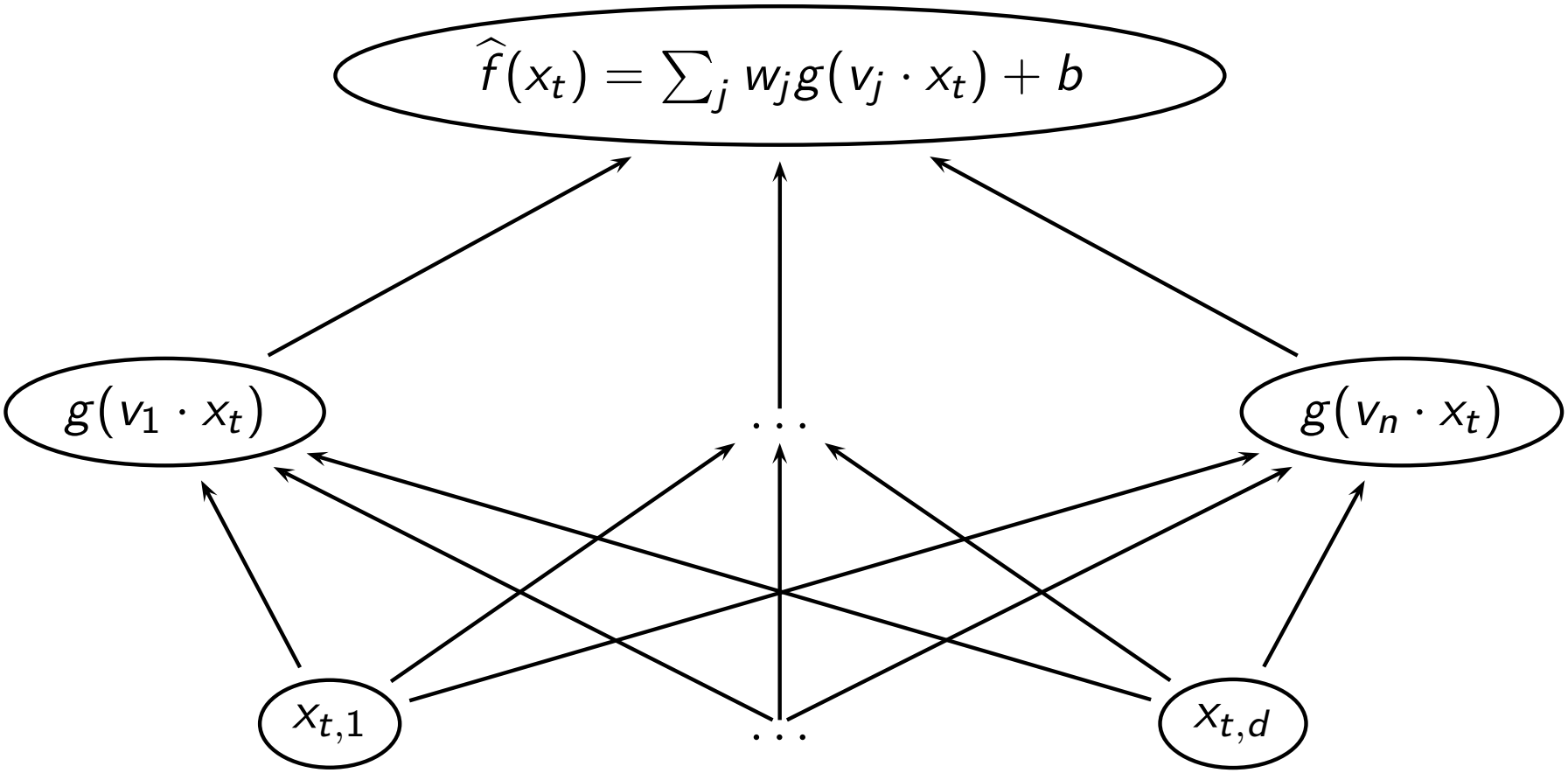
Continuous neural networks

Nicolas Le Roux
joint work with Yoshua Bengio

April 5^{th}, 2006

Usual Neural Networks



g is the transfer function: tanh, sigmoid, sign, ...

