

1 **CONSTRUCTION-FREE MEDIAN QUASI-MONTE CARLO RULES**
2 **FOR FUNCTION SPACES WITH UNSPECIFIED SMOOTHNESS**
3 **AND GENERAL WEIGHTS** *

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5 **Abstract.** We study quasi-Monte Carlo (QMC) integration of smooth functions defined over
6 the multi-dimensional unit cube. Inspired by a recent work of Pan and Owen, we study a new
7 *construction-free median QMC rule* which can exploit the smoothness and the weights of function
8 spaces adaptively. For weighted Korobov spaces, we draw a sample of r independent generating
9 vectors of rank-1 lattice rules, compute the integral estimate for each, and approximate the true
10 integral by the median of these r estimates. For weighted Sobolev spaces, we use the same approach
11 but with the rank-1 lattice rules replaced by high-order polynomial lattice rules. A major advantage
12 over the existing approaches is that we do not need to construct good generating vectors by a
13 computer search algorithm, while our median QMC rule achieves almost the optimal worst-case
14 error rate for the respective function space with *any smoothness and weights*, with a probability that
15 converges to 1 exponentially fast as r increases. Numerical experiments illustrate and support our
16 theoretical findings.

17 **Key words.** Numerical integration; quasi-Monte Carlo; rank-1 lattice rule; high-order polyno-
18 mial lattice rule; weighted function space; median; construction-free

19 **AMS subject classifications.** 65D30, 65D32, 41A55, 46E35

20 **1. Introduction.** We consider numerical integration of functions defined over
21 the s -dimensional unit cube $[0, 1]^s$. For an integrable function $f : [0, 1]^s \rightarrow \mathbb{R}$, we
22 denote the integral of f by

$$23 \quad I_s(f) := \int_{[0,1]^s} f(\mathbf{x}) \, d\mathbf{x}.$$

24 The *quasi-Monte Carlo* (QMC) method approximates $I_s(f)$ by the equally-weighted
25 average of function evaluations over a deterministic N -element point set $P_{N,s} \subset [0, 1]^s$:

$$26 \quad Q_{P_{N,s}}(f) = \frac{1}{N} \sum_{\mathbf{x} \in P_{N,s}} f(\mathbf{x}).$$

27 The *worst-case error* for a given normed function space \mathcal{F} and point set $P_{N,s}$ is

$$28 \quad e^{\text{wor}}(Q_{P_{N,s}}; \mathcal{F}) = \sup_{f \in \mathcal{F}; \|f\| \leq 1} |Q_{P_{N,s}}(f) - I_s(f)|$$

29 where $\|f\|$ is the norm of f in this space. The key to success of the QMC method
30 lies in a proper choice of the point set depending on a target class of functions. One
31 wants to construct point sets for which this worst-case error is small and converges
32 at the fastest possible rate as a function of N , for the given space \mathcal{F} . In *randomized*
33 *QMC*, the point set $P_{N,s}$ is randomized in a way that $Q_{P_{N,s}}(f)$ becomes an unbiased

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estimator of $I_s(f)$ and one wishes to minimize its variance [25, 26, 30]. In this paper, we focus on deterministic QMC point sets only.

There are two main families of QMC point sets: digital nets and sequences [10, 36] and lattice point sets [36, 43]. We refer the reader to [7, 30, 32] for further introductory details. In this paper, we consider rank-1 lattice point sets for Korobov spaces of periodic functions, and high-order polynomial lattice point sets [9, 35, 41] (which are a special type of digital nets) for Sobolev spaces of non-periodic functions. Each point set from these types is defined by an s -dimensional generating vector, with integer coordinates in the ordinary lattice case and with polynomial coordinates in the polynomial lattice case.

In both cases, the weighted spaces of functions are defined by selecting a positive smoothness parameter α and a positive weight γ_u for each subset of coordinates $u \subseteq \{1, \dots, s\}$, with $\gamma_\emptyset = 1$. The parameter α determines how smooth the admissible functions are required to be. For the Korobov spaces, it tells the minimal rate at which the Fourier coefficients of f are required to decay, and when it is an integer it corresponds to the minimal number of square-integrable mixed partial derivatives of f with respect to each coordinate; see [37, Appendix A]. For the Sobolev spaces, α is a positive integer which also imposes integrability conditions on the partial derivatives of f . The weights γ_u act as constant importance factors given to the subsets of coordinates [11, 45]. A larger γ_u means that the projection of f over the subset of coordinates in u can have a larger variation in some sense, so that more importance should be given to the uniformity of the points over this projection.

It is known that the best possible QMC point sets cannot provide a better convergence rate than $\mathcal{O}(N^{-\alpha})$ for the worst-case error for these two function spaces. On the other hand, there are effective search algorithms which, for a given α and a given selection of weights γ_u , can construct good rank-1 lattice or polynomial lattice point sets for the function spaces determined by these parameters, and for which the worst-case error converges as $\mathcal{O}(N^{-\alpha+\epsilon})$ for any $\epsilon > 0$ [10, 22]. Software that implement such algorithms is also freely available [27, 29, 38]. These algorithms typically use a greedy component-by-component (CBC) construction approach proposed originally by [20], then re-introduced and popularized by [44]. With the CBC approach, the generating vector is determined one coordinate at a time by optimizing a figure of merit that depends only on this new coordinate and the previous ones, and where the previous coordinates can no longer be changed.

In general, the number of weights γ_u to specify is $2^s - 1$. When s is large, specifying all these weights individually becomes impractical, so it is common practice to “parameterize” the weights by a smaller number of parameters, usually linear instead of exponential in s . The most popular forms of parameterizations are the product weights, the order-dependent weights, and their combination. For the product weights, one specifies a weight $\gamma_j > 0$ for each coordinate $j = 1, \dots, s$, and the γ_u 's are defined as $\gamma_u = \prod_{j \in u} \gamma_j$. For the order-dependent weights, γ_u depends only on the cardinality of u : $\gamma_u = \Gamma_{|u|}$, where $\Gamma_1, \Gamma_2, \dots, \Gamma_s$ are selected positive constants. Their combination gives the product and order-dependent (POD) weights, for which $\gamma_u = \Gamma_{|u|} \prod_{j \in u} \gamma_j$ [34]. The main reason why the most popular choices of weights have this form is that the existing search algorithms are truly efficient for large s only when the weights have this specific POD form [11, 24, 27, 29, 34, 39]. Then, by using a fast-CBC approach that speeds up the search by exploiting a fast Fourier transform [29, 39], one can find a generating vector that gives a worst-case error of $\mathcal{O}(N^{-\alpha+\epsilon})$ in $\mathcal{O}(sN \log N)$ operations for ordinary lattices and $\mathcal{O}(\alpha s N \log N)$ for polynomial lattices (with interlacing).

84 Although this form is convenient, the restriction to POD weights is limiting: for
 85 a given application, the appropriate weights may be quite far from the POD form. In
 86 this case, imposing POD implies that the point sets are constructed with the wrong
 87 weights. Moreover, even without constraints imposed on the form of weights, finding
 88 or approximating appropriate weights and the appropriate α for a given application is
 89 generally very difficult [28]. When the points are constructed with the wrong weights,
 90 the QMC method can be quite ineffective in general. These drawbacks have been
 91 addressed very partially in recent papers. In [12], the authors introduces a construc-
 92 tion algorithm that does not require the knowledge of α . It uses a CBC construction
 93 algorithm with a figure of merit that assumes $\alpha = 1$, and for each coordinate it also
 94 constructs the generating vector one binary digit at a time. The method provides a
 95 convergence rate of $\mathcal{O}(N^{-\alpha+\epsilon})$. In [4], the authors study the stability of rank-1 lattice
 96 rules and polynomial lattice rules to a (limited) misspecification of α and the weights,
 97 for product and POD weights. They obtain worst-case error bounds for function spaces
 98 determined by parameters α' and $\gamma' = \{\gamma'_u\}$ when the rules are constructed using
 99 parameters α and $\gamma = \{\gamma_u\}$ instead, under certain conditions on those weights. These
 100 results are interesting but they do not completely eliminate the need to specify the
 101 weights.

102 The method studied in this paper requires no knowledge at all on α and the
 103 weights γ_u . No value needs to be specified for any of these parameters. The algorithm
 104 is inspired by recent work from Pan and Owen [40], and works as follows. For a
 105 fixed odd integer $r > 0$, we draw r generating vectors independently and uniformly
 106 from the set of all admissible generating vectors. For each of them, we compute the
 107 corresponding QMC approximation $Q_{P_{N,s}}(f)$, then we take the median $M(f)$ of these
 108 r approximations as our final estimate of $I_s(f)$. Since the method does not require
 109 the explicit construction of a good point set, we call it a *construction-free median*
 110 *QMC rule*.

111 Our main contribution is to prove that for \mathcal{F} representing either a weighted
 112 Korobov or weighted Sobolev space determined by parameters α and γ , the error
 113 $|M(f) - I_s(f)|$ obeys the following type of probabilistic bound: For any $\epsilon > 0$ and
 114 $0 < \rho < 1$, there is a constant $c_1 = c_1(\alpha, \gamma, \epsilon) > 0$ (which depends on α , the γ_u 's and
 115 ϵ) such that

$$116 \quad \mathbb{P} \left[\sup_{f \in \mathcal{F}; \|f\| \leq 1} |M(f) - I_s(f)| \leq \frac{c_1(\alpha, \gamma, \epsilon)}{(\rho N)^{\alpha - \epsilon}} \right] \geq 1 - \rho^{(r+1)/2} / 4.$$

117 In other words, the worst-case error of the median estimator is bounded by a quantity
 118 that decreases almost at the best possible rate of $\mathcal{O}(N^{-\alpha})$, with a probability that
 119 converges to 1 exponentially fast as a function of r . That is, we have a simple method
 120 that provides essentially the best possible convergence rate with very high probability,
 121 without requiring any knowledge of α and the weights. The key reason why this is
 122 possible is that the vast majority of the choices of generating vectors turn out to be
 123 quite good and give a QMC approximation which is quite close to $I_s(f)$. Only a small
 124 minority give a large error. For the vector giving the median value to be in that small
 125 minority, there must be at least $(r + 1)/2$ generating vectors in the sample of size r
 126 that belong to this small minority, and the probability that this happens decreases
 127 towards zero exponentially in r .

128 The remainder is organized as follows. In Section 2, we recall some basic facts
 129 on lattice rules for Korobov spaces, and we prove our main result for the median
 130 estimator in this setting. In Section 3, we do the same for high-order polynomial

131 lattice rules in Sobolev spaces. In Section 4, we report numerical experiments to
 132 support our theoretical findings.

133 2. Lattice rules for Korobov spaces.

134 **2.1. Definitions.** Lattice point sets are well suited for performing numerical
 135 integration of smooth periodic functions. A rank-1 lattice point set is defined as
 136 follows:

137 **DEFINITION 2.1** (rank-1 lattice point set). *Let $N \geq 2$ be the number of points*
 138 *and $\mathbf{z} = (z_1, \dots, z_s) \in \{1, \dots, N-1\}^s$. The rank-1 lattice point set defined by N and*
 139 *the generating vector \mathbf{z} is*

$$140 \quad P_{N,s,\mathbf{z}} = \left\{ \left(\left\{ \frac{nz_1}{N} \right\}, \dots, \left\{ \frac{nz_s}{N} \right\} \right) \in [0, 1)^s \mid n = 0, 1, \dots, N-1 \right\},$$

141 where $\{x\} := x - \lfloor x \rfloor$ denotes the fractional part of a real $x \geq 0$. The QMC algorithm
 142 using $P_{N,s,\mathbf{z}}$ as a point set is called the rank-1 lattice rule with generating vector \mathbf{z} .

143 Let $f : [0, 1)^s \rightarrow \mathbb{R}$ be periodic with an absolutely convergent Fourier series

$$144 \quad f(\mathbf{x}) = \sum_{\mathbf{k} \in \mathbb{Z}^s} \hat{f}(\mathbf{k}) \exp(2\pi i \mathbf{k} \cdot \mathbf{x}),$$

145 where the dot product \cdot denotes the usual inner product of two vectors on the Euclid-
 146 ean space \mathbb{R}^s and $\hat{f}(\mathbf{k})$ denotes the \mathbf{k} -th Fourier coefficient of f :

$$147 \quad \hat{f}(\mathbf{k}) := \int_{[0,1)^s} f(\mathbf{x}) \exp(-2\pi i \mathbf{k} \cdot \mathbf{x}) \, d\mathbf{x}.$$

148 Note that $\hat{f}(\mathbf{0})$ coincides with the integral $I_s(f)$. As a class of periodic functions, we
 149 consider the following *weighted Korobov space*.

