Tools for Staffing and Scheduling in Call Centers:

Survey of work in our lab

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with Thanos Avramidis, Eric Buist, Tolga Cezik, Wyean Chan, Nabil Channouf

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- Building realistic models; Computer games vs tools for decision making;
- Optimization of staffing, scheduling, call routing, priorities, outbound call, etc.

Article are on my web page: type Pierre L'Ecuyer in Google.

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K call types. Depends on required technical skill, language, importance, etc.
I agent types (or skill groups). Each has skills to handle certain call types.
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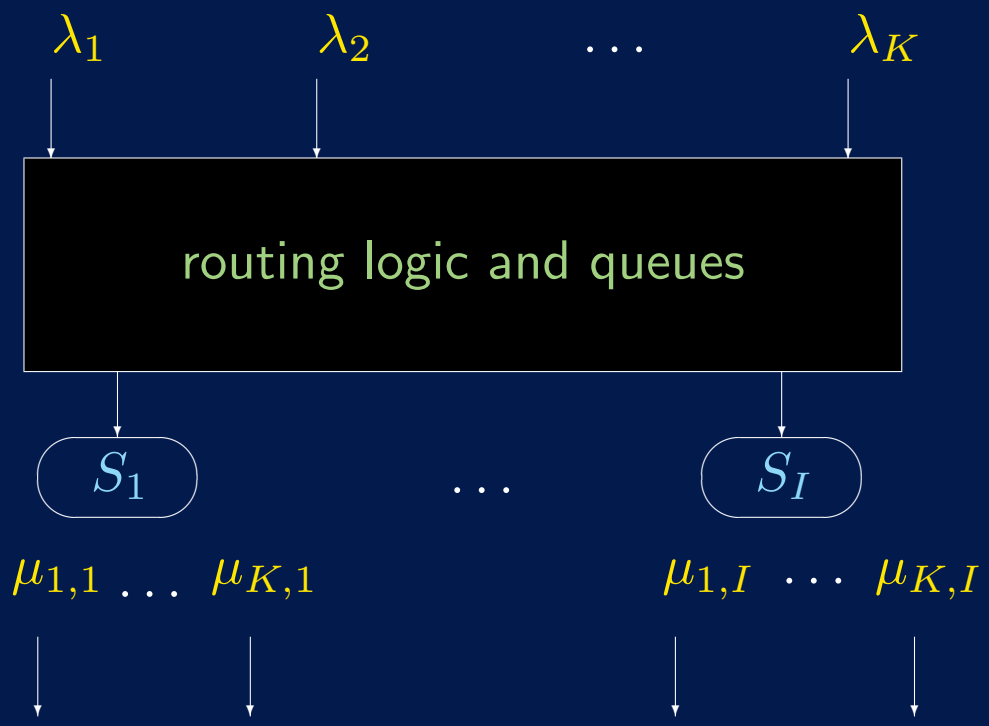

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For well-balanced systems, one or two skills per agent often gives a performance almost as good as all skills for all agents (e.g., Wallace and Whitt 2004).

Performance measures

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Constraints

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Average waiting time?

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Rules that control in real time the agent-to-call and call-to-agent assignments.
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Other: Could weight the calls, use state-dependent thresholds, etc.

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Scheduling and rostering problem:
In practice, we do not have an infinite supply of each agent type!
For a given set of agents and a given set of admissible shifts, assign a shift to each agent, to meet the performance constraints at minimal cost.

Blend systems: inbound and outbound calls

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Other ways?

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So we developed our own tool: ContactCenters, a Java library built over SSJ.

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Library based on well-supported modern programming language Java.
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Pre-compiled generic models.

Building realistic models

We need lots of detailed data about the call center operations:
arrivals, service times, breaks, hidden rules and practice, etc.