Neural networks are universal approximators

Hornik et al. (1989)

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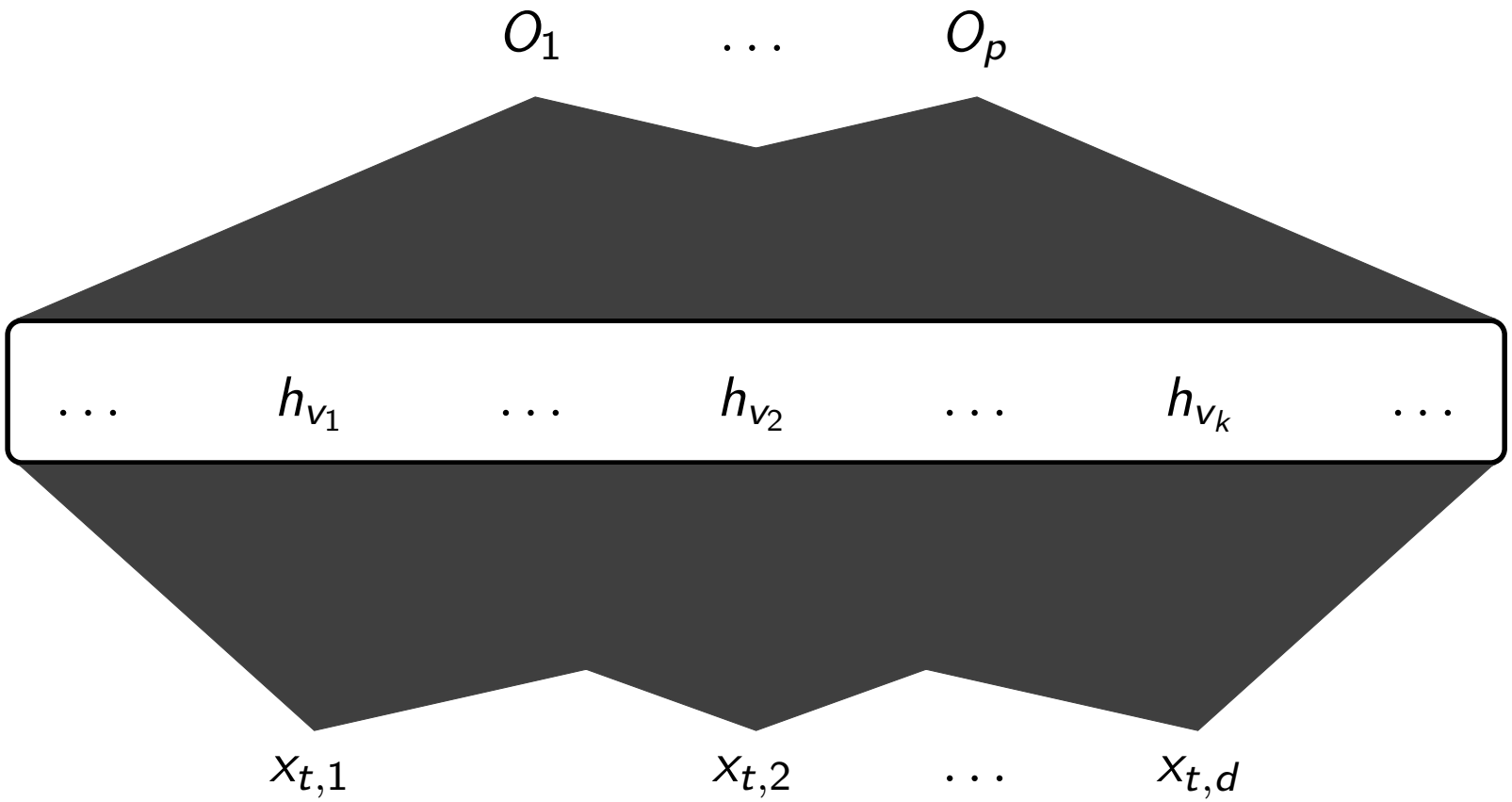
Neural Networks with an infinite number of hidden units should be able to approximate every function arbitrarily well.

A nice picture before the Maths



Going to infinity

$$\widehat{f}(x) = \sum_j w_j \;\; g(v_j \cdot x) + b \longrightarrow \widehat{f}(x) = \int_j w(j) \;\; g(v(j) \cdot x) \; dj + b$$

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But a function from $\mathbb{R}$ to $\mathbb{R}^{d+1}$ is $d+1$ functions from $\mathbb{R}$ to $\mathbb{R}$. This is very similar to Kolmogorov's representation theorem.