150 **DEFINITION 2.2** (weighted Korobov space). *Let $\alpha > 1/2$ and $\boldsymbol{\gamma} = \{\gamma_u\}_{u \subseteq \{1, \dots, s\}}$*
 151 *be a set of positive weights with $\gamma_\emptyset = 1$. For a non-empty subset $u \subseteq \{1, \dots, s\}$ and a*
 152 *vector $\mathbf{k}_u \in (\mathbb{Z} \setminus \{0\})^{|u|}$, we denote by $(\mathbf{k}_u, \mathbf{0})$ the vector $\mathbf{h} \in \mathbb{Z}^s$ such that $h_j = k_j$ if*
 153 *$j \in u$ and $h_j = 0$ otherwise, and define*

$$154 \quad r_{\alpha,\boldsymbol{\gamma}}(\mathbf{k}_u, \mathbf{0}) := \gamma_u \prod_{j \in u} \frac{1}{|k_j|^\alpha},$$

156 and set $r_{\alpha,\boldsymbol{\gamma}}(\mathbf{0}) = 1$. The weighted Korobov space, denoted by $\mathcal{F}_{s,\alpha,\boldsymbol{\gamma}}^{\text{kor}}$, is a reproducing
 157 kernel Hilbert space with reproducing kernel

$$158 \quad K_{s,\alpha,\boldsymbol{\gamma}}^{\text{kor}}(\mathbf{x}, \mathbf{y}) = \sum_{\mathbf{k} \in \mathbb{Z}^s} (r_{\alpha,\boldsymbol{\gamma}}(\mathbf{k}))^2 \exp(2\pi i \mathbf{k} \cdot (\mathbf{x} - \mathbf{y})),$$

159 and inner product

$$160 \quad \langle f, g \rangle_{s,\alpha,\boldsymbol{\gamma}}^{\text{kor}} = \sum_{\mathbf{k} \in \mathbb{Z}^s} \frac{\hat{f}(\mathbf{k}) \overline{\hat{g}(\mathbf{k})}}{(r_{\alpha,\boldsymbol{\gamma}}(\mathbf{k}))^2}.$$

161 We denote the induced norm by $\|f\|_{s,\alpha,\boldsymbol{\gamma}}^{\text{kor}} := \sqrt{\langle f, f \rangle_{s,\alpha,\boldsymbol{\gamma}}^{\text{kor}}}$.

162 One wishes to have a good generating vector \mathbf{z} such that the worst-case error of
 163 the corresponding lattice rule for $\mathcal{F}_{s,\alpha,\gamma}^{\text{kor}}$, defined by

$$164 \quad e^{\text{wor}}(Q_{P_{N,s,\mathbf{z}}}; \mathcal{F}_{s,\alpha,\gamma}^{\text{kor}}) := \sup_{\substack{f \in \mathcal{F}_{s,\alpha,\gamma}^{\text{kor}} \\ \|f\|_{s,\alpha,\gamma}^{\text{kor}} \leq 1}} |Q_{P_{N,s,\mathbf{z}}}(f) - I_s(f)|,$$

165 is small. No good explicit construction scheme for such a \mathbf{z} is known for $s \geq 3$, so that
 166 we usually resort to a computer search algorithm as mentioned earlier. By restricting
 167 each z_j to be in the set

$$168 \quad \mathbb{U}_N := \{1 \leq z \leq N-1 \mid \gcd(z, N) = 1\},$$

169 we ensure that each projection of $P_{N,s,\mathbf{z}}$ on a single coordinate contains the N dis-
 170 tinct values $\{0, 1/N, \dots, (N-1)/N\}$ (no superposed points). The CBC construction
 171 algorithm for a good generating vector \mathbf{z} starts with $z_1 = 1$, then for $j = 2, \dots, s$ it
 172 searches for the best component z_j from the set \mathbb{U}_N while keeping the earlier compo-
 173 nents z_1, \dots, z_{j-1} unchanged.

174 For our median QMC rank-1 lattice rule for weighted Korobov spaces, we select
 175 an independent random sample $\mathbf{z}_1, \dots, \mathbf{z}_r$ from the set \mathbb{U}_N^s , and we approximate $I_s(f)$
 176 by the median

$$177 \quad M_{N,s,r}(f) := \text{median}(Q_{P_{N,s,\mathbf{z}_1}}(f), \dots, Q_{P_{N,s,\mathbf{z}_r}}(f)).$$

178 Note that, for given $\mathbf{z}_1, \dots, \mathbf{z}_r$, the index ℓ for which \mathbf{z}_ℓ gives the median $M_{N,s,r}(f)$
 179 generally depends on f . The worst-case error in this case is the random variable

$$180 \quad (2.1) \quad e^{\text{wor}}(M_{N,s,r}; \mathcal{F}_{s,\alpha,\gamma}^{\text{kor}}) := \sup_{\substack{f \in \mathcal{F}_{s,\alpha,\gamma}^{\text{kor}} \\ \|f\|_{s,\alpha,\gamma}^{\text{kor}} \leq 1}} |M_{N,s,r}(f) - I_s(f)|.$$

181 In this random expression, we assume that $\mathbf{z}_1, \dots, \mathbf{z}_r$ are first picked randomly, then
 182 f is taken as the worst-case function for the median, for these given $\mathbf{z}_1, \dots, \mathbf{z}_r$.

183 **2.2. Our main results on lattice rules for Korobov spaces.** To prove our
 184 main result, we need a few more definitions.

185 **DEFINITION 2.3** (dual lattice). *For $N \geq 2$ and $\mathbf{z} \in \mathbb{U}_N^s$, the set*

$$186 \quad P_{N,s,\mathbf{z}}^\perp := \{\mathbf{k} \in \mathbb{Z}^s \mid \mathbf{k} \cdot \mathbf{z} \equiv 0 \pmod{N}\}$$

187 *is called the dual lattice of the rank-1 lattice point set $P_{N,s,\mathbf{z}}$.*

188 The following character property of the rank-1 lattice rule is well-known, see for
 189 instance [5, Lemmas 4.2 and 4.3].

190 **LEMMA 2.4** (character property). *For $N \geq 2$, $\mathbf{z} \in \mathbb{U}_N^s$ and $\mathbf{k} \in \mathbb{Z}^s$, we have*

$$191 \quad \frac{1}{N} \sum_{\mathbf{x} \in P_{N,s,\mathbf{z}}} \exp(2\pi i \mathbf{k} \cdot \mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{k} \in P_{N,s,\mathbf{z}}^\perp, \\ 0 & \text{otherwise.} \end{cases}$$

192 As our first main result, we prove a probabilistic upper bound on the worst-case
 193 error of our median rank-1 lattice rule for weighted Korobov spaces.

194 **THEOREM 2.5.** *Let $N \geq 2$ be an integer, $r > 0$ be an odd integer, and $\mathbf{z}_1, \dots, \mathbf{z}_r$*
 195 *be chosen independently and randomly from the set \mathbb{U}_N^s (with replacement). Then,*
 196 *for any $\alpha > 1/2$ and γ , the worst-case error of the median rule obeys the following*
 197 *bound:*

$$198 \quad e^{\text{wor}}(M_{N,s,r}; \mathcal{F}_{s,\alpha,\gamma}^{\text{kor}}) \leq \inf_{1/(2\alpha) < \lambda < 1} \left(\frac{1}{\eta\varphi(N)} \sum_{\emptyset \neq u \subseteq \{1, \dots, s\}} \gamma_u^{2\lambda} (2\zeta(2\alpha\lambda))^{|u|} \right)^{1/(2\lambda)}$$

199 *with a probability of at least*

$$200 \quad 1 - \binom{r}{(r+1)/2} \eta^{(r+1)/2},$$

201 *for any $0 < \eta < 1$, where φ and ζ denote the Euler totient function and the Riemann*
 202 *zeta function, respectively.*

203 We note that the result for $r = 1$, i.e., the case without taking the median, can
 204 be found, for instance, in [11, Theorem 2], and has been used together with a random
 205 choice of N in [21] to prove an improved rate of convergence of the randomized error.

206 The following inequality on medians is a key ingredient in the proof of the theorem.
 207 Although it can be regarded as a special case of Jensen's inequality on medians proven
 208 in [33], we give a short direct proof to make the paper more self-contained.

209 **LEMMA 2.6.** *For any odd integer r and real numbers a_1, \dots, a_r , it holds that*

$$210 \quad |\text{median}(a_1, \dots, a_r)| \leq \text{median}(|a_1|, \dots, |a_r|).$$

211 *Proof.* Because r is odd, the median is unique. Let $\text{median}(a_1, \dots, a_r) = a_m$ for
 212 some $m \in \{1, \dots, r\}$. If $a_m \geq 0$, then $|a_m| = a_m \leq \text{median}(|a_1|, \dots, |a_r|)$. If $a_m < 0$,
 213 there are at least $(r-1)/2$ other a_ℓ 's for which $a_\ell \leq a_m < 0$, so $|a_\ell| \geq |a_m| > 0$.
 214 Then, $|\text{median}(a_1, \dots, a_r)| = |a_m| \leq \text{median}(|a_1|, \dots, |a_r|)$. \square

215 The same inequality holds for even r by defining the median to be the arithmetic
 216 mean of the two middle values. However, in this paper, we focus on the case where r
 217 is odd for the sake of simplicity.

218 We now prove our main result.

219 *Proof of Theorem 2.5.* Since any $f \in \mathcal{F}_{s,\alpha,\gamma}^{\text{kor}}$ has an absolutely convergent Fourier
 220 series, by applying Lemma 2.4, Lemma 2.6 and the Cauchy-Schwarz inequality, it
 221 holds for given $\mathbf{z}_1, \dots, \mathbf{z}_r$ that

$$\begin{aligned} 222 \quad & e^{\text{wor}}(M_{N,s,r}; \mathcal{F}_{s,\alpha,\gamma}^{\text{kor}}) \\ 223 \quad &= \sup_{\substack{f \in \mathcal{F}_{s,\alpha,\gamma}^{\text{kor}} \\ \|f\|_{s,\alpha,\gamma}^{\text{kor}} \leq 1}} \left| \text{median}_{1 \leq \ell \leq r} \frac{1}{N} \sum_{\mathbf{x} \in P_{N,s,\mathbf{z}_\ell}} f(\mathbf{x}) - I(f) \right| \\ 224 \quad &= \sup_{\substack{f \in \mathcal{F}_{s,\alpha,\gamma}^{\text{kor}} \\ \|f\|_{s,\alpha,\gamma}^{\text{kor}} \leq 1}} \left| \text{median}_{1 \leq \ell \leq r} \frac{1}{N} \sum_{\mathbf{x} \in P_{N,s,\mathbf{z}_\ell}} \sum_{\mathbf{k} \in \mathbb{Z}^s} \hat{f}(\mathbf{k}) \exp(2\pi i \mathbf{k} \cdot \mathbf{x}) - \hat{f}(\mathbf{0}) \right| \\ 225 \quad &= \sup_{\substack{f \in \mathcal{F}_{s,\alpha,\gamma}^{\text{kor}} \\ \|f\|_{s,\alpha,\gamma}^{\text{kor}} \leq 1}} \left| \text{median}_{1 \leq \ell \leq r} \sum_{\mathbf{k} \in P_{N,s,\mathbf{z}_\ell}^\perp \setminus \{\mathbf{0}\}} \hat{f}(\mathbf{k}) \right| \end{aligned}$$

$$\begin{aligned}
226 \quad & \leq \sup_{\substack{f \in \mathcal{F}_{s,\alpha,\gamma}^{\text{kor}} \\ \|f\|_{s,\alpha,\gamma}^{\text{kor}} \leq 1}} \text{median}_{1 \leq \ell \leq r} \sum_{\mathbf{k} \in P_{N,s,z_\ell}^\perp \setminus \{\mathbf{0}\}} |\hat{f}(\mathbf{k})| \\
227 \quad & = \sup_{\substack{f \in \mathcal{F}_{s,\alpha,\gamma}^{\text{kor}} \\ \|f\|_{s,\alpha,\gamma}^{\text{kor}} \leq 1}} \text{median}_{1 \leq \ell \leq r} \sum_{\mathbf{k} \in P_{N,s,z_\ell}^\perp \setminus \{\mathbf{0}\}} \frac{|\hat{f}(\mathbf{k})|}{r_{\alpha,\gamma}(\mathbf{k})} r_{\alpha,\gamma}(\mathbf{k}) \\
228 \quad & \leq \sup_{\substack{f \in \mathcal{F}_{s,\alpha,\gamma}^{\text{kor}} \\ \|f\|_{s,\alpha,\gamma}^{\text{kor}} \leq 1}} \text{median}_{1 \leq \ell \leq r} \left(\sum_{\mathbf{k} \in P_{N,s,z_\ell}^\perp \setminus \{\mathbf{0}\}} \frac{|\hat{f}(\mathbf{k})|^2}{(r_{\alpha,\gamma}(\mathbf{k}))^2} \right)^{1/2} \\
229 \quad & \quad \times \left(\sum_{\mathbf{k} \in P_{N,s,z_\ell}^\perp \setminus \{\mathbf{0}\}} (r_{\alpha,\gamma}(\mathbf{k}))^2 \right)^{1/2} \\
230 \quad & \leq \sup_{\substack{f \in \mathcal{F}_{s,\alpha,\gamma}^{\text{kor}} \\ \|f\|_{s,\alpha,\gamma}^{\text{kor}} \leq 1}} \left(\sum_{\mathbf{k} \in \mathbb{Z}^s \setminus \{\mathbf{0}\}} \frac{|\hat{f}(\mathbf{k})|^2}{(r_{\alpha,\gamma}(\mathbf{k}))^2} \right)^{1/2} \text{median}_{1 \leq \ell \leq r} \mathcal{S}_{\alpha,\gamma}(z_\ell) \\
231 \quad (2.2) \quad & \leq \text{median}_{1 \leq \ell \leq r} \mathcal{S}_{\alpha,\gamma}(z_\ell), \\
232 \quad &
\end{aligned}$$

