hold. We also provide an example that fits none of the previous special cases, and for which our strong tractability conditions hold. Thus, our conditions are a strict generalization of the previous ones.

As mentioned in the first section, the weights are said to have the *product form* if $\gamma_{\mathfrak{u}} = \prod_{j \in \mathfrak{u}} \gamma_j$ for each $\mathfrak{u} \subseteq \mathcal{D}$, where $\gamma_j > 0$. They are *order-dependent* if $\gamma_{\mathfrak{u}}$ depends only on $|\mathfrak{u}|$, the cardinality of $\mathfrak{u}$, and they have *finite-order* if $\gamma_{\mathfrak{u}} = 0$ when $|\mathfrak{u}| > q$, for some finite constant q. These last two types of weights capture the idea of low effective dimension in the superposition sense in certain applications (see [1, 13] for the details).

For weights in the product form, it was shown in [22] that if $\sum_{j=1}^{\infty} \gamma_j < \infty$ (the weights are summable), then there is an infinite-dimensional vector $\boldsymbol{z} = (z_1, z_2, \dots)$ for which $D^*_{n,d,\boldsymbol{\gamma}}(z_1, \dots, z_d) \le C n^{-1+\delta}$ for all n and d. The same summability condition over the product weights is required for strong tractability in other spaces of functions with respect to different figures of merit, see for example [9, 10, 12, 24]. We now argue that if the weights are summable, then $C(\boldsymbol{\gamma}, \delta) < \infty$ and thus our conditions for strong tractability are satisfied. To prove this, we use [9, Lemma 3] in a slightly modified version and present the result below for completeness.

Proposition 7. *If the weights are product and satisfy the summability condition*

$$\sum_{j=1}^{\infty} \gamma_j < \infty,$$

then $C(\boldsymbol{\gamma}, \delta) = \lim_{d\to\infty} C_d(\boldsymbol{\gamma}, \delta) < \infty$.

Proof. If the weights are product we can write for any $n \ge n_0$ and for any d that

$$\sum_{|\mathfrak{u}|<\infty} \gamma_{\mathfrak{u}} (c \ln n)^{|\mathfrak{u}|} = \prod_{j=1}^{\infty} (1 + c\gamma_j \ln n) - 1.$$

Now let us denote

$$\sigma_m = \sum_{j=m+1}^{\infty} \gamma_j.$$

Because the γ_j are summable, it is clear that σ_m may be made arbitrarily small by taking m sufficiently large. Since the γ_j are all positive, we have $\sigma_m > 0$. Then

$$\begin{aligned} \ln\left(\prod_{j=1}^{\infty} (1 + c\gamma_j \ln n)\right) &= \sum_{j=1}^{\infty} \ln(1 + c\gamma_j \ln n) \\ &\le \sum_{j=1}^{m} \ln\left(1 + \sigma_m^{-1} + c\gamma_j \ln n\right) \\ &\quad + \sum_{j=m+1}^{\infty} \ln(1 + c\gamma_j \ln n) \end{aligned}$$

$$= \sum_{j=1}^{m} \ln\left(1+\sigma_m^{-1}\right) + \sum_{j=1}^{m} \ln\left(1+\frac{c\gamma_j \ln n}{1+\sigma_m^{-1}}\right) + \sum_{j=m+1}^{\infty} \ln\left(1+c\gamma_j \ln n\right).$$

By using that $\ln(1+x) \le x$, for any $x \ge 0$, we obtain

$$\begin{aligned}
\ln\left(\prod_{j=1}^{\infty}(1+c\gamma_j \ln n)\right) &\le m\ln\left(1+\sigma_m^{-1}\right) + \sum_{j=1}^{m} \frac{c\gamma_j\sigma_m \ln n}{(1+\sigma_m^{-1})\sigma_m} \\
&\quad + \sum_{j=m+1}^{\infty} c\gamma_j \ln n \\
&\le m\ln\left(1+\sigma_m^{-1}\right) + c\ln(n)\sigma_m \sum_{j=1}^{m}\gamma_j + c\ln(n)\sigma_m \\
&\le m\ln\left(1+\sigma_m^{-1}\right) + c\ln(n)\sigma_m\left(\sigma_0+1\right).
\end{aligned}$$