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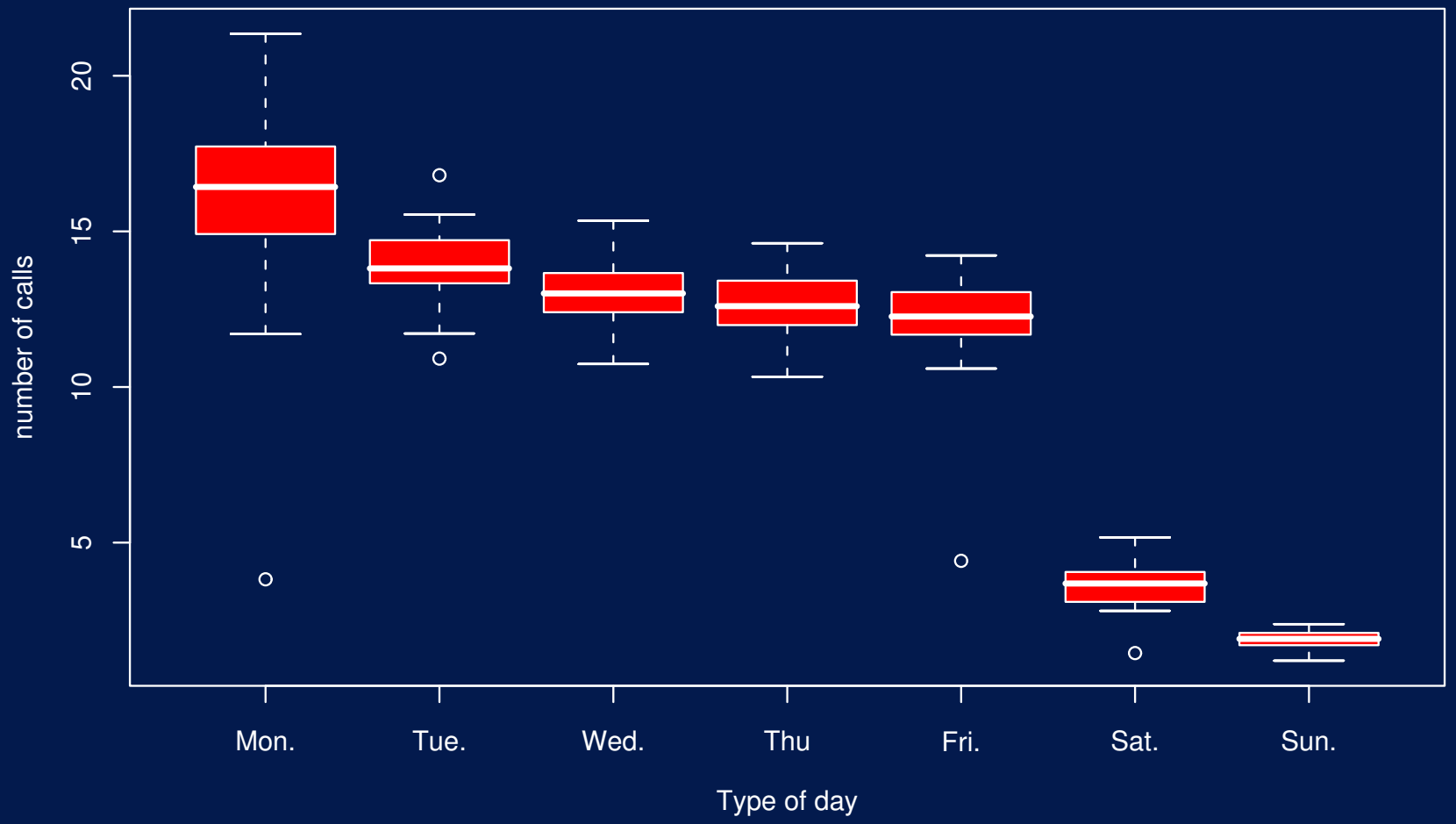
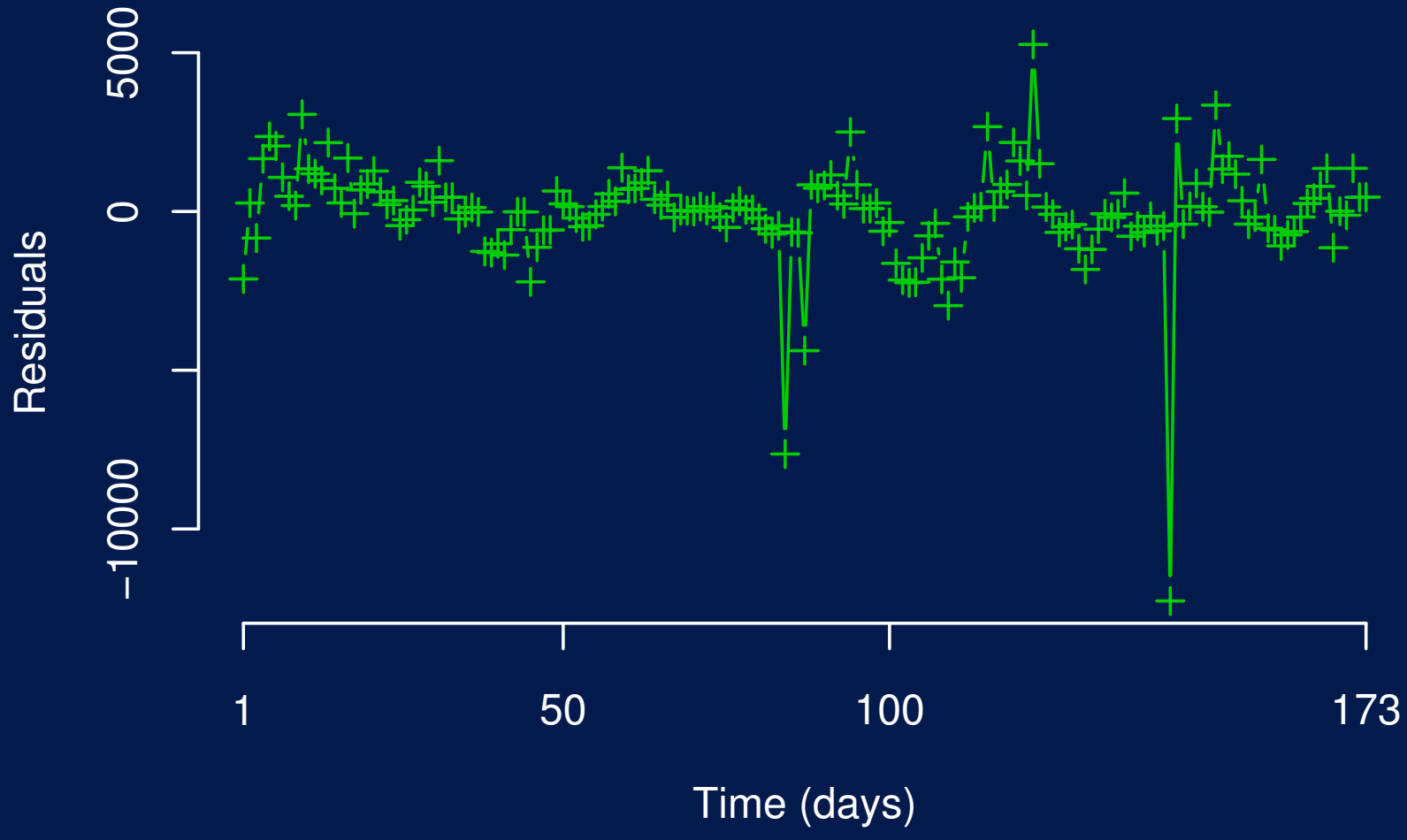
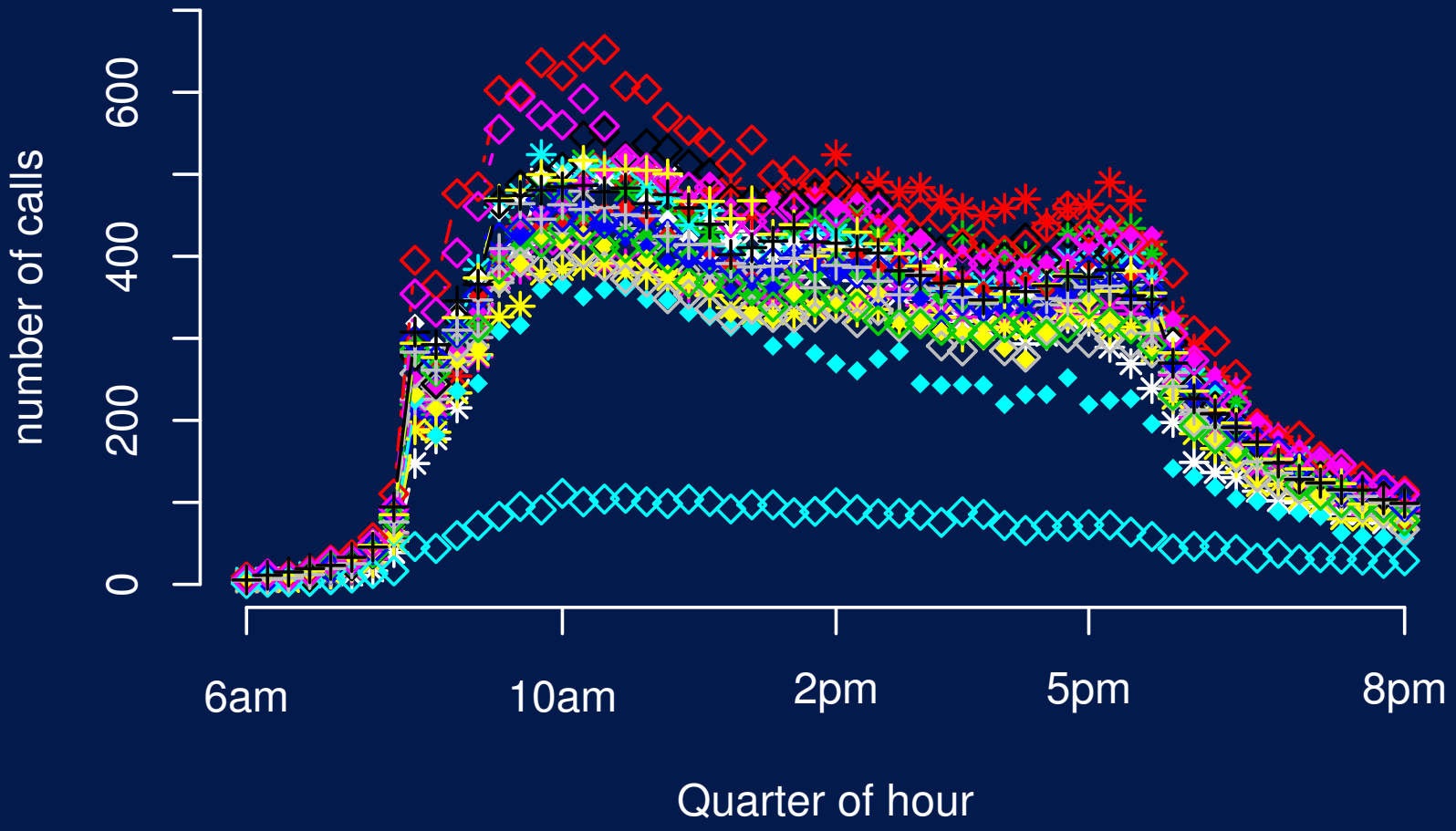
Examples: service times are not exponential; arrivals are not Poisson and stationary, perhaps breaks are not exactly as planned, perhaps agents are not available exactly as planned, etc.