233 in which

$$234 \quad \mathcal{S}_{\alpha,\gamma}(z) := \left(\sum_{\mathbf{k} \in P_{N,s,z}^\perp \setminus \{\mathbf{0}\}} (r_{\alpha,\gamma}(\mathbf{k}))^2 \right)^{1/2}.$$

235 For $1/(2\alpha) < \lambda \leq 1$, by using the subadditivity

$$236 \quad (2.3) \quad \left(\sum_i a_i \right)^\lambda \leq \sum_i a_i^\lambda, \\
237 \quad$$

238 which holds for non-negative reals $a_1, a_2, \dots > 0$, see [5, Theorem 2.2], and noting
239 that the cardinality of \mathbb{U}_N is equal to $\varphi(N)$, we have

$$\begin{aligned}
240 \quad & \frac{1}{(\varphi(N))^s} \sum_{\mathbf{z} \in \mathbb{U}_N^s} (\mathcal{S}_{\alpha,\gamma}(z))^{2\lambda} \\
241 \quad & = \frac{1}{(\varphi(N))^s} \sum_{\mathbf{z} \in \mathbb{U}_N^s} \left(\sum_{\mathbf{k} \in P_{N,s,z}^\perp \setminus \{\mathbf{0}\}} (r_{\alpha,\gamma}(\mathbf{k}))^2 \right)^\lambda \\
242 \quad & \leq \frac{1}{(\varphi(N))^s} \sum_{\mathbf{z} \in \mathbb{U}_N^s} \sum_{\mathbf{k} \in P_{N,s,z}^\perp \setminus \{\mathbf{0}\}} (r_{\alpha,\gamma}(\mathbf{k}))^{2\lambda} \\
243 \quad & = \sum_{\mathbf{k} \in \mathbb{Z}^s \setminus \{\mathbf{0}\}} (r_{\alpha,\gamma}(\mathbf{k}))^{2\lambda} \frac{1}{(\varphi(N))^s} \sum_{\mathbf{z} \in \mathbb{U}_N^s} \frac{1}{N} \sum_{\mathbf{x} \in P_{N,s,z}} \exp(2\pi i \mathbf{k} \cdot \mathbf{x}) \\
244 \quad & = \sum_{\mathbf{k} \in \mathbb{Z}^s \setminus \{\mathbf{0}\}} (r_{\alpha,\gamma}(\mathbf{k}))^{2\lambda} \frac{1}{(\varphi(N))^s} \sum_{\mathbf{z} \in \mathbb{U}_N^s} \frac{1}{N} \sum_{n=0}^{N-1} \exp(2\pi i n \mathbf{k} \cdot \mathbf{z}/N)
\end{aligned}$$

$$\begin{aligned}
245 \quad &= \sum_{\emptyset \neq u \subseteq \{1, \dots, s\}} \gamma_u^{2\lambda} \sum_{\mathbf{k}_u \in (\mathbb{Z} \setminus \{0\})^{|u|}} \frac{1}{(\varphi(N))^s} \sum_{\mathbf{z} \in \mathbb{U}_N^s} \frac{1}{N} \sum_{n=0}^{N-1} \prod_{j \in u} \frac{\exp(2\pi i n k_j z_j / N)}{|k_j|^{2\alpha\lambda}} \\
246 \quad &= \frac{1}{N} \sum_{n=0}^{N-1} \sum_{\emptyset \neq u \subseteq \{1, \dots, s\}} \frac{\gamma_u^{2\lambda}}{(\varphi(N))^{|u|}} \sum_{\mathbf{z}_u \in \mathbb{U}_N^{|u|}} \sum_{\mathbf{k}_u \in (\mathbb{Z} \setminus \{0\})^{|u|}} \prod_{j \in u} \frac{\exp(2\pi i n k_j z_j / N)}{|k_j|^{2\alpha\lambda}} \\
247 \quad &= \frac{1}{N} \sum_{n=0}^{N-1} \sum_{\emptyset \neq u \subseteq \{1, \dots, s\}} \gamma_u^{2\lambda} (T_{2\alpha\lambda}(n, N))^{|u|}, \\
248 \quad &
\end{aligned}$$

249 where we write

$$250 \quad T_{2\alpha\lambda}(n, N) := \frac{1}{\varphi(N)} \sum_{z \in \mathbb{U}_N} \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{\exp(2\pi i n k z / N)}{|k|^{2\alpha\lambda}}.$$

251 Since it follows from [23, Lemmas 2.1 & 2.2] that, for any positive integer d

$$252 \quad \frac{1}{N} \sum_{n=0}^{N-1} (T_{2\alpha\lambda}(n, N))^d \leq \frac{(2\zeta(2\alpha\lambda))^d}{\varphi(N)},$$

253 we obtain

$$\begin{aligned}
254 \quad &\frac{1}{(\varphi(N))^s} \sum_{\mathbf{z} \in \mathbb{U}_N^s} (\mathcal{S}_{\alpha, \gamma}(\mathbf{z}))^{2\lambda} \leq \sum_{\emptyset \neq u \subseteq \{1, \dots, s\}} \gamma_u^{2\lambda} \frac{1}{N} \sum_{n=0}^{N-1} (T_{2\alpha\lambda}(n, N))^{|u|} \\
255 \quad &\leq \frac{1}{\varphi(N)} \sum_{\emptyset \neq u \subseteq \{1, \dots, s\}} \gamma_u^{2\lambda} (2\zeta(2\alpha\lambda))^{|u|}. \\
256 \quad &
\end{aligned}$$

257 This gives an upper bound on the average of $(\mathcal{S}_{\alpha, \gamma}(\mathbf{z}))^{2\lambda}$ over all of the admissible
258 $\mathbf{z} \in \mathbb{U}_N^s$, which holds for any $1/(2\alpha) < \lambda \leq 1$.

259 Then, Markov's inequality ensures that for any $0 < \eta < 1$, the probability of
260 having

$$261 \quad \mathcal{S}_{\alpha, \gamma}(\mathbf{z}) > \inf_{1/(2\alpha) < \lambda < 1} \left(\frac{1}{\eta \varphi(N)} \sum_{\emptyset \neq u \subseteq \{1, \dots, s\}} \gamma_u^{2\lambda} (2\zeta(2\alpha\lambda))^{|u|} \right)^{1/(2\lambda)} =: B(\alpha, \gamma)$$

262 is at most η for a random choice of $\mathbf{z} \in \mathbb{U}_N^s$. For the median estimator $M_{N, s, r}$ to be
263 larger than this bound $B(\alpha, \gamma)$, we must have $\mathcal{S}_{\alpha, \gamma}(\mathbf{z}_\ell) > B(\alpha, \gamma)$ for at least $(r+1)/2$
264 vectors among $\mathbf{z}_1, \dots, \mathbf{z}_r$. Taking the union bound on possible sets of $(r+1)/2$ vectors
265 with $\mathcal{S}_{\alpha, \gamma}(\mathbf{z}_\ell) > B(\alpha, \gamma)$, the probability that this happens is bounded above by

$$266 \quad \binom{r}{(r+1)/2} \eta^{(r+1)/2}.$$

267 Combining this with the bound shown in (2.2) completes the proof. \square

268 *Remark 2.7.* One can easily prove by induction on k that $\binom{2k-1}{k} < 4^{k-1}$ for $k \geq 2$.
269 Indeed, this is true for $k = 2$, and for $k \geq 2$, one has

$$270 \quad \binom{2k+1}{k+1} = \frac{2(2k+1)}{(k+1)} \binom{2k-1}{k} < 4 \binom{2k-1}{k} < 4^k.$$

271 Then, for any odd $r \geq 3$, we have

$$272 \quad (2.4) \quad \binom{r}{(r+1)/2} \eta^{(r+1)/2} < (4\eta)^{(r+1)/2}/4.$$

273 Thus, the probability given in Theorem 2.5 must be larger than $1 - (4\eta)^{(r+1)/2}/4$,
274 which converges to 1 exponentially fast as a function of r for $0 < \eta < 1/4$.

275 By taking $1/(2\lambda) = \alpha - \epsilon$ and $\rho = 4\eta$ and using the previous remark, we obtain
276 the following corollary as a simplified version of Theorem 2.5.

277 **COROLLARY 2.8.** *For any odd $r \geq 3$, $\epsilon > 0$, and $0 < \rho < 1$, there is a constant*
278 *$c_1 = c_1(\alpha, \boldsymbol{\gamma}, \epsilon) > 0$ (which depends on α , the γ_u 's and ϵ) such that*

$$279 \quad \mathbb{P} \left[e^{\text{wor}}(M_{N,s,r}; \mathcal{F}_{s,\alpha,\boldsymbol{\gamma}}^{\text{kor}}) \leq \frac{c_1(\alpha, \boldsymbol{\gamma}, \epsilon)}{(\rho N)^{\alpha-\epsilon}} \right] \geq 1 - \rho^{(r+1)/2}/4.$$

280 *Proof.* For prime N , we have $\varphi(N) = N - 1$ and the corollary follows from
281 Theorem 2.5 and the bound (2.4) in Remark 2.7. For a general N , we know from [42,
282 Theorem 15] that

$$283 \quad \frac{1}{\varphi(N)} \leq \frac{1}{N} \left[e^C \log \log N + \frac{2.50637}{\log \log N} \right]$$

284 for any $N \geq 3$, where $C = 0.57721 \dots$ is the Euler's constant. From this, the proba-
285 bilistic bound follows. \square

286 *Remark 2.9.* Under some additional conditions on the weights $\boldsymbol{\gamma}$, this bound de-
287 pends only polynomially on the dimension s , and can even be independent on the
288 dimension s . We refer to [11, Theorem 3] for the case of general weights and [11,
289 Theorem 4] for the case of product weights.

290 *Remark 2.10.* As mentioned in Section 1, our median rank-1 lattice rule is moti-
291 vated by the observation that most of the possible generating vectors $\mathbf{z} \in \mathbb{U}_N^s$ are a
292 good choice, but the remaining ones are bad. To show this, let us first point out that
293 $\mathcal{S}_{\alpha,\boldsymbol{\gamma}}(\mathbf{z})$ coincides with the worst-case error of the rank-1 lattice rule with the given
294 \mathbf{z} , see [7, Theorem 5.12]. As already seen in the proof of Theorem 2.5, a proportion
295 of the generating vectors \mathbf{z} which satisfy the bound of order $N^{-\alpha+\epsilon}$, i.e.,

$$296 \quad \mathcal{S}_{\alpha,\boldsymbol{\gamma}}(\mathbf{z}) \leq \inf_{1/(2\alpha) < \lambda < 1} \left(\frac{1}{\eta \varphi(N)} \sum_{\emptyset \neq u \subseteq \{1, \dots, s\}} \gamma_u^{2\lambda} (2\zeta(2\alpha\lambda))^{|u|} \right)^{1/(2\lambda)},$$

297 is greater than or equal to $1 - \eta$, for any $0 < \eta < 1$. On the other hand, the averaging
298 argument in the proof of Theorem 2.5 with $\lambda = 1$ gives

$$299 \quad \frac{1}{(\varphi(N))^s} \sum_{\mathbf{z} \in \mathbb{U}_N^s} (\mathcal{S}_{\alpha,\boldsymbol{\gamma}}(\mathbf{z}))^2 \leq \frac{1}{\varphi(N)} \sum_{\emptyset \neq u \subseteq \{1, \dots, s\}} \gamma_u^2 (2\zeta(2\alpha))^{|u|}.$$

300 This implies that, for each N , there exists a small, distinct set of “bad” generating
301 vectors whose $\mathcal{S}_{\alpha,\boldsymbol{\gamma}}(\mathbf{z})$ values are quite large so that the average of the *squared* worst-
302 case error over all the possible generating vectors is merely of order N^{-1} . Such bad
303 vectors may include those with all components being the same.