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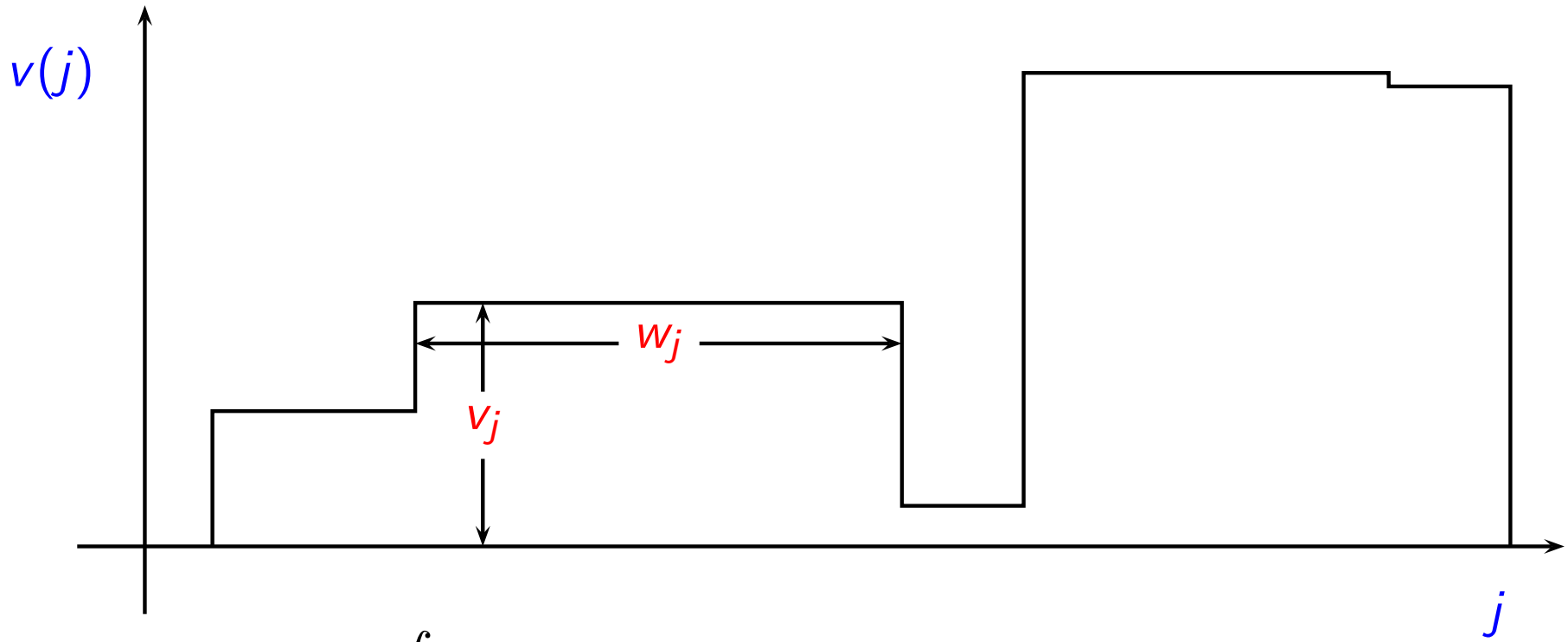

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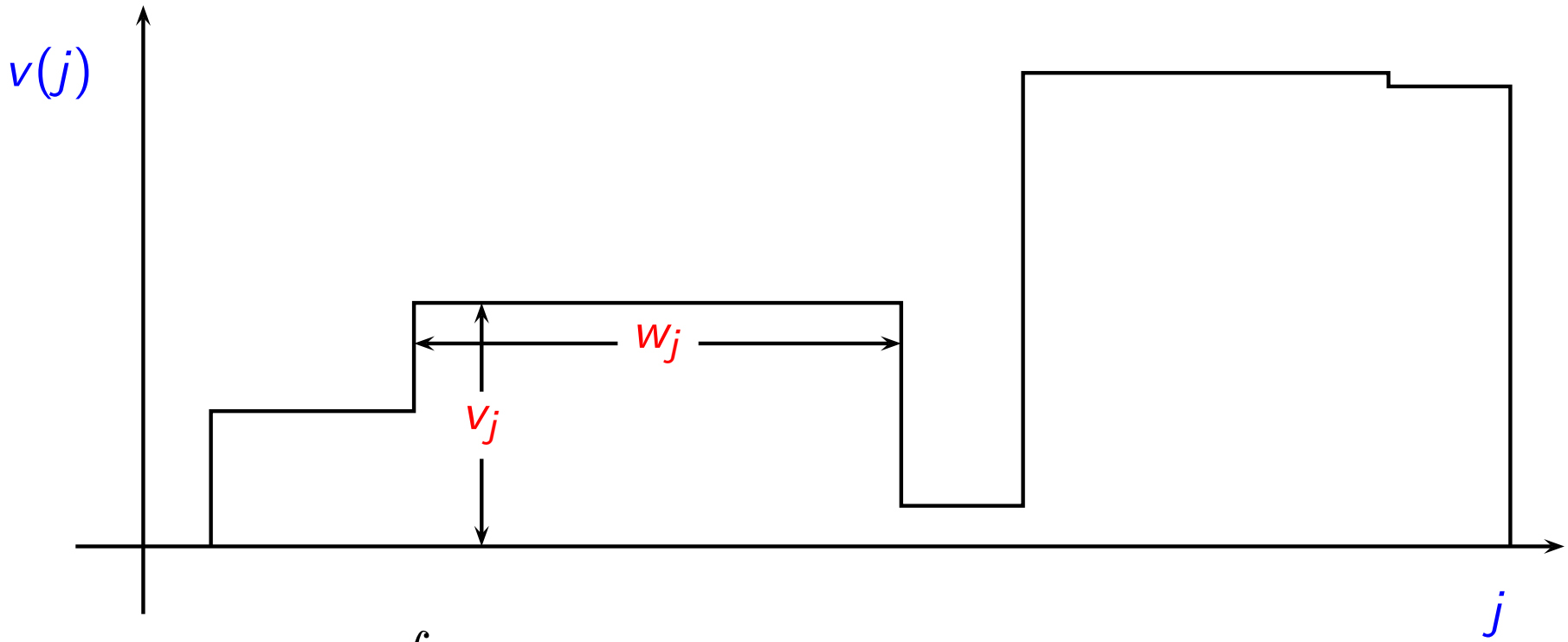