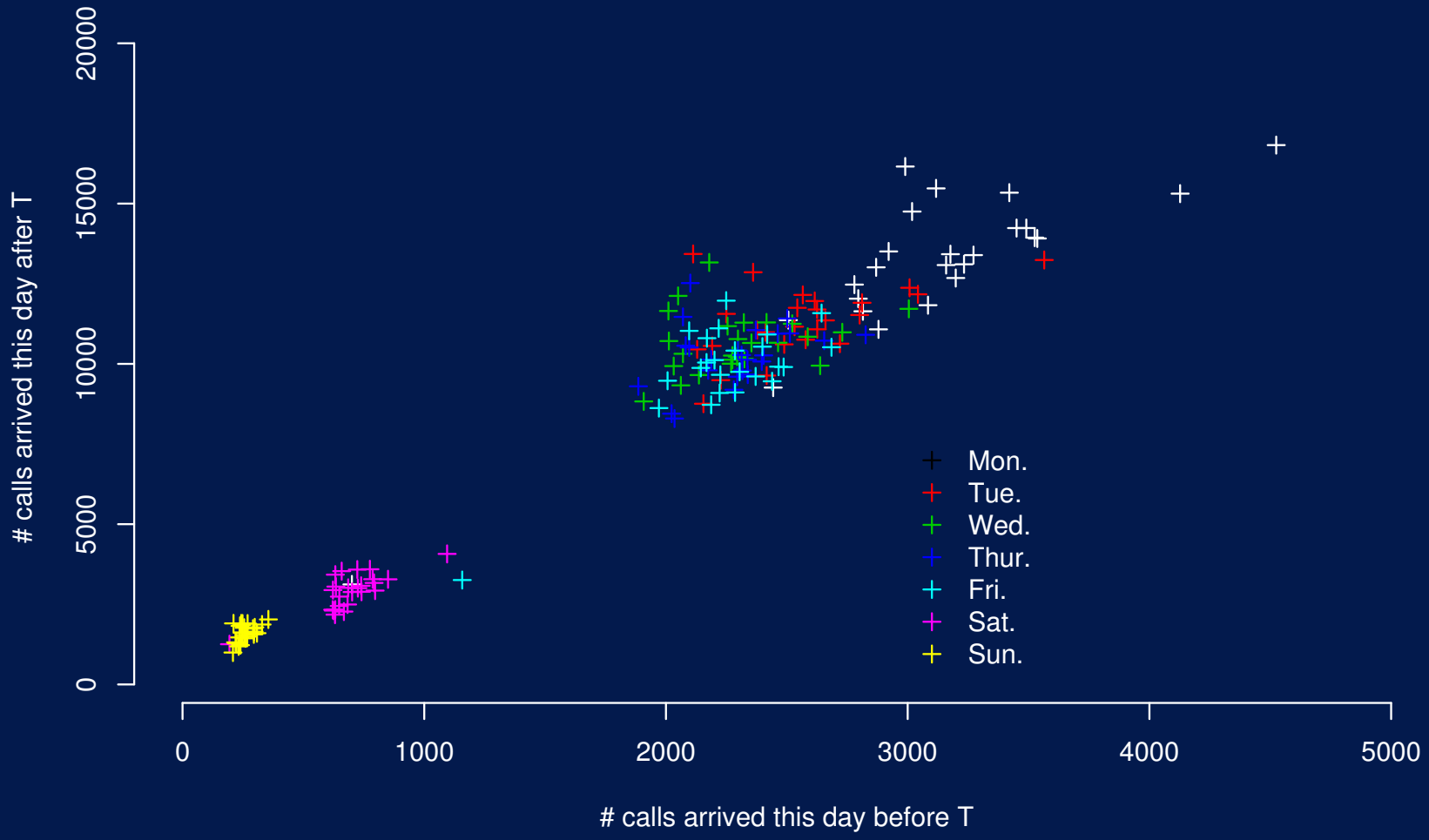
Arrival counts on different days



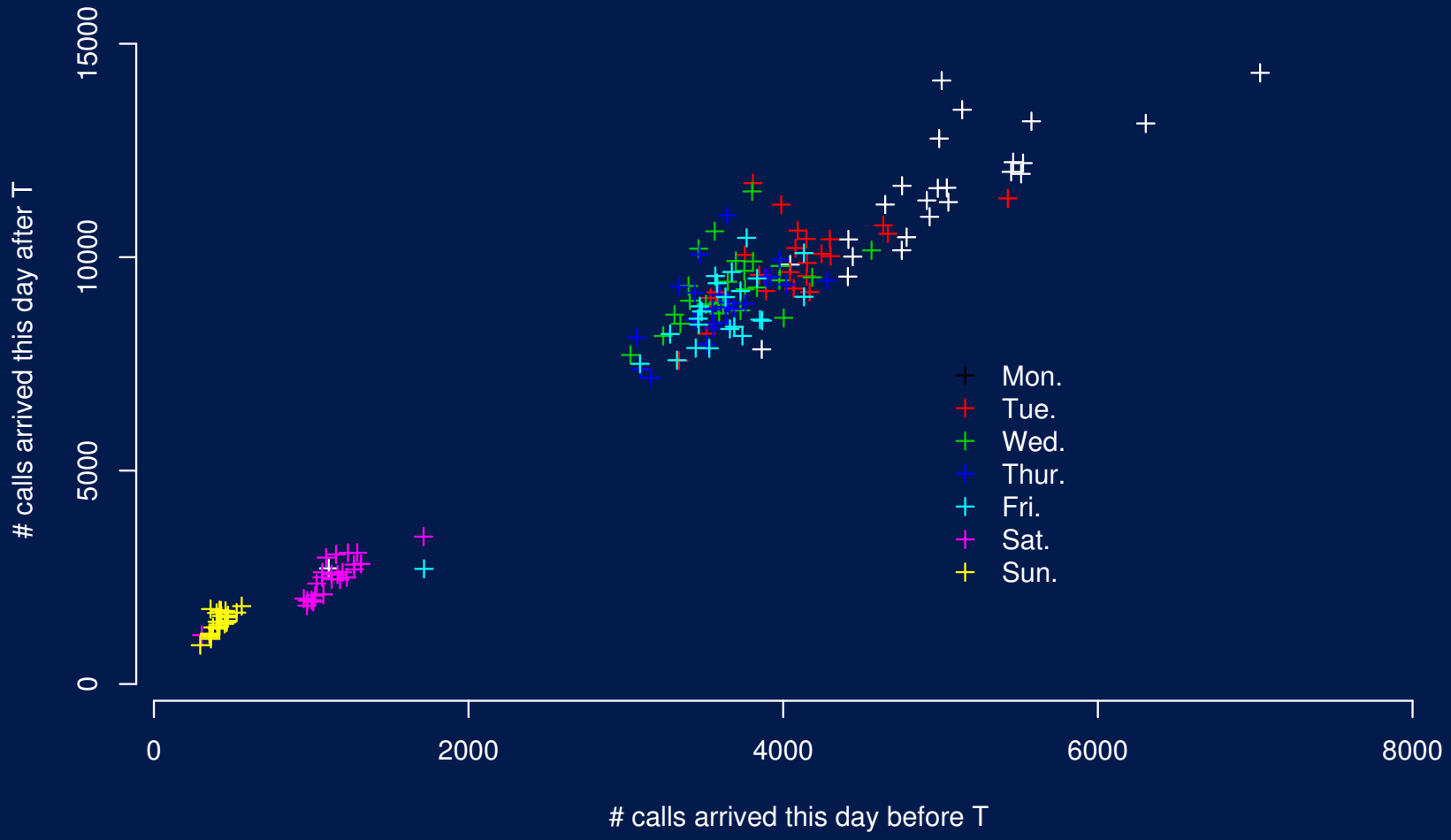
Daily residuals after removing day-effect, Box-Ljung test: p-value = 7.534e-06