Hence we have

$$\prod_{j=1}^{\infty}(1+c\gamma_j \ln n) \le \left(1+\sigma_m^{-1}\right)^m n^{c\sigma_m(\sigma_0+1)}.$$

By choosing m such that $c\sigma_m(\sigma_0+1) \le \delta$, we obtain

$$\prod_{j=1}^{\infty}(1+c\gamma_j \ln n) \le \left(1+\sigma_m^{-1}\right)^m n^{\delta}.$$

Now, by taking $\overline{C}(\boldsymbol{\gamma},\delta) = (1+\sigma_m^{-1})^m$, we obtain

$$\sum_{|\mathfrak{u}|<\infty} \gamma_{\mathfrak{u}}(c\ln n)^{|\mathfrak{u}|} \le \overline{C}(\boldsymbol{\gamma},\delta) n^{\delta} \quad \text{for all } n \ge n_0.$$

By combining this with (10), we obtain that $C_d(\boldsymbol{\gamma},\delta) \le \overline{C}(\boldsymbol{\gamma},\delta) < \infty$ for any d and hence the result follows. □

For order-dependent weights, we shall denote $\Gamma_\ell = \gamma_{\mathfrak{u}}$ when $|\mathfrak{u}| = \ell$, for $\ell = 1,\ldots,d$. For these types of weights, we have

$$C_d(\boldsymbol{\gamma},\delta) = \sup_{n\ge n_0} n^{-\delta} \sum_{j=1}^{d} \binom{d}{j} \Gamma_j (c\ln n)^j \le \tilde{C}_d(\boldsymbol{\gamma},\delta) = \sum_{j=1}^{d} \binom{d}{j} \Gamma_j \left(\frac{cj}{e\delta}\right)^j$$

and Theorem 6 says that strong tractability holds if $C_d(\boldsymbol{\gamma}, \delta)$ or $\tilde{C}(\boldsymbol{\gamma}, \delta)$ is bounded uniformly in d. In [21], the bound on $e_{n,d}(\boldsymbol{z})$ in Corollary 3 is developed for the special case of order-dependent weights, $c = 2$, and prime n, but tractability is not discussed for that case.

As an example of such weights, suppose that $\Gamma_j = [j!d(d-1)\cdots(d-j+1)]^{-1}$ for $j = 1, \ldots, d$. Then by using (24), we have

$$\tilde{C}(\boldsymbol{\gamma}, \delta) = \lim_{d\to\infty} \sum_{j=1}^{d} \binom{d}{j} \Gamma_j \left(\frac{cj}{e\delta}\right)^j \le \sum_{j=1}^{\infty} \frac{1}{j!} \left(\frac{c}{\delta}\right)^j = \exp\left(\frac{c}{\delta}\right) < \infty,$$

so we obtain strong tractability.

For finite-order weights, it is known that tractability (but not strong tractability) always holds; see [26] for instance. This can also be seen via our general conditions, as follows. Suppose that $\gamma_{\mathfrak{u}} = 0$ whenever $|\mathfrak{u}| > q$, for an integer q. By replacing each $\gamma_{\mathfrak{u}}$ by $\Gamma := \max_{\mathfrak{u}} \gamma_{\mathfrak{u}}$ in (13), we obtain

$$\sum_{\mathfrak{u}\subseteq\mathcal{D}} \gamma_{\mathfrak{u}} \left(\frac{c|\mathfrak{u}|}{e\delta}\right)^{|\mathfrak{u}|} \le \Gamma \left(\frac{cq}{e\delta}\right)^q \sum_{j=1}^{q} \binom{d}{j} = \mathcal{O}(d^q),$$

which implies tractability regardless of the weights. Strong tractability conditions for finite-order weights in other spaces of functions were given in [12]. We will now show that if these strong tractability conditions hold, then our conditions also hold.