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Neural networks approximate v with a piecewise constant function.

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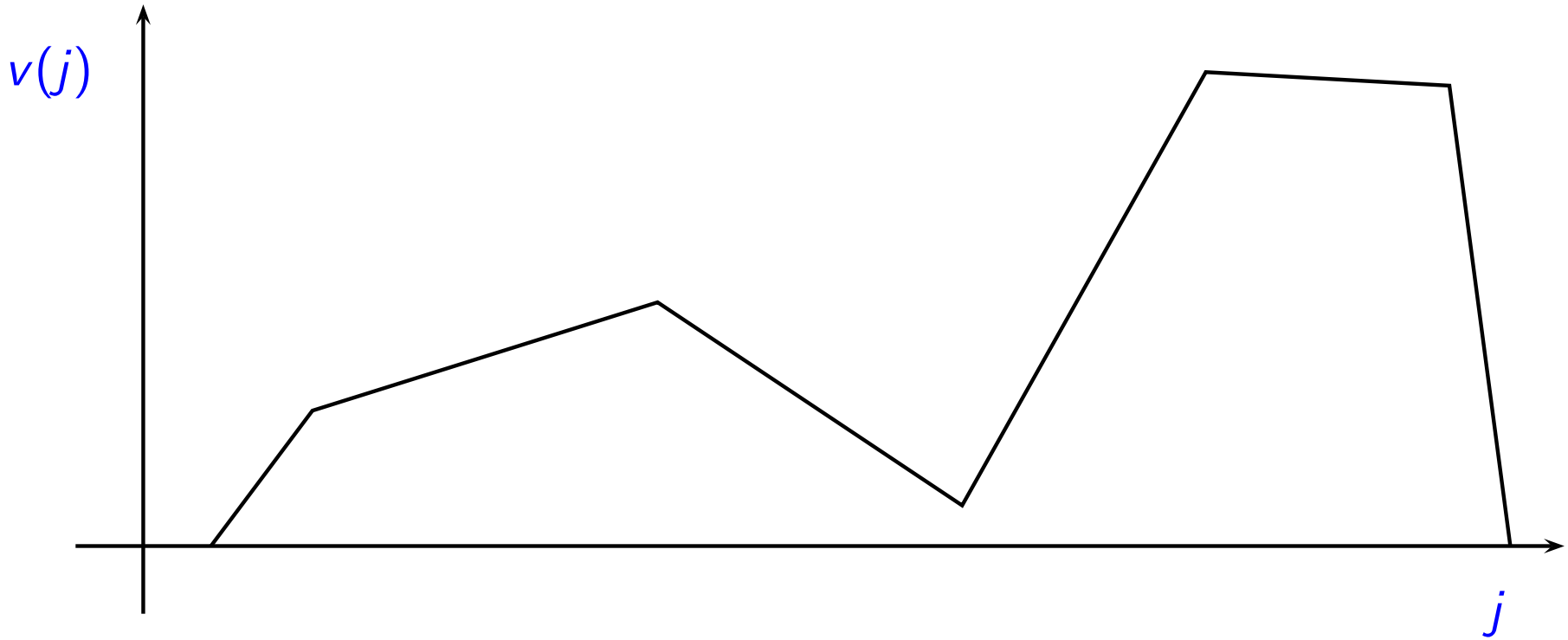
3. A piecewise constant trajectory only crosses a finite number of points in the space of input weights.
4. We could imagine trajectories that "fill" the space a bit more.