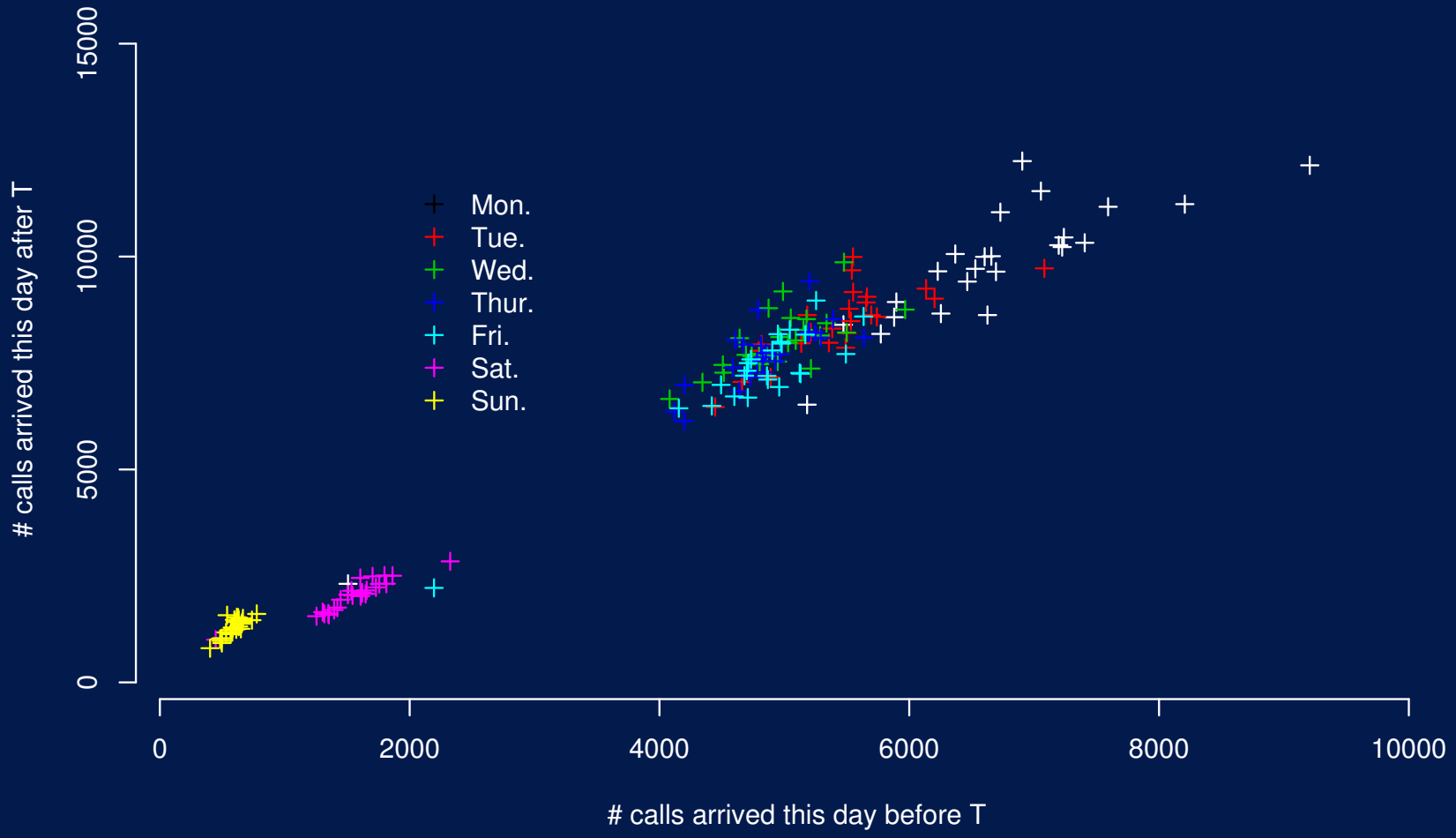
Monday, arrival counts by quarter hour



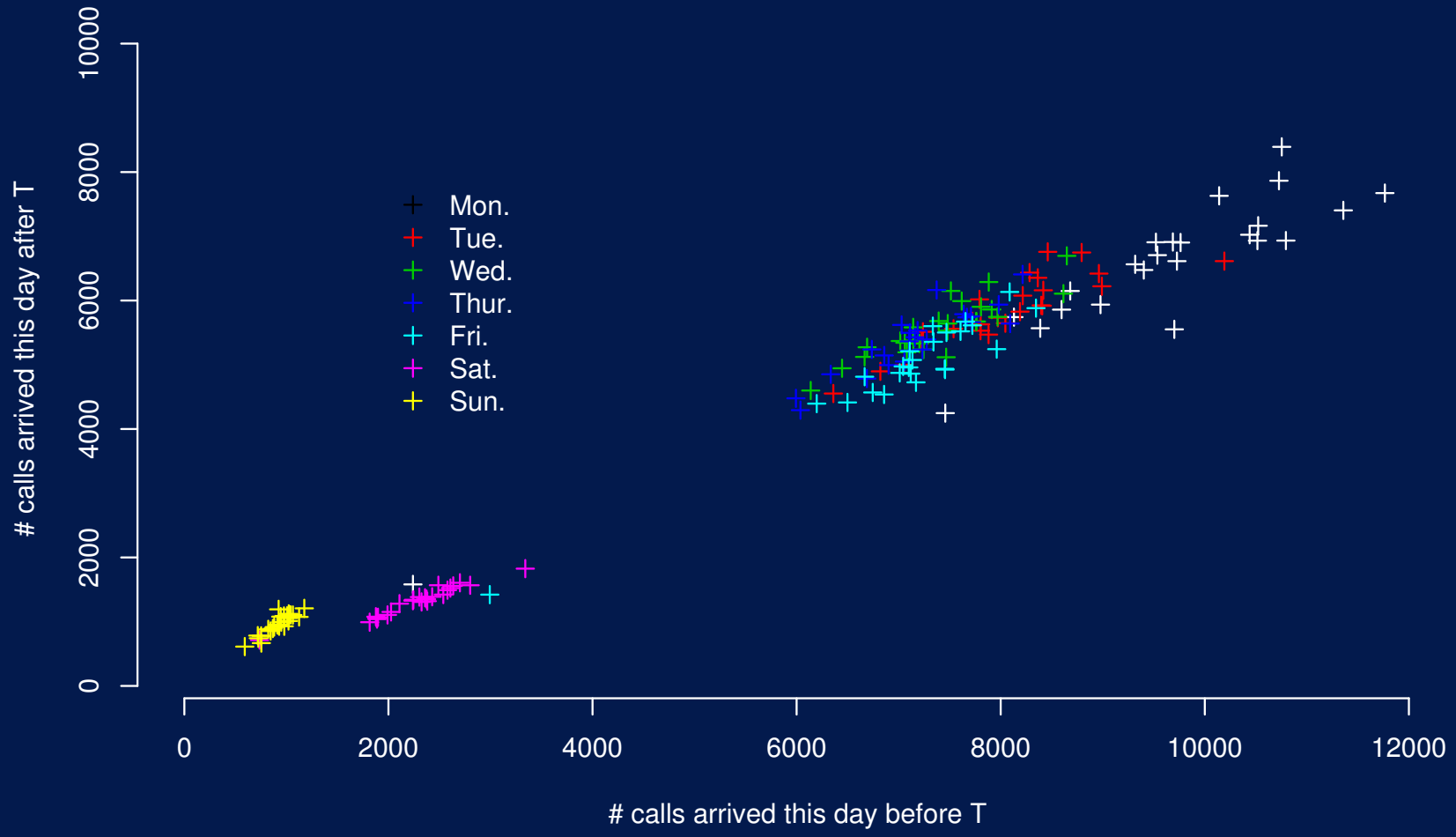
Scatter plot, all days T=10 am



Scatter plot, all days T=11 am



Scatter plot, all days T=12 am



Scatter plot, all days T=2 pm

Arrival process modeling

Arrival processes can be nonstationary, doubly stochastic, dependent, etc.
Examples, for single type: nonstationary Poisson with random inflation factor each day; arrival rate $B\lambda(t)$ at time t, where $\mathbb{E}[B] = 1$.
(Whitt, Avramidis et al. 2004).

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Service time distributions;
Patience time distributions;
Etc.