304 *Remark 2.11.* Theorem 2.5 (or its corollary) gives a probabilistic error bound on
305 the worst-case error (2.1), together with a lower bound on the probability that this

306 error bound holds. The exact value of this probability depends on the probability
 307 distribution of $Q_{P_{N,s,z}}(f)$ when z is drawn uniformly from \mathbb{U}_N^s , and on the choices
 308 of r and of the other parameters in the error bound. To get some insight on how
 309 it behaves, we will simplify the setting slightly and look at a one-side error bound
 310 for a fixed f : we want to estimate the probability that the median $M_{N,s,r}(f)$ does
 311 not exceed some arbitrary constant y larger than the mean $I_s(f)$. Suppose that this
 312 y is the q -quantile y_q of the distribution of $Q_{P_{N,s,z}}(f)$ for some $q \in (3/4, 1)$, i.e.,
 313 $q = \mathbb{P}[Q_{P_{N,s,z}}(f) \leq y_q]$. Then the median $M_{N,s,r}(f)$ is larger than y_q if and only if at
 314 least $(r+1)/2$ values are larger than y_q , and the probability that this happens is

$$315 \quad (2.5) \quad p_+(r, q) = \sum_{i=(r+1)/2}^r \binom{r}{i} (1-q)^i q^{r-i}.$$

317 Figure 1 plots $\log_{10} p_+(r, q)$ as a function of $r \in \{3, 5, \dots, 49\}$ for $q = 0.5$, $q = 0.75$ and
 318 $q = 0.9$, respectively. We see that $p_+(r, 0.5) = 0.5$ for any r and that $p_+(r, q) \approx 10^{-\gamma r}$
 319 where $\gamma \approx 0.071$ for $q = 0.75$ and $\gamma \approx 0.231$ for $q = 0.9$. These plots provide some
 320 insight on the choice of r . In particular, for fixed $q > 0.5$, doubling r squares the
 321 probability $p_+(r, q)$. Suppose for example that we want $p_+(r, q) \leq 10^{-4}$, to have a
 322 reasonable assurance that $M_{N,s,r}(f) \leq y_q$. The plot shows that the minimal value of
 323 r for this is about $r = 13$ for $q = 0.9$, and about $r = 49$ for $q = 0.75$. For a given
 324 $f \in \mathcal{F}_{s,\alpha,\gamma}^{\text{kor}}$ and fixed N , a larger q means a larger y_q , but for a fixed q we can reduce
 325 y_q and bring it close to $I_s(f)$ by increasing N . From Theorem 2.5 with $r = 1$ and
 326 $\eta = 1 - q < 1/4$ (or its corollary with $\rho = 4(1 - q) < 1$), we have that $|y_q - I_s(f)|$
 327 is $\mathcal{O}(N^{-\alpha+\epsilon})$. In summary, for a fixed $q > 3/4$, we can decrease the error bound
 328 by increasing N and increase the probability that the bound holds by increasing r .
 329 We can also increase both q and N in a way such that y_q remains about the same;
 330 then the same $p_+(r, q)$ can be obtained with a reduced r . What we just said is for
 331 the upper bound $M_{N,s,r}(f) \leq y_q$, but essentially the same discussion can be made
 332 concerning the assurance that $M_{N,s,r}(f) > y_{1-q}$. In applications, the values of y_q
 333 and y_{1-q} are unknown, but our reasoning suggests that a moderate value of r , say no
 334 more than 25, should be sufficient in practice, together with a large N (as large as the
 335 computing budget allows). The results of our numerical experiments support this.

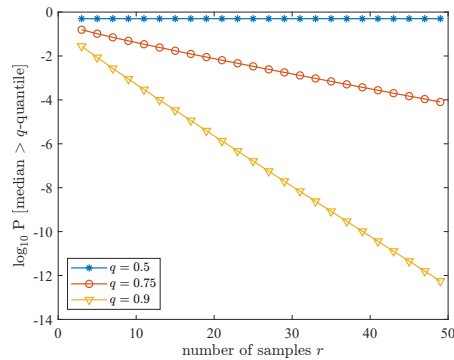


FIG. 1. Probability $p_+(r, q)$ (on a \log_{10} scale) as a function of r for $q = 0.5$ (blue), $q = 0.75$ (red) and $q = 0.9$ (yellow).

336 *Remark 2.12.* It is known that rank-1 lattice rules also work for non-periodic

337 functions by applying the tent transformation

$$338 \quad \pi(x) = 1 - |2x - 1|$$

339 component-wise to every point in the set $P_{N,s,\mathbf{z}}$ [18, 8, 2, 16]. The same probabilistic
 340 upper bound, shown in Theorem 2.5, holds for the worst-case error of the median rule
 341 built up of the tent-transformed rank-1 lattice rules in the so-called *weighted half-*
 342 *period cosine spaces* with any parameter α and weights γ . As shown in [8, Lemma 1],
 343 the half-period cosine space coincides with an unanchored Sobolev space with smooth-
 344 ness 1 when $\alpha = 1$.

345 **3. High-order polynomial lattice rules for Sobolev spaces.** We now con-
 346 sider high-order polynomial lattice point sets as defined in [9]. These point sets are
 347 well suited for performing numerical integration of smooth non-periodic functions.
 348 In what follows let b be a prime, and \mathbb{F}_b be the finite field of order b , which we
 349 identify with the set $\{0, 1, \dots, b-1\}$. Let \mathbb{N} be the set of positive integers and
 350 $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For $k \in \mathbb{N}_0$ having the b -adic finite expansion $k = \kappa_0 + \kappa_1 b + \dots$, we
 351 write $k(x) = \kappa_0 + \kappa_1 x + \dots \in \mathbb{F}_b[x]$. With these ingredients, we have the following
 352 definition from [9].

353 **DEFINITION 3.1** (high-order polynomial lattice point sets). *Let $m, n \in \mathbb{N}$ with*
 354 *$m \leq n$, $p \in \mathbb{F}_b[x]$ with $\deg(p) = n$ and $\mathbf{q} = (q_1, \dots, q_s) \in (\mathbb{F}_b[x])^s$ with $\deg(q_j) < n$.*
 355 *The high-order polynomial lattice point set defined by m, n, p and \mathbf{q} consists of $N = b^m$*
 356 *points and is given by*

$$357 \quad P_{m,n,s,p,\mathbf{q}} = \left\{ \left(\nu_n \left(\frac{h(x)q_1(x)}{p(x)} \right), \dots, \nu_n \left(\frac{h(x)q_s(x)}{p(x)} \right) \right) \in [0, 1)^s \mid h = 0, \dots, b^m - 1 \right\},$$

358 where $\nu_n : \mathbb{F}_b((x^{-1})) \rightarrow [0, 1)$ is defined by

$$359 \quad \nu_n \left(\sum_{i=w}^{\infty} \frac{a_i}{x^i} \right) := \sum_{i=\max(1,w)}^n \frac{a_i}{b^i}.$$

360 *The QMC algorithm using $P_{m,n,s,p,\mathbf{q}}$ as a point set is called the high-order polynomial*
 361 *lattice rule with modulus p and generating vector \mathbf{q} . The order of this rule is defined*
 362 *as $\lfloor n/m \rfloor$.*

363 Typically, n will be a multiple of m . When $n = m$, this gives the digital net
 364 construction introduced in [35] and called polynomial lattice rule in [10]. Note that [26,
 365 31] introduced the term “polynomial lattice rule” with a slightly different definition,
 366 in which the coordinates of the points have an infinite periodic expansion and the
 367 modulus p has degree m . The construction in Definition 3.1 essentially builds a
 368 polynomial lattice point set with b^n points and uses only the first b^m points.

369 Instead of the weighted Korobov space $\mathcal{F}_{s,\alpha,\gamma}^{\text{kor}}$, we consider the following Sobolev-
 370 type Banach space as our target space for high-order polynomial lattice rules.

371 **DEFINITION 3.2** (weighted Sobolev space). *Let $\alpha \in \mathbb{N}$, $\alpha \geq 2$, $1 \leq q \leq \infty$ and*
 372 *let $\gamma = \{\gamma_u\}_{u \subseteq \{1, \dots, s\}}$ be a set of positive weights with $\gamma_\emptyset = 1$. The weighted Sobolev*
 373 *space, denoted by $\mathcal{F}_{s,\alpha,\gamma,q}^{\text{sob}}$, is a Banach space consisting of non-periodic (in the sense*
 374 *of not necessarily periodic) smooth functions with the norm*

$$375 \quad \|f\|_{s,\alpha,\gamma,q}^{\text{sob}} :=$$

$$376 \quad \sup_{u \subseteq \{1, \dots, s\}} \gamma_u^{-1} \left(\sum_{v \subseteq u} \sum_{\tau_{u \setminus v} \in \{1, \dots, \alpha\}^{|u \setminus v|}} \int_{[0,1]^{|v|}} \left| \int_{[0,1]^{s-|v|}} f^{(\tau_{u \setminus v}, \alpha_v, \mathbf{0})}(\mathbf{x}) \, d\mathbf{x}_{-v} \right|^q \, d\mathbf{x}_v \right)^{1/q},$$

377 where $(\tau_{u \setminus v}, \alpha_v, \mathbf{0})$ denotes the vector $\mathbf{h} \in \mathbb{N}_0^s$ such that $h_j = \tau_j$ if $j \in u \setminus v$, $h_j = \alpha$
 378 if $j \in v$, and $h_j = 0$ otherwise, and $f^{(\tau_{u \setminus v}, \alpha_v, \mathbf{0})}(\mathbf{x})$ denotes the mixed derivative of
 379 order $(\tau_{u \setminus v}, \alpha_v, \mathbf{0})$ of f . Moreover, we write $\mathbf{x}_v = (x_j)_{j \in v}$ and $\mathbf{x}_{-v} = (x_j)_{j \in \{1, \dots, s\} \setminus v}$.

380 This Sobolev space was introduced by [6] in the context of partial differential
 381 equations with random coefficients. (The original function space in [6] contains the
 382 additional parameter $r \in [1, \infty]$, and the definition of the norm has been corrected in
 383 <https://arxiv.org/abs/1309.4624>. In this paper we choose $r = \infty$, which makes the
 384 norm smallest over $r \in [1, \infty]$.) The parameter α determines the differentiability of
 385 the non-periodic functions. As for the rank-1 lattice rules for the weighted Korobov
 386 spaces, it is desirable to have good modulus p and generating vector \mathbf{q} such that the
 387 worst-case error of the corresponding high-order polynomial lattice rule for $\mathcal{F}_{s, \alpha, \gamma}^{\text{sob}}$ is
 388 small. Originally in [6], interlaced polynomial lattice rules [13, 15] were used instead
 389 of high-order polynomial lattice rules, and it was shown that the worst-case error
 390 bound of order $N^{-\alpha+\epsilon}$ with arbitrarily small $\epsilon > 0$ can be achieved by the CBC
 391 algorithm applied to interlaced polynomial lattice rules. The major advantage of
 392 interlaced polynomial lattice rules over high-order polynomial lattice rules lies in the
 393 construction cost for the CBC algorithm: for the product weights, constructing an
 394 interlaced rule requires $\mathcal{O}(\alpha s N \log N)$ operations with $\mathcal{O}(N)$ memory [13], whereas
 395 constructing a high-order rule require $\mathcal{O}(s N^\alpha \log N)$ operations with $\mathcal{O}(N^\alpha)$ memory
 396 [1].

397 However, interlaced polynomial lattice rules are not necessarily a better choice
 398 than high-order polynomial lattice rules. To construct an interlaced polynomial lat-
 399 tice rule, which relies on the digit interlacing method due to Dick [3], we must select
 400 an integer interlacing factor d , the construction cost increases linearly with d , and
 401 the resulting rule cannot exploit the smoothness of functions beyond d . This means
 402 that if $\alpha > d$, the worst-case error bound is only of order $N^{-d+\epsilon}$. High-order poly-
 403 nomial lattice rules do not explicitly require such a factor. We only need to specify
 404 the maximum precision n of the points. This n can be set as large as possible, for in-
 405 stance, with $b = 2$, we can take $n = 53$ for the double-precision floating-point format.
 406 This way, high-order polynomial lattice rules can be possibly made adaptive to the
 407 smoothness α of functions, addressing the drawback of interlaced polynomial lattice
 408 rules. Since we do not apply CBC in this paper, we prefer high-order rules over the
 409 interlaced ones.