Piecewise affine approximations



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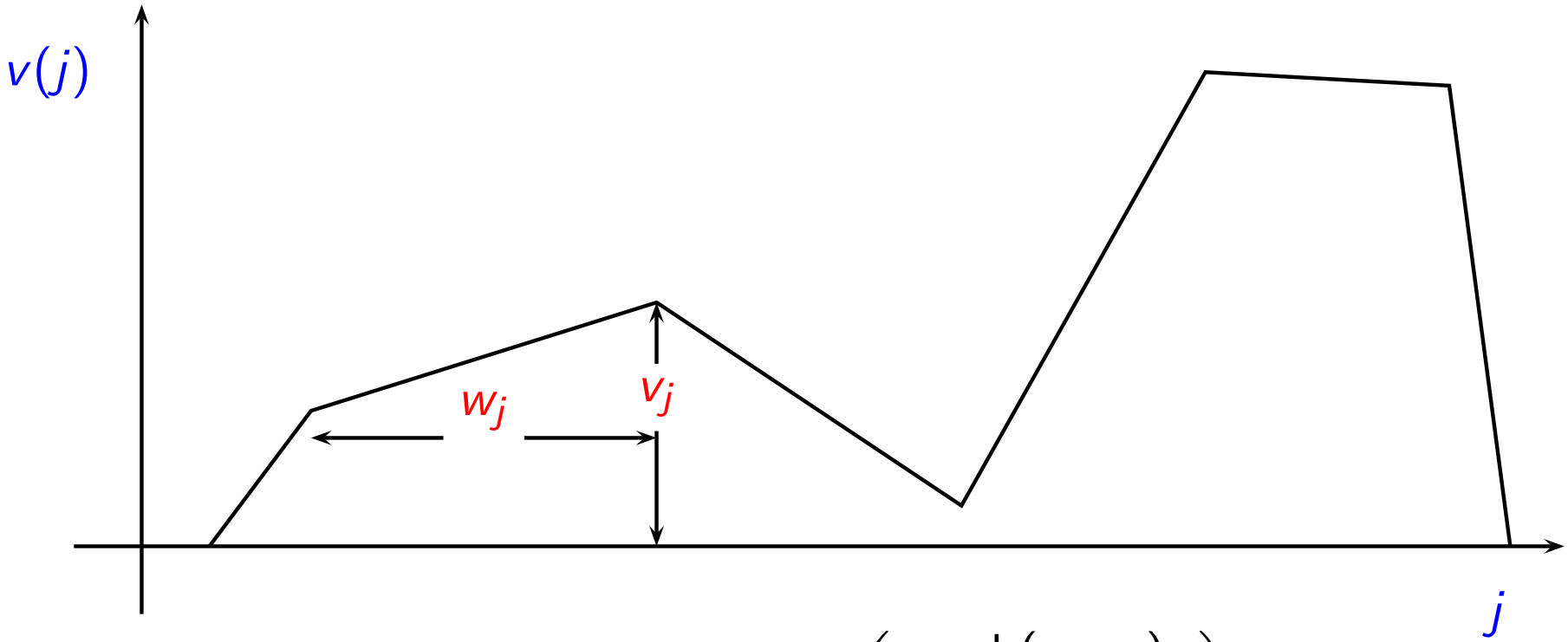