Steady-state approximations of SL

Single skill: M/M/s (Erlang-C Analysis).

$\mu =$ service rate; $\lambda =$ arrival rate; $\rho = \lambda/\mu =$ load

$s =$ staffing (number of servers)

Delay probability:

$$P(\text{wait} > 0) = \frac{\dfrac{\rho^s}{s!}\dfrac{s}{s-\rho}}{\dfrac{\rho^s}{s!}\dfrac{s}{s-\rho} + \displaystyle\sum_{i=0}^{s-1}\frac{\rho^i}{i!}}$$

Large system: square-root safety staffing.

Halfin-Whitt (1981) limit: sequence of M/M/s queues;
Queue n has $s_n = n$ servers, arrival rate λ_n, and load $\rho_n = \lambda_n/\mu$.
$W_n =$ steady-state waiting time.

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Theorem. For $n \to \infty$, if $\lambda_n \to \infty$ and $(1 - \rho_n/n)\sqrt{n} \to \beta$ for $0 < \beta < 1$, then

$$P(W_n > 0) \to \delta = \frac{1}{1 + \beta\Phi(\beta)/\phi(\beta)}$$

where Φ and ϕ are the standard normal c.d.f. and p.d.f.

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$$n = \rho + \beta\sqrt{\rho}$$

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agents. Simpler than inverting M/M/s formula.

Time-dependent arrival rate.

Pointwise Stationary (PS) approximation (Green and Kolesar 1991, Whitt 1991):

1. Use steady-state approximation at each time point t;
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Improvement: use arrival rate at t for performance at $t + \delta$, for some delay δ (e.g., Ingolfsson et al.).

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Blocking probabilities per skill group and loss rates per call type are computed by solving a system of nonlinear equations.

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The SL for type k is approximated using a birth-and-death CTMC model at that queue, for each k.

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Allows abandonments and general routing.

Not very realistic, but useful for rough-cut first-step in staffing optimization.

Optimization: Integer Programming Formulation

Call types	$k = 1, \ldots, K$;
Agent types (or skill groups)	$i = 1, \ldots, I$;
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We have $\mathbf{y} = \mathbf{A}\mathbf{x}$ where $\mathbf{A}$ is block diagonal with i blocks $\tilde{\mathbf{A}}$, and element (p, q) of $\tilde{\mathbf{A}}$ is 1 if shift q covers period p, 0 otherwise.

$g_{k,p}(\mathbf{y})$ = long-run SL for call type k in period p.
E.g., fraction of calls answered within 20 seconds, or $s_{k,p}$ seconds:

$$g_{k,p}(\mathbf{y}) = \frac{E[\text{num. of calls in period } p \text{ answered within the limit}]}{E[\text{num. of arrivals in period } p]}.$$

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These functions $g_\bullet$ (ratios of expectations) are unknown and often very complicated. Each may depend on entire vector $\mathbf{y}$. They can be either

- approximated via simplified queueing models, or
- estimated by simulation.

Here, the routing rules are assumed fixed; we do not optimize them.

Scheduling problem

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Neglects the transient and overflow effect across periods.

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Can be alleviated by skill-transfer decision variables.

Skill-transfer decision variables (Bhulai, Koole, and Pot 2005):
$z_{p,j,i}$ = number of agents of type j that work as type-i agents in period p (they use only part of their skills).

$$\min \quad \mathbf{c}^{\mathsf{t}}\mathbf{x} = \sum_{i=1}^{I}\sum_{q=1}^{Q} c_{i,q}x_{i,q}$$

subject to

$$\sum_{q:\tilde{A}_{p,q}=1} x_{i,q} + \sum_{j\in\mathcal{S}_i^+} z_{p,j,i} - \sum_{j\in\mathcal{S}_i^-} z_{p,i,j} \geq y_{i,p} \quad \forall p,\, i$$

all $x_{i,q}, z_{p,j,i} \geq 0$ and integer.

Sample-path optimization via simulation

[Atlason, Epelman, and Henderson, 2004; Cezik and L'Ecuyer 2005]
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For a ω, the empirical SL's over the n simulation runs are:

$G_{n,k,p}(\mathbf{y}, \omega)$ for call type k in period p;
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To compute them at different values of $\mathbf{y}$, we simply use simulation with common random numbers.

Empirical scheduling optimization problem (sample version of the problem):

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Similar formulation for the staffing problem.

Neighborhood search for staffing problem

[Avramidis et al. 2006]
Stage 1: Start with feasible solution and decrease cost using neighborhood search with "loss-delay" approximation to evaluate SL and select next solution.

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Stage 2: simulation-based local adjustment to stage-1 solution to account for two mutually exclusive cases:

- the incumbent is infeasible and we seek a nearby feasible solution;
- the incumbent is feasible and we seek further cost reduction.

Variance reduction: examples

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Stratification on U in k strata: for $s = 1, \ldots, k$, generate n_s indep. r.v.'s $U^{(s)} \sim \mathrm{Unif}((s-1)/k,\, s/k)$ and denote $X_{s,1}, \ldots, X_{s,n_s}$ the corresponding observations of $G(s_0)$.