410 In what follows, we assume that the polynomial p is irreducible and we write

$$411 \quad G_n := \{q \in \mathbb{F}_b[x] \mid q \neq 0 \text{ and } \deg(q) < n\}.$$

412 Analogously to the rank-1 lattice case in Section 2, we consider the following median
 413 high-order polynomial lattice rule for weighted Sobolev spaces. For an odd integer r ,
 414 we draw $\mathbf{q}_1, \dots, \mathbf{q}_r$ randomly and independently from the set G_n^s , and we approximate
 415 $I_s(f)$ by

$$416 \quad M_{m, n, s, p, r}(f) := \text{median}(Q_{P_{m, n, s, p, \mathbf{q}_1}}(f), \dots, Q_{P_{m, n, s, p, \mathbf{q}_r}}(f)).$$

417 The worst-case error is the random variable

$$418 \quad e^{\text{wor}}(M_{m, n, s, p, r}; \mathcal{F}_{s, \alpha, \gamma, q}^{\text{sob}}) := \sup_{\substack{f \in \mathcal{F}_{s, \alpha, \gamma, q}^{\text{sob}} \\ \|f\|_{s, \alpha, \gamma, q}^{\text{sob}} \leq 1}} |M_{m, n, s, p, r}(f) - I_s(f)|.$$

419 **3.1. Main results for polynomial lattice point sets in Sobolev spaces.**

420 We first need a few definitions and lemmas.

421 **DEFINITION 3.3** (dual polynomial lattice). *Let $m, n \in \mathbb{N}$ with $m \leq n$, $p \in$*
 422 *$\mathbb{F}_b[x]$ with $\deg(p) = n$ and $\mathbf{q} \in G_n^s$. For $k \in \mathbb{N}_0$ with the b -adic finite expansion*
 423 *$k = \kappa_0 + \kappa_1 b + \dots$, we define*

$$424 \quad \text{tr}_n(k) = \sum_{i=0}^{n-1} \kappa_i x^i \in G_n \cup \{0\}.$$

425 *This operator is applied component-wise to a vector. Then the set*

$$426 \quad P_{m,n,s,p,\mathbf{q}}^\perp = \{\mathbf{k} \in \mathbb{N}_0^s \mid \text{tr}_n(\mathbf{k}) \cdot \mathbf{q} \equiv a \pmod{p} \text{ with } \deg(a) < n - m\},$$

427 *is called the dual net of the high-order polynomial lattice point set $P_{m,n,s,p,\mathbf{q}}$.*

428 **DEFINITION 3.4** (Walsh functions). *Let us write $\omega_b := \exp(2\pi i/b)$. For $k \in \mathbb{N}_0$,*
 429 *we denote the b -adic expansion of k by $k = \kappa_0 + \kappa_1 b + \dots$. The k -th Walsh function*
 430 *$\text{wal}_k: [0, 1) \rightarrow \mathbb{C}$ is defined by*

$$431 \quad \text{wal}_k(x) := \omega_b^{\kappa_0 \xi_1 + \kappa_1 \xi_2 + \dots},$$

432 *where the b -adic expansion of $x \in [0, 1)$ is denoted by $x = \xi_1/b + \xi_2/b^2 + \dots$, which is*
 433 *understood to be unique in the sense that infinitely many of the ξ_i are different from*
 434 *$b - 1$.*

435 *For $s \geq 2$ and $\mathbf{k} = (k_1, \dots, k_s) \in \mathbb{N}_0^s$, the s -dimensional \mathbf{k} -th Walsh function*
 436 *$\text{wal}_{\mathbf{k}}: [0, 1)^s \rightarrow \mathbb{C}$ is defined by*

$$437 \quad \text{wal}_{\mathbf{k}}(\mathbf{x}) := \prod_{j=1}^s \text{wal}_{k_j}(x_j).$$

438 It is well-known that the system of Walsh functions is a complete orthogonal system in
 439 $L_2([0, 1)^s)$, see [10, Appendix A]. The following character property of the high-order
 440 polynomial lattice point set is analogous to what is stated in Lemma 2.4.

441 **LEMMA 3.5** (character property). *For $m, n \in \mathbb{N}$ with $m \leq n$, $p \in \mathbb{F}_b[x]$ with*
 442 *$\deg(p) = n$ and $\mathbf{q} \in G_n^s$, we have*

$$443 \quad \frac{1}{b^m} \sum_{\mathbf{x} \in P_{m,n,s,p,\mathbf{q}}} \text{wal}_{\mathbf{k}}(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{k} \in P_{m,n,s,p,\mathbf{q}}^\perp, \\ 0 & \text{otherwise.} \end{cases}$$

444 For any $f \in \mathcal{F}_{s,\alpha,\gamma,\mathbf{q}}^{\text{sob}}$, we have the following absolutely convergent Walsh series

$$445 \quad f(\mathbf{x}) = \sum_{\mathbf{k} \in \mathbb{N}_0^s} \tilde{f}(\mathbf{k}) \text{wal}_{\mathbf{k}}(\mathbf{x}),$$

446 where $\tilde{f}(\mathbf{k})$ denotes the \mathbf{k} -th Walsh coefficient of f :

$$447 \quad \tilde{f}(\mathbf{k}) := \int_{[0,1)^s} f(\mathbf{x}) \overline{\text{wal}_{\mathbf{k}}(\mathbf{x})} \, d\mathbf{x}.$$

448 Note that $\tilde{f}(\mathbf{0})$ coincides with the integral $I_s(f)$. The following result on the decay of
 449 Walsh coefficients for $f \in \mathcal{F}_{s,\alpha,\gamma,\mathbf{q}}^{\text{sob}}$ was shown in [6, Theorem 3.5].

450 LEMMA 3.6 (decay of Walsh coefficients). Let $\alpha \in \mathbb{N}$, $\alpha \geq 2$, $1 \leq q \leq \infty$ and
 451 $\boldsymbol{\gamma} = \{\gamma_u\}_{u \subseteq \{1, \dots, s\}}$ be a set of positive weights with $\gamma_\emptyset = 1$. For any $f \in \mathcal{F}_{s, \alpha, \boldsymbol{\gamma}, q}^{\text{sob}}$, a
 452 non-empty subset $u \subseteq \{1, \dots, s\}$ and $\mathbf{k}_u \in \mathbb{N}^{|u|}$, it holds that

$$453 \quad |\tilde{f}(\mathbf{k}_u, \mathbf{0})| \leq \|f\|_{s, \alpha, \boldsymbol{\gamma}, q}^{\text{sob}} \gamma_u C_\alpha^{|u|} b^{-\mu_\alpha(\mathbf{k}_u)},$$

454 where

$$455 \quad C_\alpha := \left(1 + \frac{1}{b} + \frac{1}{b(b+1)}\right)^{\alpha-2} \left(3 + \frac{2}{b} + \frac{2b+1}{b-1}\right) \max\left(\frac{2}{(2 \sin \frac{\pi}{b})^\alpha}, \max_{1 \leq \tau < \alpha} \frac{1}{(2 \sin \frac{\pi}{b})^\tau}\right)$$

456 and $\mu_\alpha(\mathbf{k}_u) := \sum_{j \in u} \mu_\alpha(k_j)$, with

$$457 \quad \mu_\alpha(k) = \sum_{i=1}^{\min(\alpha, c)} a_i$$

458 for $k \in \mathbb{N}$ whose b -adic expansion is given by $k = \kappa_1 b^{a_1-1} + \kappa_2 b^{a_2-1} + \dots + \kappa_c b^{a_c-1}$
 459 such that $c \geq 1$, $\kappa_1, \dots, \kappa_c \in \{1, \dots, b-1\}$ and $a_1 > \dots > a_c$.

460 As the second main result of this paper, we show a probabilistic upper bound
 461 on the worst-case error of our median high-order polynomial lattice rule for weighted
 462 Sobolev spaces.

463 THEOREM 3.7. Let $m, n \in \mathbb{N}$ with $m \leq n$, $p \in \mathbb{F}_b[x]$ be irreducible with $\deg(p) =$
 464 n , r be odd and $\mathbf{q}_1, \dots, \mathbf{q}_r$ be chosen independently and randomly from the set G_n^s .
 465 Then, for any integer $\alpha \geq 2$ and $\boldsymbol{\gamma}$, the worst-case error is bounded above by

$$466 \quad e^{\text{wor}}(M_{m, n, s, p, r}; \mathcal{F}_{s, \alpha, \boldsymbol{\gamma}, q}^{\text{sob}}) \leq \inf_{1/\alpha < \lambda < 1} \left(\frac{2}{\eta(b^{\min(m, \lambda n)} - 1)} \sum_{\emptyset \neq u \subseteq \{1, \dots, s\}} \gamma_u^\lambda C_\alpha^{\lambda|u|} A_{\alpha, \lambda}^{|u|} \right)^{1/\lambda}$$

467 with a probability of at least

$$468 \quad 1 - \binom{r}{(r+1)/2} \eta^{(r+1)/2},$$

469 for any $0 < \eta < 1$, where we write

$$470 \quad (3.1) \quad A_{\alpha, \lambda} = \sum_{\tau=1}^{\alpha-1} \prod_{i=1}^{\tau} \frac{b-1}{b^{\lambda i} - 1} + \frac{b^{\lambda \alpha} - 1}{b^{\lambda \alpha} - b} \prod_{i=1}^{\alpha} \frac{b-1}{b^{\lambda i} - 1}.$$

472 *Proof.* Throughout this proof, we write

$$473 \quad \tilde{r}_{\alpha, \boldsymbol{\gamma}}(\mathbf{k}_u, \mathbf{0}) = \gamma_u C_\alpha^{|u|} b^{-\mu_\alpha(\mathbf{k}_u)},$$

474 for a non-empty subset $u \subseteq \{1, \dots, s\}$ and $\mathbf{k}_u \in \mathbb{N}^{|u|}$. Since any $f \in \mathcal{F}_{s, \alpha, \boldsymbol{\gamma}, q}^{\text{sob}}$ has
 475 an absolutely convergent Walsh series, by applying Lemma 3.5, Lemma 2.6, Hölder's
 476 inequality and Lemma 3.6 in this order, it holds that

$$477 \quad e^{\text{wor}}(M_{m, n, s, p, r}; \mathcal{F}_{s, \alpha, \boldsymbol{\gamma}, q}^{\text{sob}})$$

$$478 \quad = \sup_{\substack{f \in \mathcal{F}_{s, \alpha, \boldsymbol{\gamma}, q}^{\text{sob}} \\ \|f\|_{s, \alpha, \boldsymbol{\gamma}, q}^{\text{sob}} \leq 1}} \left| \text{median}_{1 \leq \ell \leq r} \frac{1}{b^m} \sum_{\mathbf{x} \in P_{m, n, s, p, \mathbf{q}_\ell}} f(\mathbf{x}) - I(f) \right|$$

$$\begin{aligned}
479 \quad &= \sup_{\substack{f \in \mathcal{F}_{s,\alpha,\gamma,q}^{\text{sob}} \\ \|f\|_{s,\alpha,\gamma,q}^{\text{sob}} \leq 1}} \left| \text{median}_{1 \leq \ell \leq r} \frac{1}{b^m} \sum_{\mathbf{x} \in P_{m,n,s,p,q_\ell}} \sum_{\mathbf{k} \in \mathbb{N}_0^s} \tilde{f}(\mathbf{k}) \text{wal}_{\mathbf{k}}(\mathbf{x}) - \tilde{f}(\mathbf{0}) \right| \\
480 \quad &= \sup_{\substack{f \in \mathcal{F}_{s,\alpha,\gamma,q}^{\text{sob}} \\ \|f\|_{s,\alpha,\gamma,q}^{\text{sob}} \leq 1}} \left| \text{median}_{1 \leq \ell \leq r} \sum_{\mathbf{k} \in P_{m,n,s,p,q_\ell}^\perp \setminus \{\mathbf{0}\}} \tilde{f}(\mathbf{k}) \right| \\
481 \quad &\leq \sup_{\substack{f \in \mathcal{F}_{s,\alpha,\gamma,q}^{\text{sob}} \\ \|f\|_{s,\alpha,\gamma,q}^{\text{sob}} \leq 1}} \text{median}_{1 \leq \ell \leq r} \sum_{\mathbf{k} \in P_{m,n,s,p,q_\ell}^\perp \setminus \{\mathbf{0}\}} \frac{|\tilde{f}(\mathbf{k})|}{\tilde{r}_{\alpha,\gamma}(\mathbf{k})} \tilde{r}_{\alpha,\gamma}(\mathbf{k}) \\
482 \quad &\leq \sup_{\substack{f \in \mathcal{F}_{s,\alpha,\gamma,q}^{\text{sob}} \\ \|f\|_{s,\alpha,\gamma,q}^{\text{sob}} \leq 1}} \text{median}_{1 \leq \ell \leq r} \left(\sup_{\mathbf{k} \in P_{m,n,s,p,q_\ell}^\perp \setminus \{\mathbf{0}\}} \frac{|\tilde{f}(\mathbf{k})|}{\tilde{r}_{\alpha,\gamma}(\mathbf{k})} \right) \left(\sum_{\mathbf{k} \in P_{m,n,s,p,q_\ell}^\perp \setminus \{\mathbf{0}\}} \tilde{r}_{\alpha,\gamma}(\mathbf{k}) \right) \\
483 \quad &\leq \sup_{\substack{f \in \mathcal{F}_{s,\alpha,\gamma,q}^{\text{sob}} \\ \|f\|_{s,\alpha,\gamma,q}^{\text{sob}} \leq 1}} \left(\sup_{\mathbf{k} \in \mathbb{N}_0^s \setminus \{\mathbf{0}\}} \frac{|\tilde{f}(\mathbf{k})|}{\tilde{r}_{\alpha,\gamma}(\mathbf{k})} \right) \text{median}_{1 \leq \ell \leq r} \tilde{\mathcal{S}}_{\alpha,\gamma,p}(q_\ell) \\
484 \quad (3.2) \quad &\leq \text{median}_{1 \leq \ell \leq r} \tilde{\mathcal{S}}_{\alpha,\gamma,p}(q_\ell), \\
485 \quad &
\end{aligned}$$