Proposition 8. *If the weights are finite-order and satisfy the summability condition*

$$\sum_{|\mathfrak{u}|<\infty} \gamma_{\mathfrak{u}} < \infty,$$

then $C(\boldsymbol{\gamma}, \delta) = \lim_{d\to\infty} C_d(\boldsymbol{\gamma}, \delta) < \infty$.

Proof. Since the weights are finite-order, we have $\gamma_{\mathfrak{u}} = 0$ for any set $\mathfrak{u}$ with $|\mathfrak{u}| > q$, where q is fixed. We then obtain

$$n^{-\delta} \sum_{\mathfrak{u}\subseteq\mathcal{D}} \gamma_{\mathfrak{u}} (c \ln n)^{|\mathfrak{u}|} \le n^{-\delta} (c \ln n)^q \sum_{|\mathfrak{u}|\le q} \gamma_{\mathfrak{u}}.$$

Clearly, there exists a constant $C_1 > 0$ such that $n^{-\delta}(c \ln n)^q \le C_1$, which implies that $C_d(\boldsymbol{\gamma}, \delta) \le C_1 \sum_{|\mathfrak{u}|\le q} \gamma_{\mathfrak{u}}$ for any dimension d. Using now the summability assumption from the hypothesis, the result follows. □

We now give an example where the weights satisfy condition (13), but fit none of the previous special cases just discussed.

Example 9. In general, we can write

$$\tilde{C}_d(\boldsymbol{\gamma}, \delta) = \sum_{j=1}^{d} B(j)$$

where

$$B(j) := \sum_{\mathfrak{u} \subseteq \{1,\ldots,j-1\}} \gamma_{\mathfrak{u}\cup\{j\}} \left(\frac{c(|\mathfrak{u}|+1)}{e\delta}\right)^{|\mathfrak{u}|+1}.$$

Take a constant $\alpha > 2$ and define

$$\gamma_{\mathfrak{u}\cup\{j\}} = j^{-\alpha(|\mathfrak{u}|+1)}$$

for $j = 1, 2, \ldots$ and $\mathfrak{u} \subseteq \{1, \ldots, j-1\}$. Replacing these weights in the expression of $B(j)$, we obtain

$$\begin{aligned} B(j) &= \sum_{\mathfrak{u} \subseteq \{1,\ldots,j-1\}} j^{-\alpha(|\mathfrak{u}|+1)} \left(\frac{c(|\mathfrak{u}|+1)}{e\delta}\right)^{|\mathfrak{u}|+1} \\ &\leq \frac{cj^{-\alpha+1}}{e\delta} \sum_{\mathfrak{u} \subseteq \{1,\ldots,j-1\}} \left(\frac{cj^{-\alpha+1}}{e\delta}\right)^{|\mathfrak{u}|} \\ &\leq \frac{cj^{-\alpha+1}}{e\delta} \left(1 + \frac{cj^{-\alpha+1}}{e\delta}\right)^{j-1}. \end{aligned}$$

One can check that

$$\lim_{j\to\infty} \left(1 + \frac{cj^{-\alpha+1}}{e\delta}\right)^{j-1} = e^{1/(\alpha-1)}.$$

Consequently,

$$B(j) \leq \frac{c}{e\delta} j^{-\alpha+1}.$$

Hence

$$\tilde{C}_d(\boldsymbol{\gamma}, \delta) = \sum_{j=1}^{d} B(j) \leq \frac{c}{e\delta} \sum_{j=1}^{d} j^{-\alpha+1} \leq \frac{c\zeta(\alpha-1)}{e\delta} < \infty.$$

In the above, we used that $\alpha > 2$ so $\zeta(\alpha-1) < \infty$. Then, $C_d(\boldsymbol{\gamma}, \delta)$ is also bounded uniformly in d, and therefore our strong tractability conditions hold.