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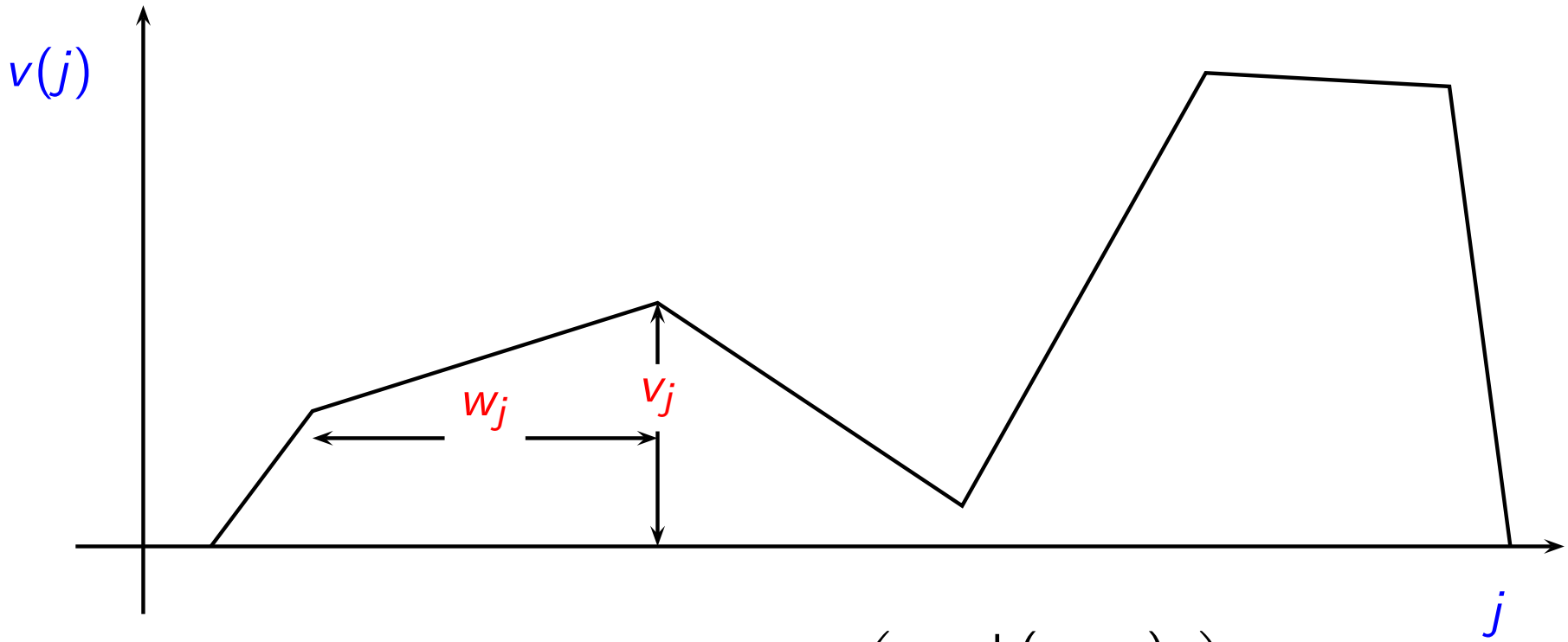

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Seeing v as a function, we could introduce smoothness wrt j using constraints on successive values of v.

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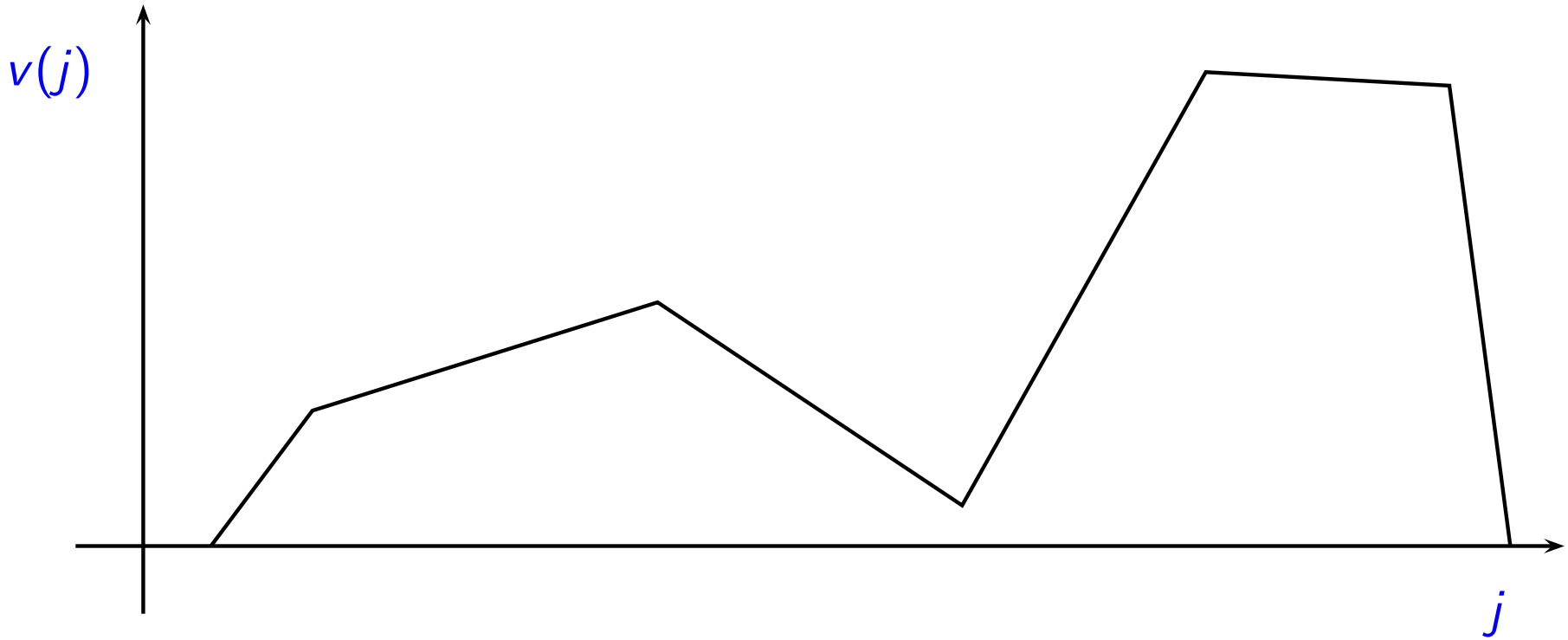
Theorem: rate of convergence of affine neural networks