486 where

$$487 \quad \tilde{\mathcal{S}}_{\alpha,\gamma,p}(\mathbf{q}) = \sum_{\mathbf{k} \in P_{m,n,s,p,\mathbf{q}}^\perp \setminus \{\mathbf{0}\}} \tilde{r}_{\alpha,\gamma}(\mathbf{k}).$$

488 For $1/(\alpha) < \lambda \leq 1$, by using the subadditivity (2.3), we have

$$\begin{aligned}
489 \quad &\frac{1}{|G_n|^s} \sum_{\mathbf{q} \in G_n^s} (\tilde{\mathcal{S}}_{\alpha,\gamma,p}(\mathbf{q}))^\lambda = \frac{1}{(b^n - 1)^s} \sum_{\mathbf{q} \in G_n^s} \left(\sum_{\mathbf{k} \in P_{m,n,s,p,\mathbf{q}}^\perp \setminus \{\mathbf{0}\}} \tilde{r}_{\alpha,\gamma}(\mathbf{k}) \right)^\lambda \\
490 \quad &\leq \frac{1}{(b^n - 1)^s} \sum_{\mathbf{q} \in G_n^s} \sum_{\mathbf{k} \in P_{m,n,s,p,\mathbf{q}}^\perp \setminus \{\mathbf{0}\}} (\tilde{r}_{\alpha,\gamma}(\mathbf{k}))^\lambda \\
491 \quad &= \sum_{\mathbf{k} \in \mathbb{N}_0^s \setminus \{\mathbf{0}\}} (\tilde{r}_{\alpha,\gamma}(\mathbf{k}))^\lambda \frac{1}{(b^n - 1)^s} \sum_{\substack{\mathbf{q} \in G_n^s \\ \text{tr}_n(\mathbf{k}) \cdot \mathbf{q} \equiv a \pmod{p} \\ \deg(a) < n-m}} 1. \\
492 \quad &
\end{aligned}$$

493 If $p \mid \text{tr}_n(\mathbf{k})$, the condition $\text{tr}_n(\mathbf{k}) \cdot \mathbf{q} \equiv a \pmod{p}$ trivially holds with $a = 0$ for all
494 $\mathbf{q} \in G_n^s$. Otherwise if $p \nmid \text{tr}_n(\mathbf{k})$, i.e., if there exists a non-empty subset $u \subseteq \{1, \dots, s\}$
495 such that $p \nmid \text{tr}_n(k_j)$ for all $j \in u$ and $p \mid \text{tr}_n(k_j)$ for $j \notin u$, the condition $\text{tr}_n(\mathbf{k}) \cdot \mathbf{q} \equiv a$
496 \pmod{p} is equivalent to $\text{tr}_n(\mathbf{k}_u) \cdot \mathbf{q}_u \equiv a \pmod{p}$, which itself is equivalent to

$$497 \quad \text{tr}_n(k_j)q_j \equiv a - \text{tr}_n(\mathbf{k}_{u \setminus \{j\}}) \cdot \mathbf{q}_{u \setminus \{j\}} \pmod{p},$$

498 for any $j \in u$. As we have $p \nmid \text{tr}_n(k_j)$ and we assume that p is irreducible, there
499 exists at most one $q_j \in G_n$ which satisfies the above equality for each $a \in \mathbb{F}_b[x]$ with
500 $\deg(a) < n - m$ and $\mathbf{q}_{u \setminus \{j\}} \in G_n^{|u|-1}$. Therefore, the number of $\mathbf{q} \in G_n^s$ which satisfy
501 $p \nmid \text{tr}_n(\mathbf{k})$ and $\text{tr}_n(\mathbf{k}) \cdot \mathbf{q} \equiv a \pmod{p}$ with $\deg(a) < n - m$ is bounded above by

502 the product of the number of possible choices for $a \in \mathbb{F}_b[x]$, which is b^{n-m} , and the
 503 number of possible choices for $\mathbf{q}_{\{1,\dots,s\} \setminus \{j\}}$, which is $(b^n - 1)^{s-1}$. Thus it follows that

$$\begin{aligned}
 504 \quad & \frac{1}{|G_n|^s} \sum_{\mathbf{q} \in G_n^s} (\tilde{\mathcal{S}}_{\alpha,\gamma,p}(\mathbf{q}))^\lambda \\
 505 \quad & \leq \sum_{\substack{\mathbf{k} \in \mathbb{N}_0^s \setminus \{\mathbf{0}\} \\ p|\text{tr}_n(\mathbf{k})}} (\tilde{r}_{\alpha,\gamma}(\mathbf{k}))^\lambda + \sum_{\substack{\mathbf{k} \in \mathbb{N}_0^s \setminus \{\mathbf{0}\} \\ p \nmid \text{tr}_n(\mathbf{k})}} (\tilde{r}_{\alpha,\gamma}(\mathbf{k}))^\lambda \frac{b^{n-m}(b^n - 1)^{s-1}}{(b^n - 1)^s} \\
 506 \quad & \leq \sum_{\mathbf{k} \in \mathbb{N}_0^s \setminus \{\mathbf{0}\}} (\tilde{r}_{\alpha,\gamma}(b^n \mathbf{k}))^\lambda + \frac{1}{b^m - 1} \sum_{\mathbf{k} \in \mathbb{N}_0^s \setminus \{\mathbf{0}\}} (\tilde{r}_{\alpha,\gamma}(\mathbf{k}))^\lambda \\
 507 \quad & = \sum_{\emptyset \neq u \subseteq \{1,\dots,s\}} \sum_{\mathbf{k}_u \in \mathbb{N}^{|u|}} (\tilde{r}_{\alpha,\gamma}(b^n \mathbf{k}_u, \mathbf{0}))^\lambda + \frac{1}{b^m - 1} \sum_{\emptyset \neq u \subseteq \{1,\dots,s\}} \sum_{\mathbf{k}_u \in \mathbb{N}^{|u|}} (\tilde{r}_{\alpha,\gamma}(\mathbf{k}_u, \mathbf{0}))^\lambda \\
 508 \quad & = \sum_{\emptyset \neq u \subseteq \{1,\dots,s\}} \gamma_u^\lambda C_\alpha^{\lambda|u|} \left(\sum_{k \in \mathbb{N}} b^{-\lambda \mu_\alpha(b^n k)} \right)^{|u|} \\
 509 \quad & \quad + \frac{1}{b^m - 1} \sum_{\emptyset \neq u \subseteq \{1,\dots,s\}} \gamma_u^\lambda C_\alpha^{\lambda|u|} \left(\sum_{k \in \mathbb{N}} b^{-\lambda \mu_\alpha(k)} \right)^{|u|} \\
 510 \quad & \leq \sum_{\emptyset \neq u \subseteq \{1,\dots,s\}} \gamma_u^\lambda \frac{C_\alpha^{\lambda|u|} A_{\alpha,\lambda}^{|u|}}{b^{\lambda n|u|}} + \frac{1}{b^m - 1} \sum_{\emptyset \neq u \subseteq \{1,\dots,s\}} \gamma_u^\lambda C_\alpha^{\lambda|u|} A_{\alpha,\lambda}^{|u|} \\
 511 \quad & \leq \frac{2}{b^{\min(m,\lambda n)} - 1} \sum_{\emptyset \neq u \subseteq \{1,\dots,s\}} \gamma_u^\lambda C_\alpha^{\lambda|u|} A_{\alpha,\lambda}^{|u|}, \\
 512 \quad &
 \end{aligned}$$

513 where we have used the results of [14, Lemma 7] on the sums of $b^{-\lambda \mu_\alpha(k)}$ and $b^{-\lambda \mu_\alpha(b^n k)}$
 514 in the third inequality, which involve $A_{\alpha,\lambda}$ given in (3.1). This gives a bound on the
 515 average of $(\tilde{\mathcal{S}}_{\alpha,\gamma,p}(\mathbf{q}))^\lambda$ which holds for any $1/\alpha < \lambda \leq 1$.

516 Then, Markov's inequality ensures that, for any $0 < \eta < 1$, the event

$$517 \quad \tilde{\mathcal{S}}_{\alpha,\gamma,p}(\mathbf{q}) > \inf_{1/\alpha < \lambda < 1} \left(\frac{2}{\eta(b^{\min(m,\lambda n)} - 1)} \sum_{\emptyset \neq u \subseteq \{1,\dots,s\}} \gamma_u^\lambda C_\alpha^{\lambda|u|} A_{\alpha,\lambda}^{|u|} \right)^{1/\lambda} =: \tilde{B}(\alpha, \gamma)$$

518 happens with a probability of at most η under a random choice of $\mathbf{q} \in G_n^s$. For
 519 the median estimator $M_{m,n,s,p,r}$ to be larger than this bound $\tilde{B}(\alpha, \gamma)$, we must have
 520 $\tilde{\mathcal{S}}_{\alpha,\gamma}(\mathbf{q}_\ell) > \tilde{B}(\alpha, \gamma)$ for at least $(r+1)/2$ vectors among $\mathbf{q}_1, \dots, \mathbf{q}_r$. The probability
 521 that this happens is bounded above by

$$522 \quad \binom{r}{(r+1)/2} \eta^{(r+1)/2}.$$

523 Combining this with the bound shown in (3.2) completes the proof. \square

524 As pointed out in [6, Section 3.1], $\alpha \geq 2$ is required to ensure the convergence of
 525 the infinite sum

$$526 \quad \sum_{\mathbf{k} \in P_{m,n,s,p,q}^\perp \setminus \{\mathbf{0}\}} b^{-\mu_\alpha(\mathbf{k})},$$

527 for any irreducible p and $\mathbf{q} \in G_n^s$. Thus, the case $\alpha = 1$ is not covered by our result.

528 Using Remark 2.7, we obtain the following corollary:

529 COROLLARY 3.8. Let $\alpha \geq 2$, γ be a set of weights, and $n \geq \alpha m = \alpha \log_b N$. Then
 530 for any odd $r \geq 3$, $\epsilon > 0$ and $0 < \rho < 1$, there is a constant $c_1 = c_1(\alpha, \gamma, \epsilon) > 0$ (which
 531 depends on α , the γ_u 's and ϵ) such that

$$532 \quad \mathbb{P} \left[e^{\text{wor}}(M_{m,n,s,p,r}; \mathcal{F}_{s,\alpha,\gamma}^{\text{sob}}) \leq \frac{c_1(\alpha, \gamma, \epsilon)}{(\rho N)^{\alpha-\epsilon}} \right] \geq 1 - \rho^{(r+1)/2}/4.$$

533 *Proof.* Take $1/\lambda = \alpha - \epsilon$. Under the assumption on n , we have $n \geq \alpha m > m/\lambda$
 534 and then $(b^{\min(m,\lambda n)})^{-1/\lambda} = b^{-\min(m/\lambda, n)} = N^{-1/\lambda} = N^{-\alpha+\epsilon}$. Then the result follows
 535 from Theorem 3.7 and the bound (2.4) in Remark 2.7. \square

536 Thus, provided that we take n large enough, we get a convergence rate of almost
 537 $\mathcal{O}(N^{-\alpha})$ (with high probability) for any $\alpha \geq 2$. In other words, our median high-
 538 order polynomial lattice rule exploits the smoothness of functions adaptively. Note
 539 that Remark 2.11 also applies here.