4. Constructing the lattice rules by the CBC method

Here we show how generating vectors $\boldsymbol{z}$ that satisfy the bound of Theorem 4 can be found efficiently by the well known component-by-component (CBC) technique. This method was used for instance in [4, 11, 20, 21, 22]. Further details can be found in those papers and the references therein. The CBC method does not necessarily find an optimal $\boldsymbol{z}$, but it returns a "good" one (in the sense that its discrepancy does not exceed the bound) with a limited (and reasonable) amount of computing effort.

A nice and useful feature of this construction method is that it provides a generating vector $\boldsymbol{z}$ whose truncation to its first m coordinates gives a good rank-1 lattice, for any $m < d$. It also permits one to extend the lattice in dimension, by adding coordinates to $\boldsymbol{z}$ whenever needed. Conceptually, this can be viewed as a way to construct an infinite-dimensional vector $\boldsymbol{z}$.

The case where $d = 1$ is trivial, because $e_{n,1}(z) = 0$ for any $z \in \mathcal{Z}_n$. We also have $R_n(\boldsymbol{z}, \mathfrak{u}) = 0$ for all subsets $\mathfrak{u} \subseteq \mathcal{D}$ with $|\mathfrak{u}| = 1$; see [21]. We use the notation

$$e_{n,m}(z_1, \ldots, z_m) = \sum_{\mathfrak{u} \subseteq \{1,2,\ldots,m\}} \gamma_{\mathfrak{u}} R_n((z_1, \ldots, z_m), \mathfrak{u}).$$

Algorithm 1 CBC construction algorithm.

1. Set z_1, the value of the first component of $\boldsymbol{z}$, say to $z_1 = 1$.
2. For $m = 2, 3, \ldots, d$:
   find $z_m \in \mathcal{Z}_n$ that minimizes $e_{n,m}(z_1, \ldots, z_m)$ with $z_1, \ldots, z_{m-1}$ fixed (this can be done by an exhaustive search).

Return $\boldsymbol{z} = (z_1, \ldots, z_d)$.

We will prove that this algorithm returns a generating vector $\boldsymbol{z}$ for which $e_{n,d}(\boldsymbol{z})$ does not exceed the bound in Theorem 2. This implies that the corresponding discrepancy satisfies the bound in Theorem 4, and converges as $\mathcal{O}(n^{-1+\delta})$ if $C(\boldsymbol{\gamma}, \delta) < \infty$. The proof is by induction on d and the idea is to show that if a vector $\boldsymbol{z}$ satisfies the bound in an arbitrary dimension $d \geq 1$, then there is always a choice of the $(d+1)$-th coordinate such that the $(d+1)$-dimensional vector $\boldsymbol{z}$ also satisfies the bound.

Theorem 10. *Let $n \geq 3$ be an arbitrary integer. Suppose that for an arbitrary dimension $d \geq 1$, we have a vector $\boldsymbol{z} \in \mathcal{Z}_n^d$ such that*

$$e_{n,d}(\boldsymbol{z}) \leq \frac{2}{n} \sum_{\mathfrak{u} \subseteq \mathcal{D}} \gamma_{\mathfrak{u}} (c \ln n)^{|\mathfrak{u}|}.$$

Then there exists a $z_{d+1} \in \mathcal{Z}_n$ such that

$$e_{n,d+1}(\boldsymbol{z}, z_{d+1}) \leq \frac{2}{n} \sum_{\mathfrak{u} \subseteq \mathcal{D}_1} \gamma_{\mathfrak{u}} (c \ln n)^{|\mathfrak{u}|},$$

where $\mathcal{D}_1 := \mathcal{D} \cup \{d+1\}$. Such a z_{d+1} can be found by minimizing $e_{n,d+1}(\boldsymbol{z}, z_{d+1})$ over the set $\mathcal{Z}_n$.