Stratified estimator:

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where $\sigma_s^2 = \mathrm{Var}[X \mid S = s]$ (must be estimated), is minimized by taking (if we neglect rounding)

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We have

$$\mathrm{Var}[\bar{X}_n] \quad = \quad \mathrm{Var}[\bar{X}_{\mathrm{so},n}] + \frac{1}{nk}\sum_{s=1}^{k}(\sigma_s - \bar{\sigma})^2 + \frac{1}{nk}\sum_{s=1}^{k}(\mu_s - \mu)^2.$$

Combining with a control variate.

Let $A =$ number of arrivals during the day. CV estimator conditional on $U = u$:

$$X_c(u) = X - \beta(u)[A - \mathbb{E}[A \mid U = u]] = X - \beta(u)C(u).$$

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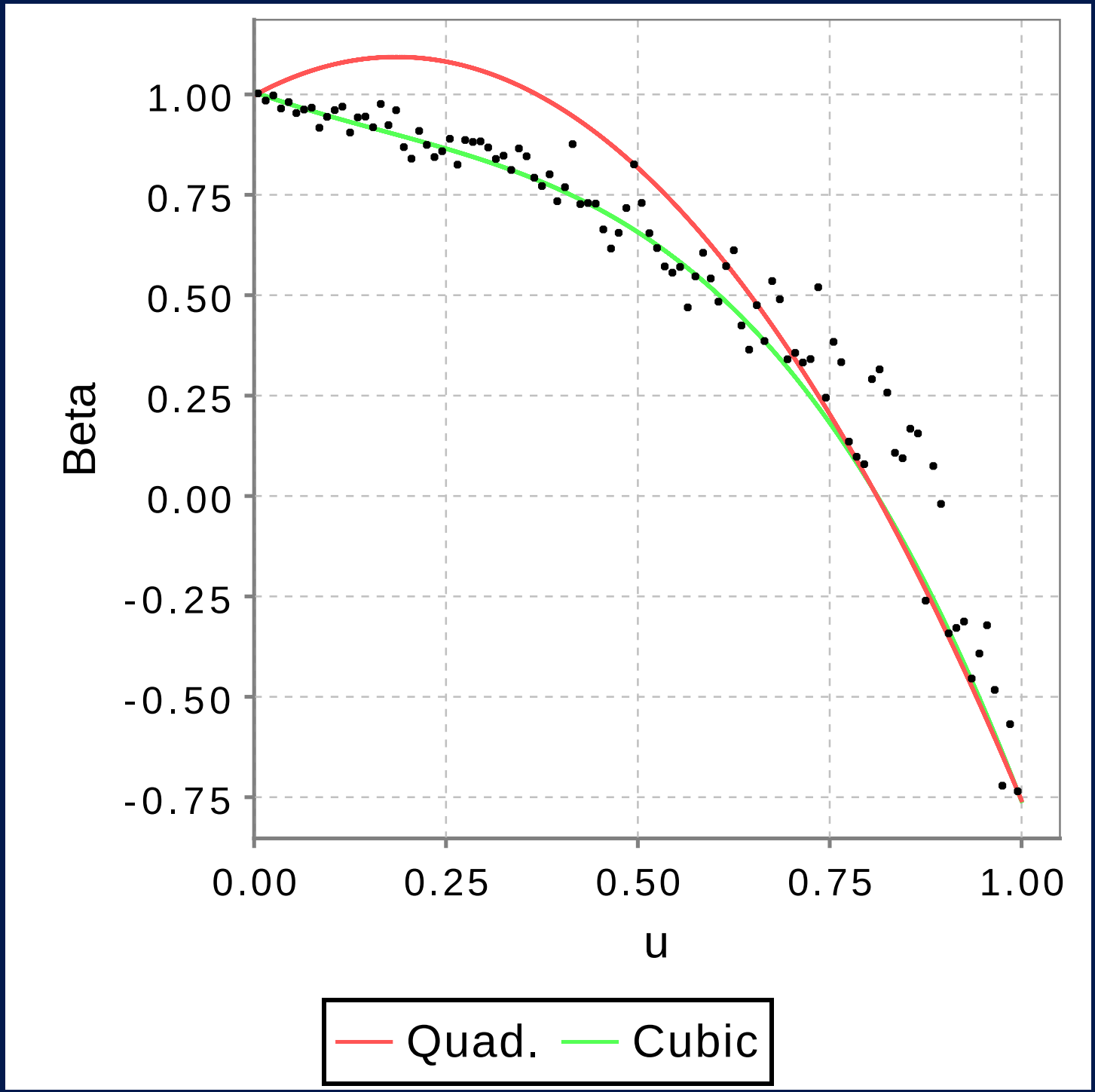
Variance in stratum s is now

$$\sigma_s^2 = \mathrm{Var}[X_c(U^{(s)})] = \frac{1}{k}\int_{(s-1)/k}^{s/k} \mathrm{Var}[X - \beta^*(u)C \mid U = u]du$$

where $U^{(s)} \sim \mathrm{Unif}((s-1)/k,\, s/k)$. So the optimal allocation changes!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With well-synchronized CRN, we can prove that $\mathrm{Var}[\Delta_i] = O(\delta^{-1-\epsilon})$.

Ongoing and Future Work in Our Group

- Develop a Java library for simulation of call centers.
- Variance reduction methods for simulation.
- Integrate with optimization algorithms for staffing, scheduling, rostering, routing.
- Integrate with approximation formulas and CTMC models.
- Better models for inbound traffic and other aspects.
- Currently, our work is driven by the interests of Bell Canada.

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