540 **4. Numerical experiments.** We conclude this paper with numerical experi-
 541 ments both for rank-1 lattice rules and high-order polynomial lattice rules. The goal
 542 is to illustrate how the worst-case error for the median rule truly behaves on some
 543 concrete examples. In particular, we want to illustrate the fact that most of the
 544 possible generating vectors are a good choice, while a small minority are bad.

545 4.1. Lattice rules for periodic functions.

546 EXAMPLE 4.1. For our first example, we consider a weighted Korobov space with
 547 integer smoothness parameter $\alpha \geq 1$ and product weights $\gamma_u = \prod_{j \in u} \gamma_j$. The worst-
 548 case error of the rank-1 lattice rule with generating vector \mathbf{z} for that space has the
 549 explicit form

$$550 \quad \mathcal{S}_{\alpha,\gamma}(\mathbf{z}) = \left(-1 + \frac{1}{N} \sum_{\mathbf{x} \in P_{N,s,\mathbf{z}}} \prod_{j=1}^s \left[1 + \gamma_j^2 \frac{(-1)^{\alpha+1} (2\pi)^{2\alpha}}{(2\alpha)!} B_{2\alpha}(x_j) \right] \right)^{1/2},$$

551 where $B_{2\alpha}$ denotes the Bernoulli polynomial of degree 2α ; see [28] and [7, Section 5].
 552 In this artificial simple case, we know the exact optimal weights that must be taken in
 553 a CBC search for \mathbf{z} , so we can compare the median estimator with the best possible
 554 case of a CBC search.

555 We take two primes $N = 251$ and $N = 2039$, both for $s = 50$ dimensions, with
 556 $\alpha = 2$ and $\gamma_j = 1/j^3$. For each of those N , we drew 10^5 generating vectors \mathbf{z}
 557 randomly and uniformly from $\{1, \dots, N-1\}^s$, and computed $\mathcal{S}_{\alpha,\gamma}(\mathbf{z})$ for each. The
 558 left panels of Figure 2 show a histogram of the 10^5 realizations of $\log_2 \mathcal{S}_{\alpha,\gamma}(\mathbf{z})$ for each
 559 of these two cases. Each histogram provides a good estimate of the true distribution
 560 of $\log_2 \mathcal{S}_{\alpha,\gamma}(\mathbf{z})$, which is a discrete distribution because \mathbf{z} is drawn from a finite set.
 561 Interestingly, the distributions are very asymmetric and are far from smooth on the
 562 right side: some rectangles are very high while others are zero in the same area. The
 563 largest observed values are -2.4353 for $N = 251$ and -2.4967 for $N = 2039$. We can
 564 estimate from this data the q -quantiles y_q of the distribution of $\mathcal{S}_{\alpha,\gamma}(\mathbf{z})$, similar to
 565 those of the distribution of $Q_{P_{N,s,\mathbf{z}}}(f)$ in Remark 2.11. For $q = 0.75$, the corresponding
 566 empirical q -quantiles are -8.3907 for $N = 251$ and -12.0306 for $N = 2039$, while,
 567 for $q = 0.9$, they are -7.0975 for $N = 251$ and -10.3101 for $N = 2039$. These
 568 quantiles are much less than the worst observed values. These empirical results agree
 569 with the fact that only a very small proportion of the vectors \mathbf{z} are bad. Suppose we
 570 draw r random realizations of \mathbf{z} and want the median of the r corresponding values

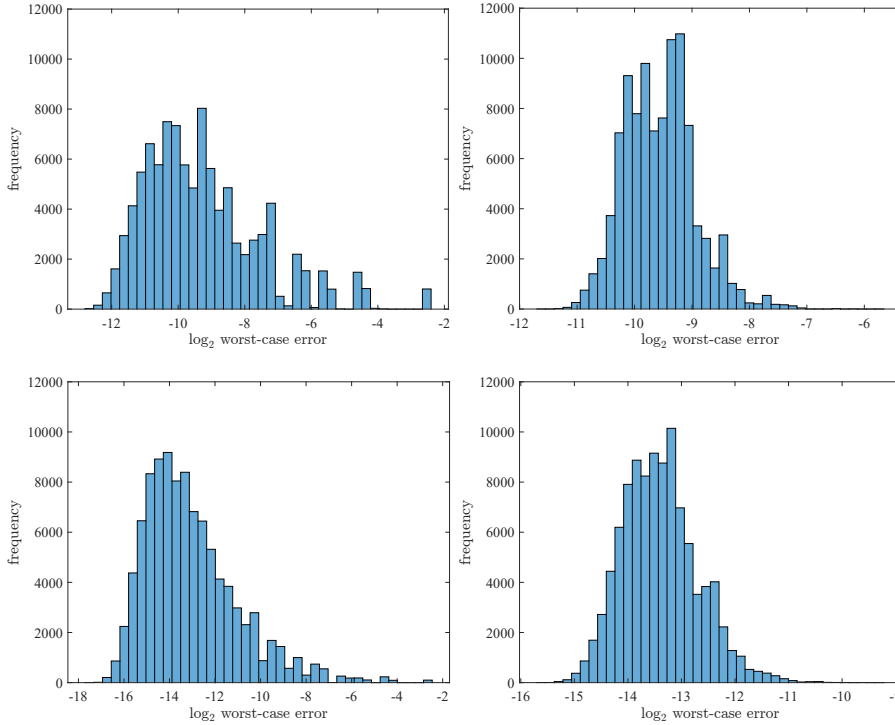


FIG. 2. Histograms of the \log_2 of the worst-case error $\mathcal{S}_{\alpha,\gamma}(\mathbf{z})$ with $\alpha = 2$ and $\gamma_j = 1/j^3$ for rank-1 lattice rules with randomly chosen generating vectors with $N = 251$ (upper panels) and $N = 2039$ (lower panels). The left panels are for a single choice ($r = 1$), while for the right panels we take the median of the worst-case error for rank-1 lattice rules with $r = 11$ randomly chosen generating vectors.

571 of $\mathcal{S}_{\alpha,\gamma}(\mathbf{z})$ to be larger than $y = 10^{-3} \approx 2^{-10}$ with a probability smaller than 10^{-4} .
 572 For $N = 2039$ this y equals y_q for $q \approx 0.9$, and Figure 1 shows that we can achieve
 573 approximately the target probability of 10^{-4} with $r = 13$. For a larger N , the required
 574 r is even smaller. Note that for $N = 251$, $y_p = 2^{-10}$ corresponds to some $q < 0.5$, for
 575 which the target probability of 10^{-4} cannot be achieved even for a very large r , as
 576 shown in Figure 1.

577 For the remainder of our experiments reported in this paper, we took $r = 11$.
 578 The right panels of Figure 2 show histograms of 10^5 independent realizations of
 579 $\log_2[\text{median}(\mathcal{S}_{\alpha,\gamma}(\mathbf{z}_1), \dots, \mathcal{S}_{\alpha,\gamma}(\mathbf{z}_r))]$ for randomly chosen $\mathbf{z}_1, \dots, \mathbf{z}_r$ with $r = 11$,
 580 corresponding to the cases $N = 251$ and $N = 2039$. We see that the distributions
 581 have much less variance and are more symmetric than for a single random \mathbf{z} , confir-
 582 ming the fact that taking the median successfully filters (adaptively) the bad vector
 583 generators. Recall that the standard deviation of the empirical median as a function
 584 of r generally decreases as $\mathcal{O}(r^{-1/2})$. That is, increasing r decreases the noise rather
 585 slowly. For the following examples, we made additional experiments with $r = 31$
 586 to see if it would make the error plots less noisy, and we did not see much visible
 587 difference.

588 EXAMPLE 4.2. For our second example, we perform a numerical integration of

589 the smooth, periodic function

$$590 \quad f_{\beta, \omega}^{\text{per}}(\mathbf{x}) = \prod_{j=1}^s [1 + \omega_j (g_{\beta}(x_j) - 1)],$$

591 with parameters β and ω_j , where the univariate function $g_{\beta} : [0, 1] \rightarrow \mathbb{R}$ is defined by

$$592 \quad g_{\beta}(x) = (2\beta + 1) \binom{2\beta}{\beta} x^{\beta} (1 - x)^{\beta}.$$

593 Note that $I_s(f_{\beta, \omega}^{\text{per}}) = 1$. The function g_{β} has been used for periodization of non-
 594 periodic functions, and our test function $f_{\beta, \omega}^{\text{per}}$ belongs to the Korobov space with
 595 $\alpha = \beta$ when β is a positive integer, see [7, Section 5.10]. In what follows, we take
 596 $s = 50$ and consider the four cases that correspond to $\beta = 2$ or $\beta = 5$, and $\omega_j = 1/j^{\beta+1}$
 597 or $\omega_j = 1/(s - j + 1)^{\beta+1}$. We compare our median lattice rule with $r = 11$, a
 598 QMC rule using non-randomized Sobol' points provided by MATLAB, and the rank-
 599 1 lattice rule with generating vector constructed by the fast CBC algorithm with
 600 $\mathcal{S}_{\alpha, \gamma}(\mathbf{z})$ as a criterion, with $\alpha = 2$ and the product weights $\gamma_j = 1/j^3$. These weights
 601 are not optimal, but they are a good heuristic choice when $\omega_j = 1/j^{\beta+1}$. When
 602 $\omega_j = 1/(s - j + 1)^{\beta+1}$, on the other hand, the weights decrease in the opposite
 603 direction as they should: they are very large for the unimportant coordinates and
 604 small for the important ones. We do this to show how badly the CBC construction
 605 method can work when we have the wrong weights, whereas the median estimator
 606 does not need any knowledge about the weights to perform well. We choose N to be
 607 a power of 2 for Sobol' points and to be a prime close to a power of 2 for lattice point
 608 sets.

609 The results for the four cases are shown in the corresponding panels of Figure 3.
 610 Both our median lattice rule and the rank-1 lattice rule constructed by the CBC
 611 algorithm can exploit the periodicity of the integrand and achieve a higher-order
 612 rate of convergence than $\mathcal{O}(1/N)$. With a good choice of the weights in the CBC
 613 algorithm, the resulting rank-1 lattice rule performs better than our median lattice
 614 rule, as shown in the left panels. However, as the right panels clearly depict, if
 615 the relative importance of each of individual variables is not correctly specified, the
 616 performance of the rank-1 lattice rule with the CBC algorithm can deteriorate and
 617 even become inferior to the QMC rule using the Sobol' points when N is not large. In
 618 contrast, our median lattice rule performs quite stably regardless of smoothness and
 619 weights.

620 To show that our median lattice rule performs well for functions of non-product
 621 forms, let us consider the additional test functions given by

$$622 \quad f_{\beta}^{\text{per, cyc}}(\mathbf{x}) = \frac{1}{5} \sum_{\ell=1}^5 \prod_{j=1}^{s/5} g_{\beta}(x_{j+s(\ell-1)/5}) \quad \text{and} \quad f_{\beta}^{\text{per, mod}}(\mathbf{x}) = \frac{1}{5} \sum_{\ell=1}^5 \prod_{j=1}^{s/5} g_{\beta}(x_{\ell+5(j-1)}),$$

623 respectively, with $\beta = 5$ and $s = 20$. We have that $I_s(f_{\beta}^{\text{per, cyc}}) = I_s(f_{\beta}^{\text{per, mod}}) = 1$ and
 624 these two integrands belong to the Korobov space with $\alpha = \beta$. The results for the two
 625 integrands are shown in the corresponding panels of Figure 4. For large N , both our
 626 median lattice rule and the rank-1 lattice rule constructed by the CBC algorithm are
 627 superior to the QMC rule using the Sobol' points. Although the difference between
 628 $f_{\beta}^{\text{per, cyc}}$ and $f_{\beta}^{\text{per, mod}}$ lies only in the ordering of variables, the convergence behavior
 629 of the rank-1 lattice rule constructed by the CBC algorithm is not consistent for these