Proof. We have

$$\begin{aligned} e_{n,d+1}(\boldsymbol{z}, z_{d+1}) &= \sum_{\mathfrak{u} \subseteq \mathcal{D}_1} \gamma_{\mathfrak{u}} R_n((\boldsymbol{z}, z_{d+1}), \mathfrak{u}) \\ &= \sum_{\mathfrak{u} \subseteq \mathcal{D}} \gamma_{\mathfrak{u}} R_n(\boldsymbol{z}, \mathfrak{u}) + \sum_{\substack{\mathfrak{u} \subseteq \mathcal{D}_1 \\ d+1 \in \mathfrak{u}}} \gamma_{\mathfrak{u}} R_n((\boldsymbol{z}, z_{d+1}), \mathfrak{u}). \end{aligned}$$

To simplify some of the notations that follow, we denote

$$D_k(z) = \sum_{-n/2 < h \leq n/2}{}' \frac{e^{2\pi i h k z/n}}{|h|}, \quad 0 \leq k \leq n-1.$$

Then by using (14), we obtain

$$\begin{aligned} e_{n,d+1}(\boldsymbol{z}, z_{d+1}) &= e_{n,d}(\boldsymbol{z}) + \sum_{\substack{\mathfrak{u} \subseteq \mathcal{D}_1 \\ d+1 \in \mathfrak{u}}} \gamma_{\mathfrak{u}} R_n((\boldsymbol{z}, z_{d+1}), \mathfrak{u}) \\ &= e_{n,d}(\boldsymbol{z}) + \sum_{\substack{\mathfrak{u} \subseteq \mathcal{D}_1 \\ d+1 \in \mathfrak{u}}} \gamma_{\mathfrak{u}} \left(\frac{1}{n} \sum_{k=0}^{n-1} \prod_{j \in \mathfrak{u}} D_k(z_j) \right) \\ &= e_{n,d}(\boldsymbol{z}) + \frac{1}{n} \sum_{\substack{\mathfrak{u} \subseteq \mathcal{D}_1 \\ d+1 \in \mathfrak{u}}} \gamma_{\mathfrak{u}} S_n^{|\mathfrak{u}|} \\ &\quad + \frac{1}{n} \sum_{\substack{\mathfrak{u} \subseteq \mathcal{D}_1 \\ d+1 \in \mathfrak{u}}} \gamma_{\mathfrak{u}} \sum_{k=1}^{n-1} D_k(z_{d+1}) \left(\prod_{j \in \mathfrak{u} \setminus \{d+1\}} D_k(z_j) \right), \end{aligned}$$

where in the last step, the $k = 0$ term was separated out. Next, we average $e_{n,d+1}(\boldsymbol{z}, z_{d+1})$ over all possible values of $z_{d+1} \in \mathcal{Z}_n$, which gives

$$\mathrm{Avg}(e_{n,d+1}(\boldsymbol{z}, z_{d+1})) := \frac{1}{\varphi(n)} \sum_{z_{d+1} \in \mathcal{Z}_n} e_{n,d+1}(\boldsymbol{z}, z_{d+1}).$$

As the dependence of $e_{n,d+1}(\boldsymbol{z}, z_{d+1})$ on z_{d+1} is only through the $D_k(z_{d+1})$ factor, we next focus on

$$\frac{1}{\varphi(n)} \sum_{z_{d+1} \in \mathcal{Z}_n} D_k(z_{d+1}) = T_n(k),$$

where $T_n(k)$ was defined by (18). It is easy to see that $|D_k(z)| \leq S_n$. Consequently, by using (20), we obtain

$$\mathrm{Avg}(e_{n,d+1}(\boldsymbol{z}, z_{d+1})) \leq e_{n,d}(\boldsymbol{z}) + \frac{1}{n} \sum_{\substack{\mathfrak{u} \subseteq \mathcal{D}_1 \\ d+1 \in \mathfrak{u}}} \gamma_{\mathfrak{u}} S_n^{|\mathfrak{u}|} + \frac{c \ln n}{n} \sum_{\substack{\mathfrak{u} \subseteq \mathcal{D}_1 \\ d+1 \in \mathfrak{u}}} \gamma_{\mathfrak{u}} S_n^{|\mathfrak{u}|-1}.$$