Affine neural networks converge in $O(n^{-2})$ whereas usual neural networks converge in $O(n^{-1})$ (when n grows to infinity).

Piecewise affine approximations (again)



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1. Having a complex function V requires lots of pieces.
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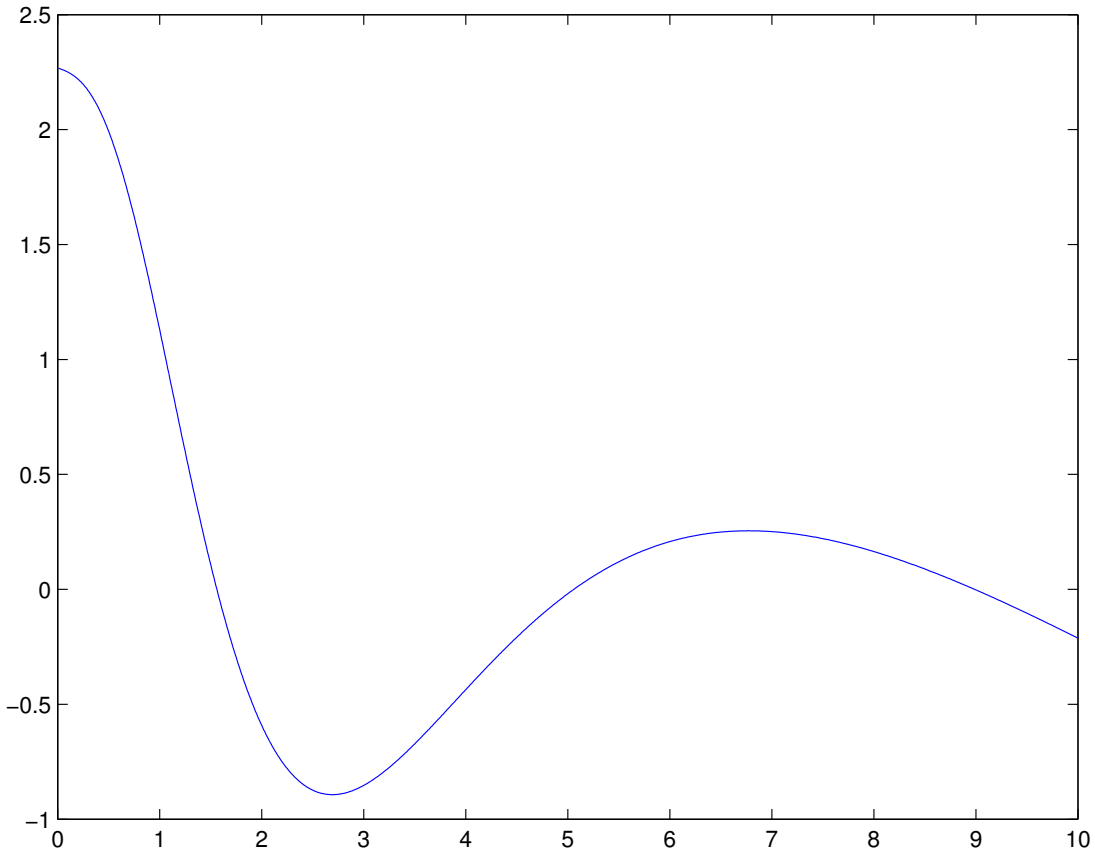
3. Maybe we could use other parametrizations inducing constraints on the pieces.
4. Instead of optimizing each input weight $v(j)$ independently, we could parametrize them as the output of a neural network.