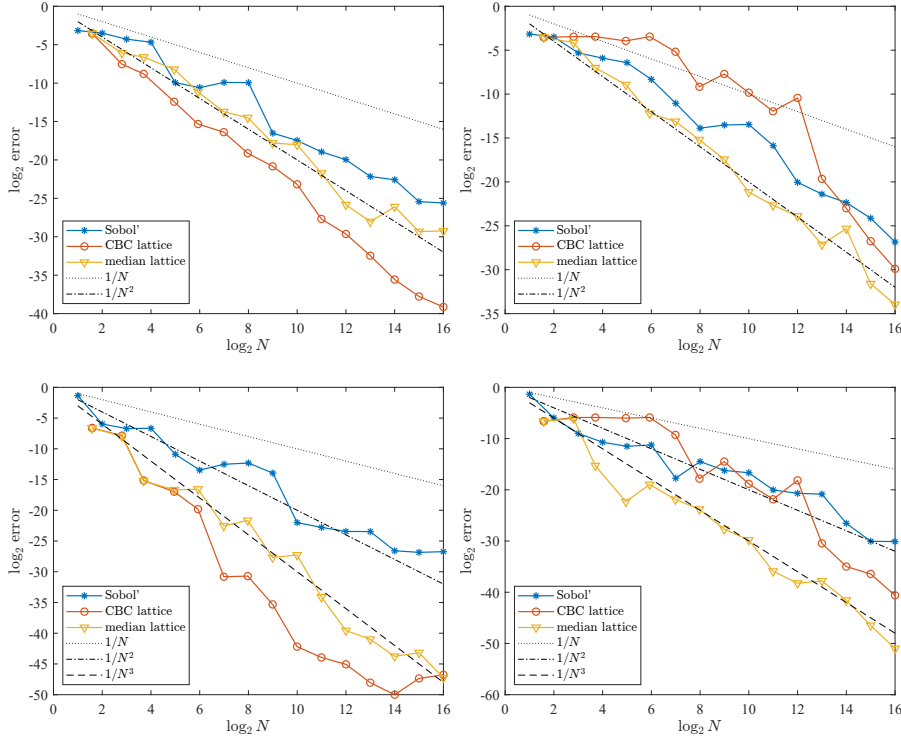


FIG. 3. Comparison of the integration error by our median lattice rule (yellow), QMC rule using Sobol' points (blue), and rank-1 lattice rule with the fast CBC algorithm (orange). The results are shown for the test function $f_{\beta,\omega}^{\text{per}}$ with the choices $\beta = 2$ and $\omega_j = 1/j^3$ (upper left), $\beta = 2$ and $\omega_j = 1/(s-j+1)^3$ (upper right), $\beta = 5$ and $\omega_j = 1/j^6$ (lower left), and $\beta = 5$ and $\omega_j = 1/(s-j+1)^6$ (lower right).

630 functions and a strange zig-zag pattern shows up for $f_{\beta}^{\text{per,mod}}$. On the contrary, our
 631 median lattice rule is not subject to the difference between the ordering of variables
 632 and performs almost equivalently.

633 4.2. High-order polynomial lattice rules for non-periodic functions.

634 Our next examples concern high-order polynomial lattice rules. Here we fix the precision to $n = 52$ and always use the primitive polynomial $p(x) = x^{52} + x^3 + 1$, found
 635 in [17], as the modulus of the polynomial lattice point sets.
 636

637 EXAMPLE 4.3. We first consider the two following one-dimensional test functions:

$$638 \quad f_1^{\text{nonper}}(x) = x^3(1/4 + \log x) \quad \text{and} \quad f_2^{\text{nonper}}(x) = xe^{x/4}.$$

639 We can see that the third derivative of f_1^{nonper} is in $L_q([0, 1])$ for any $1 \leq q < \infty$,
 640 whereas the fourth derivative is not in $L_1([0, 1])$, implying that $f_1^{\text{nonper}} \in \mathcal{F}_{1,3,\gamma,q}^{\text{sob}}$ but
 641 $f_1^{\text{nonper}} \notin \mathcal{F}_{1,4,\gamma,1}^{\text{sob}}$. Thus f_1^{nonper} has a finite smoothness. On the other hand, f_2^{nonper}
 642 is obviously infinitely differentiable, so that $f_2^{\text{nonper}} \in \mathcal{F}_{1,\alpha,\gamma,q}^{\text{sob}}$ for any $\alpha \geq 2$ and
 643 $1 \leq q \leq \infty$. Note that $I_1(f_1^{\text{nonper}}) = 0$ and $I_1(f_2^{\text{nonper}}) = 16 - 12e^{1/4}$. We compare
 644 our median high-order polynomial lattice rule with $r = 11$ and QMC rules using order
 645 2 and order 3 Sobol' points constructed by the interlacing procedure of [3], with the
 646 direction numbers provided in MATLAB (taken from [19]). To construct a Sobol'

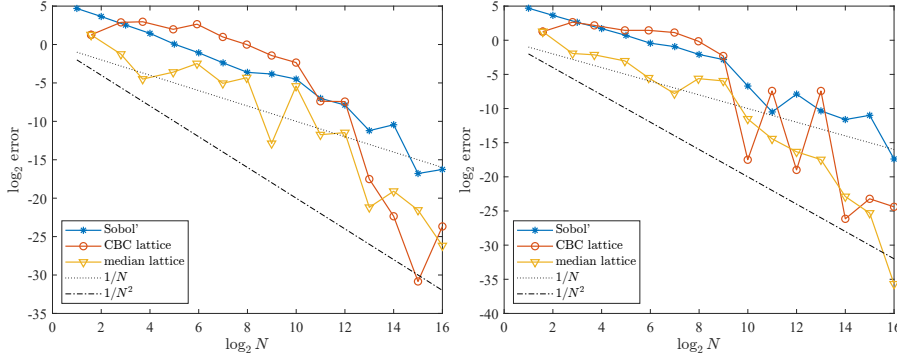


FIG. 4. Comparison of the integration error by our median lattice rule (yellow), QMC rule using Sobol' points (blue), and rank-1 lattice rule with the fast CBC algorithm (orange). The results are shown for the test functions $f_{\beta}^{\text{per,cyc}}$ (left) and $f_{\beta}^{\text{per,mod}}$ (right) with the choices $\beta = 5$ and $s = 20$.

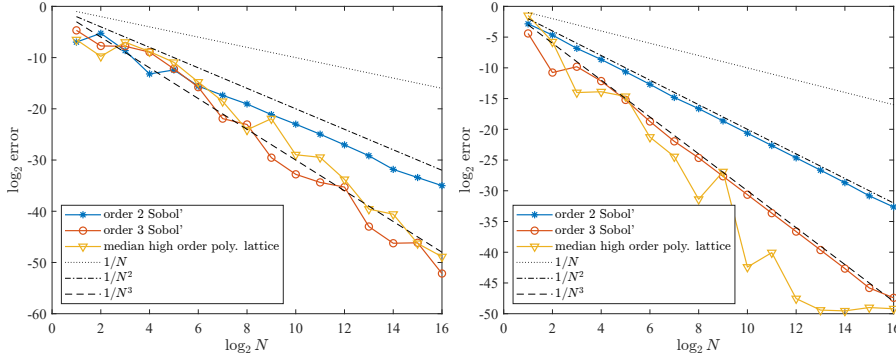


FIG. 5. Comparison of the one-dimensional integration error by our median high-order polynomial lattice rule (yellow), QMC rules using order 2 Sobol' points (blue) and order 3 Sobol' points (orange). The results are shown for the test functions f_1^{nonper} (left) and f_2^{nonper} (right), respectively.

647 point set of order d by interlacing, we first construct a ds -dimensional Sobol point set
 648 with 2^m points (with $s = 1$ in this case) and then apply the digit interlacing procedure
 649 defined in [3] to obtain the digits of the s -dimensional points. This procedure extracts
 650 the first m digits of the ds -dimensional points and reorders them in a special way to
 651 obtain the first dm digits of the s -dimensional points.

652 The results for the two one-dimensional functions are shown in Figure 5, respec-
 653 tively. As we can see from the result for f_1^{nonper} , the QMC rule using order 2 Sobol'
 654 points cannot fully exploit the smoothness of the function and the error decays at the
 655 rate of N^{-2} . On the other hand, the QMC rule using order 3 Sobol' points and our
 656 median high-order polynomial lattice rule can exploit the smoothness and achieves
 657 the convergence rate of N^{-3} . For the infinitely differentiable function f_2^{nonper} , the
 658 plot suggests that our median high-order polynomial lattice rule may converge even
 659 faster than N^{-3} . These numerical results show the major advantage of our proposed
 660 rule in terms of adaptivity in smoothness.

661 EXAMPLE 4.4. Finally, we consider the two multivariate non-periodic test func-

662 tions

$$663 \quad f_{3,\omega}^{\text{nonper}}(\mathbf{x}) = \exp\left(-\sum_{j=1}^s \omega_j x_j\right) \quad \text{and} \quad f_{3,\omega,\text{flip}}^{\text{nonper}}(\mathbf{x}) = \exp\left(-\sum_{j=1}^s \omega_j x_{s-j+1}\right),$$

664 with $s = 10$ and $\omega_j = 1/(4j^4)$. It is obvious that both $f_{3,\omega}^{\text{nonper}}$ and $f_{3,\omega,\text{flip}}^{\text{nonper}}$ are infinitely
 665 differentiable and belong to $\mathcal{F}_{s,\alpha,\gamma,q}^{\text{sob}}$ with arbitrary $\alpha \geq 2$ and $1 \leq q \leq \infty$. Note that
 666 $f_{3,\omega,\text{flip}}^{\text{nonper}}$ is defined by reordering the variables of $f_{3,\omega}^{\text{nonper}}$ so that x_j is replaced by
 667 x_{s-j+1} , and that we have

$$668 \quad I_s(f_{3,\omega}^{\text{nonper}}) = I_s(f_{3,\omega,\text{flip}}^{\text{nonper}}) = \prod_{j=1}^s \frac{1 - \exp(-\omega_j)}{\omega_j}.$$

669 The variables are ordered by decreasing order of importance in the first function, and
 670 by increasing order in the second one. As our median high-order polynomial lattice
 671 rule, based on random choices of generating vectors, does not care about the ordering
 672 of variables, it should perform the same for $f_{3,\omega}^{\text{nonper}}$ and $f_{3,\omega,\text{flip}}^{\text{nonper}}$.

673 The results are shown in Figure 6. Here again, we compare our median high-
 674 order polynomial lattice rule with $r = 11$ and QMC rules using order 2 and order 3
 675 interlaced Sobol' points. For the function $f_{3,\omega}^{\text{nonper}}$, our median high-order polynomial
 676 lattice rule can exploit the smoothness better than the QMC rule using order 2 Sobol'
 677 points. The QMC rule using order 3 Sobol' points exploits the smoothness of the
 678 integrand best and the error decays at the rate of N^{-3} and outperforms our median
 679 high-order polynomial lattice rule approximately by a constant factor for small N ,
 680 but this rate breaks down at around $\log_2 N = 13$ and our median rule catches up at
 681 $\log_2 N = 16$.

682 For $f_{3,\omega,\text{flip}}^{\text{nonper}}$, the situation changes. Our median high-order polynomial lattice
 683 rule is now comparable to the QMC rule using order 3 Sobol' points when N is small,
 684 and performs better for larger N . The error decays approximately at the rate of
 685 $N^{-2.5}$, which the QMC rule using order 2 Sobol' points cannot attain. The slowdown
 686 of the convergence for the QMC rule using order 3 Sobol' points might be due to
 687 the misspecification of important variables. Although x_s, x_{s-1}, \dots are the order of
 688 the relatively important variables for $f_{3,\omega,\text{flip}}^{\text{nonper}}$, we use the later coordinates of order 3
 689 Sobol' points, whose lower-dimensional projections are not well-distributed compared
 690 to the earlier coordinates. In this sense, the median high-order polynomial lattice rule
 691 is more robust and adaptive to the integrand at hand.

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695

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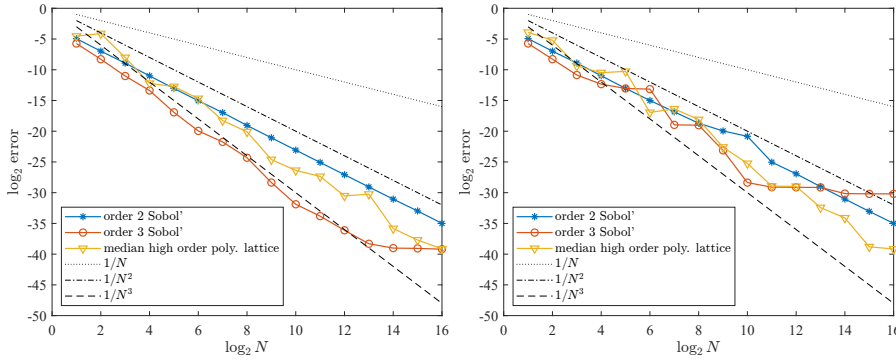


FIG. 6. Comparison of the multi-dimensional integration error by our median high-order polynomial lattice rule (yellow), QMC rules using order 2 Sobol' points (blue) and order 3 Sobol' points (orange). The results are shown for the test functions $f_{3,\omega}^{\text{nonper}}$ (left) and $f_{3,\omega,\text{flip}}^{\text{nonper}}$ (right) with the choice $\omega_j = 1/(4j^4)$.

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