We also have $S_n \leq 2 \ln n$, which together with the induction hypothesis leads to

$$\begin{aligned}
\mathrm{Avg}(e_{n,d+1}(\boldsymbol{z}, z_{d+1})) \;&\leq\; e_{n,d}(\boldsymbol{z}) + \frac{1}{n} \sum_{\substack{\mathfrak{u} \subseteq \mathcal{D}_1 \\ d+1 \in \mathfrak{u}}} \gamma_{\mathfrak{u}} (2 \ln n)^{|\mathfrak{u}|} \\
&\quad + \frac{c \ln n}{n} \sum_{\substack{\mathfrak{u} \subseteq \mathcal{D}_1 \\ d+1 \in \mathfrak{u}}} \gamma_{\mathfrak{u}} (2 \ln n)^{|\mathfrak{u}|-1} \\
&\leq\; \frac{2}{n} \sum_{\mathfrak{u} \subseteq \mathcal{D}} \gamma_{\mathfrak{u}} (c \ln n)^{|\mathfrak{u}|} + \frac{2}{n} \sum_{\substack{\mathfrak{u} \subseteq \mathcal{D}_1 \\ d+1 \in \mathfrak{u}}} \gamma_{\mathfrak{u}} (c \ln n)^{|\mathfrak{u}|},
\end{aligned}$$

where in the last step we used the fact that $c > 2$, as in the proof of Theorem 2. Clearly, there has to be at least one $z_{d+1} \in \mathcal{Z}_n$ such that $e_{n,d+1}(\boldsymbol{z}, z_{d+1}) \leq \mathrm{Avg}(e_{n,d+1}(\boldsymbol{z}, z_{d+1}))$ and this z_{d+1} may be chosen by minimising $e_{n,d+1}(\boldsymbol{z}, z_{d+1})$ over the set $\mathcal{Z}_n$. From the last inequality on the average, it is clear now that for the chosen z_{d+1}, we have

$$e_{n,d+1}(\boldsymbol{z}, z_{d+1}) \leq \frac{2}{n} \sum_{\mathfrak{u} \subseteq \mathcal{D}_1} \gamma_{\mathfrak{u}} (c \ln n)^{|\mathfrak{u}|},$$

which is the desired result. □

From this result, we can deduce that the CBC algorithm returns a $\boldsymbol{z}$ that achieves the same bound on $e_{n,d}(\boldsymbol{z})$ and on the discrepancy as in Corollary 3 and Theorem 4.

Corollary 11. *For any integer $n \geq 3$ and any $d \geq 2$, the CBC algorithm given earlier returns a vector $\boldsymbol{z} = (z_1, \ldots, z_d) \in \mathcal{Z}_n^d$ such that for $2 \leq m \leq d$,*

$$e_{n,m}(z_1, \ldots, z_m) \leq \frac{2}{n} \sum_{\mathfrak{u} \subseteq \{1,2,\ldots,m\}} \gamma_{\mathfrak{u}} (c \ln n)^{|\mathfrak{u}|}.$$

If $C(\boldsymbol{\gamma}, \delta) < \infty$, then $D^*_{n,d,\boldsymbol{\gamma}}(\boldsymbol{z}) = \mathcal{O}(n^{-1+\delta})$, as in Theorem 4.