Input weights function as the output of a neural network



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5. The prior over $w_{v,k}$ determines the amplitude of the $v(j)$'s.

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5. Let's stop a bit to rest.

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4. Example: handling of sequential data.

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6. $K_{\mathrm{sign}}(x, y) = A - B\|x - y\|$
7. This kernel has no hyperparameter.

Results on USPS with 6000 training samples

Algorithm	$wd = 10^{-3}$	$wd = 10^{-6}$	$wd = 10^{-12}$	Test
K_{sign}	2.27±0.13	1.80±0.08	1.80±0.08	4.07
G. $\sigma = 1$	58.27±0.50	58.54±0.27	58.54±0.27	58.29
G. $\sigma = 2$	7.71±0.10	7.78±0.21	7.78±0.21	12.31
G. $\sigma = 4$	1.72±0.11	2.09±0.09	2.10±0.09	4.07
G. $\sigma = 6$	1.67±0.10	2.78±0.25	3.33±0.35	3.58
G. $\sigma = 7$	1.72±0.10	3.04±0.26	4.39±0.49	3.77

Results on MNIST with 6000 training samples

Algorithm	$wd = 10^{-3}$	$wd = 10^{-6}, 10^{-9}, 10^{-12}, 0$	Test
K_{sign}	5.51 ± 0.22	4.54 ± 0.50	4.09
G. $\sigma = 1$	77.55 ± 0.40	77.55 ± 0.40	80.03
G. $\sigma = 2$	10.51 ± 0.46	10.51 ± 0.45	12.44
G. $\sigma = 3$	3.64 ± 0.10	3.64 ± 0.10	4.1
G. $\sigma = 5$	3.01 ± 0.12	3.01 ± 0.12	3.33
G. $\sigma = 7$	3.15 ± 0.09	3.18 ± 0.10	3.48

Results on LETTERS with 6000 training samples

Algorithm	$wd = 10^{-3}$	$wd = 10^{-6}$	$wd = 10^{-9}$	Test
K_{sign}	5.36 ± 0.10	5.22 ± 0.09	5.22 ± 0.09	5.5
G. $\sigma = 2$	5.47 ± 0.14	5.93 ± 0.15	5.92 ± 0.14	5.8
G. $\sigma = 4$	4.97 ± 0.10	11.06 ± 0.29	12.50 ± 0.35	5.3
G. $\sigma = 6$	6.27 ± 0.17	8.47 ± 0.20	17.61 ± 0.40	6.63
G. $\sigma = 8$	8.45 ± 0.19	6.11 ± 0.15	18.69 ± 0.34	9.25

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2. We introduced an affine-by-part parametrization of that function which corresponds to a continuous number of hidden units.
3. In the extreme case where all the input weights are present, we showed it is a kernel machine whose kernel can be computed analytically and possesses no hyperparameter.

Future work

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3. Recursively define the input weight function as the output of a neural network.

Now is the time for ...

Questions?

Computing $\int_E \text{sign}(v \cdot x + b)\text{sign}(v \cdot y + b)\, dvdb$

1. $\text{sign}(x)$ function is invariant with respect to the norm of x.
2. $\text{sign}(v \cdot x + b)\text{sign}(v \cdot y + b) = \text{sign}(v \cdot x + b)(v \cdot y + b)]$.
3. When b ranges from $-M$ to $+M$, for M large enough, $x'vv'y + b$ is negative on an interval of size $|v \cdot (x - y)|$.
4. $\int_{b=-M}^{+M} \text{sign}(x'vv'y + b)\, db = 2M - 2|v \cdot (x - y)|$.
5. Integrating this term on the unit hypersphere yields a kernel of the form $K(x, y) = A - B\|x - y\|$.

Additive and multiplicative invariance of the covariance matrix

1. In SVM and kernel regression, the elements of the weights vector α sum to 0.
2. The final solution involves $K\alpha$.
3. Thus, adding a term δ to every element of the covariance matrix yields the solution $(K + \frac{\delta}{n} ee')\alpha = K\alpha + \frac{\delta}{n} e(e'\alpha) = K\alpha$.
4.

$$\begin{aligned} C(K, \alpha, b, \lambda) &= \mathcal{L}(K\alpha + b, Y) + \lambda \alpha' K \alpha \\ C\left(\frac{K}{c}, c\alpha, b, \frac{\lambda}{c}\right) &= \mathcal{L}\left(\frac{K}{c} c\alpha + b, Y\right) + \frac{\lambda}{c} c\alpha' \frac{K}{c} c\alpha \\ &= C(K, c\alpha, b, \lambda) \end{aligned}$$