Proof. Recalling that $e_{n,1}(z) = 0$ for any $z \in \mathcal{Z}_n$, the result follows immediately from Theorem 10. □

The computational costs of the CBC construction were fully analysed in [21]. The only difference is that here we consider a composite n. Briefly, the cost of the ordinary (not fast) CBC construction is $\mathcal{O}(n^2 d 2^d)$ for arbitrary weights, but this cost can be considerably reduced for particular classes of weights. For order-dependent weights, the cost of the construction is $\mathcal{O}(n^2 d^2)$ plus $\mathcal{O}(nd)$ for storage. For finite-order weights, the cost of the construction is $\mathcal{O}(n^2 d^{q+1})$, while for product weights the involved cost is $\mathcal{O}(n^2 d)$ plus $\mathcal{O}(n)$ for storage (see [22] in this case). The fast CBC construction proposed in [18, 19] is available as indicated in [21, 22]. This fast CBC construction is based on writing the CBC algorithm in terms of a matrix-vector multiplication and using a fast algorithm to do these multiplications. The operation count of $\mathcal{O}(n^2)$ of a usual matrix-vector multiplication is reduced to $\mathcal{O}(n \ln n)$ and this reduction applies also when n is composite, as shown in [18], but with a possible larger involved constant depending on the number of divisors and the number of prime factors of n. The cost of the construction also need to account for storage, as explained in [18, 19, 21, 22].

5. Bounds on the worst-case error in Korobov spaces

In [4], the authors showed how to construct good lattice rules with a prime number of points for integrands belonging to Korobov spaces with general weights. Their framework is a reproducing kernel Hilbert space of periodic functions for which the Fourier coefficients $\hat{f}(\boldsymbol{h})$ satisfy

$$|\hat{f}(\boldsymbol{0})|^2 + \sum_{\boldsymbol{h} \in \mathbb{Z}^d} \gamma_{\mathfrak{u}}^{-1} \prod_{j \in \mathfrak{u}} |h_j|^{\alpha} |\hat{f}(\boldsymbol{h})|^2 < \infty,$$

for a given $\alpha > 1$, where $\mathfrak{u} = \{j : h_j \neq 0\}$. The square worst-case error in these Korobov spaces is given by

$$\left(e_{n,d}^{(\mathrm{Kor})}(\boldsymbol{z}, \boldsymbol{\gamma}, \alpha)\right)^2 := \frac{1}{n} \sum_{k=0}^{n-1} \sum_{\mathfrak{u} \subseteq \mathcal{D}} \gamma_{\mathfrak{u}} \prod_{j \in \mathfrak{u}} \left(\sum_{h \in \mathbb{Z}}{}' \frac{e^{2\pi \mathrm{i} h k z_j / n}}{|h|^\alpha} \right). \tag{25}$$

These authors proved the existence of lattice rules with a prime number of points that are good with respect to (25), and they showed that these rules can be constructed by a CBC algorithm. We now show that our techniques can be adapted with minor modifications to extend the results from [4] to the non-prime case. First, we observe that from the general theory on lattice rules (see [16, Chapter 5] or [23, Chapter 4]), for $\alpha > 1$, we have

$$\frac{1}{n} \sum_{k=0}^{n-1} \prod_{j \in \mathfrak{u}} \left(\sum_{h \in \mathbb{Z}}{}' \frac{e^{2\pi \mathrm{i} h k z_j / n}}{|h|^\alpha} \right) = \sum_{\substack{\boldsymbol{h} \in (\mathbb{Z} \setminus \{0\})^d \\ \boldsymbol{h} \cdot \boldsymbol{z}_{\mathfrak{u}} \equiv 0 \,(\bmod\, n)}} \prod_{j \in \mathfrak{u}} \frac{1}{|h_j|^\alpha} \leq C \sum_{\substack{\boldsymbol{h} \in E_{n,\mathfrak{u}} \\ \boldsymbol{h} \cdot \boldsymbol{z}_{\mathfrak{u}} \equiv 0 \,(\bmod\, n)}} \prod_{j \in \mathfrak{u}} \frac{1}{|h_j|^\alpha}, \tag{26}$$

where $C > 0$ is a constant. The inequality in (26) is obtained by observing that since $\alpha > 1$, then the vectors $\boldsymbol{h}$ that are not in $E_{n,\mathfrak{u}}$ bring a negligible contribution to the sum when n is large. By applying Jensen's inequality (21) in the last term of (26) and replacing in (25), we obtain

$$\left(e_{n,d}^{(\mathrm{Kor})}(\boldsymbol{z}, \boldsymbol{\gamma}, \alpha)\right)^2 \leq C \sum_{\mathfrak{u} \subseteq \mathcal{D}} \gamma_{\mathfrak{u}} \left(\sum_{\substack{\boldsymbol{h} \in E_{n,\mathfrak{u}} \\ \boldsymbol{h} \cdot \boldsymbol{z}_{\mathfrak{u}} \equiv 0 \,(\bmod\, n)}} \prod_{j \in \mathfrak{u}} \frac{1}{|h_j|} \right)^\alpha.$$

Using (14), we next obtain

$$\left(e_{n,d}^{(\mathrm{Kor})}(\boldsymbol{z}, \boldsymbol{\gamma}, \alpha)\right)^2 \leq C \sum_{\mathfrak{u} \subseteq \mathcal{D}} \gamma_{\mathfrak{u}} \left(R_n(\boldsymbol{z}, \mathfrak{u})\right)^\alpha = C \sum_{\mathfrak{u} \subseteq \mathcal{D}} \left(\gamma_{\mathfrak{u}}^{1/\alpha} R_n(\boldsymbol{z}, \mathfrak{u})\right)^\alpha.$$

Another application of Jensen's inequality allows us to obtain

$$\left(e_{n,d}^{(\mathrm{Kor})}(\boldsymbol{z}, \boldsymbol{\gamma}, \alpha)\right)^2 \leq C \left(\sum_{\mathfrak{u} \subseteq \mathcal{D}} \gamma_{\mathfrak{u}}^{1/\alpha} R_n(\boldsymbol{z}, \mathfrak{u}) \right)^\alpha. \tag{27}$$

We now see that the sum in the right-hand-side of (27) is the same as $e_{n,d}(\boldsymbol{z})$ as defined by (15) but with the weights $\gamma_{\mathfrak{u}}$ replaced by $\gamma_{\mathfrak{u}}^{1/\alpha}$. Consequently,

our techniques can be used to extend the results from [4] to the non-prime case by bounding the sum on the right of (27) and proving existence and construction results as in Theorem 2 and Theorem 10. Then, we can conclude in the same manner as in Theorem 4 that for any $\delta > 0$, if

$$\lim_{d\to\infty} \sup_{n\geq n_0} \left(n^{-\delta} \sum_{\mathfrak{u}\subseteq\mathcal{D}} \gamma_{\mathfrak{u}}^{1/\alpha} (c \ln n)^{|\mathfrak{u}|} \right) < \infty$$

(which is the same as (13) with $\gamma_{\mathfrak{u}}$ replaced by $\gamma_{\mathfrak{u}}^{1/\alpha}$), then there exists a constant $C(\boldsymbol{\gamma}, \alpha, \delta) > 0$ independent of d such that for any integer $n \geq 3$, we have

$$\left(e_{n,d}^{(\mathrm{Kor})}(\boldsymbol{z}, \boldsymbol{\gamma}, \alpha)\right)^2 \leq C(\boldsymbol{\gamma}, \alpha, \delta) n^{-\alpha+\delta}.$$

This last inequality shows that $e_{n,d}^{(\mathrm{Kor})}(\boldsymbol{z}, \boldsymbol{\gamma}, \alpha)$ is almost $\mathcal{O}(n^{-\alpha/2})$ as obtained in [4] for prime n.

Acknowledgments

This paper was written when the first author was a postdoctoral fellow at the University of Montreal. The work has been supported by an NSERC-Canada Discovery Grant and a Canada Research Chair to the second author. The first author also acknowledges support from the Research Program of the Research Foundation-Flanders (FWO-Vlaanderen) G.0597.06. We thank the two anonymous reviewers, whose many suggestions led to a significant improvement of the paper.

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