

New formulations and valid inequalities for a bilevel pricing problem

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Abstract

Consider the problem of maximizing the toll revenue collected on a multi-commodity transportation network. This fits a bilevel framework where a leader sets tolls, while users respond by selecting cheapest paths to their destination. We propose novel formulations of the problem, together with valid inequalities yielding improved algorithms.

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1. Introduction

In traditional pricing models, market response is characterized by an exogenous demand curve. Alternatively, one may integrate the user optimization process directly into the model. In the context of network tolling, this interaction fits the bilevel framework analyzed in [7], where a highway authority anticipates the reaction of users to its toll policy. In this setting, the leader selects the right balance of prices, large enough to generate revenue, but small enough not to deter users to access the toll network. The resulting mathematical program, closely related to the topics of envy-free pricing [5] or optimal product line design [6], involves bilinear functions at both levels of decision making. Even in the single-commodity instance, it subsumes classical NP-hard problems such as the traveling salesman problem (see [8]) or MAX-3-SAT (see [4,9]). The aim of the present paper is to construct improved algorithms based on tight formulations and new valid inequalities.

2. Arc formulations of the toll setting problem

Let us consider a multi-commodity network $G = \{K, N, A \cup B\}$ defined by a set K of origin–destination pairs, a node set N and two arc sets A and B representing, respectively, the sets of toll and toll-free arcs. There is a travel time c_{ij} associated with arc $(i, j) \in A \cup B$ and a toll t_{ij} associated with arc $(i, j) \in A$. We assume that travel time and cost are expressed in a common unit. Throughout the paper, the terms ‘cost’ and ‘length’ are synonymous, so that the expression ‘shortest path’ make sense. For each commodity $k \in K$, the demand for travel between the origin node $o(k)$ and the destination $d(k)$ is n^k . We let x_{ij}^k (resp. y_{ij}^k) denote the proportions of flow from commodity k assigned to toll (resp. toll-free) arc (i, j) , and assume that users are assigned

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to shortest paths with respect to the cost vector $c + t$. The toll optimization problem is then formulated as the bilinear bilevel program

$$\begin{aligned} \text{TOP : } & \max_{t, x, y \geq 0} \sum_{k \in K} \sum_{(i, j) \in A} t_{ij} n^k x_{ij}^k \\ \forall k \in K & \begin{cases} (x^k, y^k) \in \arg \min_{\bar{x}, \bar{y} \geq 0} \sum_{(i, j) \in A} (c_{ij} + t_{ij}) \bar{x}_{ij}^k + \sum_{(i, j) \in B} c_{ij} \bar{y}_{ij}^k \\ \text{s.t.} & \sum_{j: (j, i) \in A} \bar{x}_{ji}^k + \sum_{j: (j, i) \in B} \bar{y}_{ji}^k - \sum_{j: (i, j) \in A} \bar{x}_{ij}^k - \sum_{j: (i, j) \in B} \bar{y}_{ij}^k = b_i^k \quad \forall i \in N, \end{cases} \end{aligned}$$

where the node demand b_i^k is equal to -1 if $i = o(k)$, to $+1$ if $i = d(k)$, and to 0 otherwise. To avoid trivial solutions leading to infinite revenue, we assume that there always exists a path composed solely of toll-free arcs for every $k \in K$. Since the maximization involves the lower level vectors x and y , the above formulation implicitly assumes that, whenever shortest paths are not unique, the one yielding the highest revenue for the leader is adopted. This assumption is not unrealistic since, given two equivalent paths, the one generating the highest revenue could be made the most attractive through an arbitrarily small reduction of one of its tolls. Now, replacing each lower level optimization problem by its optimality conditions (primal-dual feasibility, equality of the primal and dual objectives), the toll optimization problem can be reformulated as the single-level program (see [7]).

$$\begin{aligned} \max_{t, x, y \geq 0, \lambda} & \sum_{k \in K} \sum_{(i, j) \in A} t_{ij} n^k x_{ij}^k & (1) \\ \text{s.t.} & \sum_{j: (j, i) \in A} x_{ji}^k + \sum_{j: (j, i) \in B} y_{ji}^k - \sum_{j: (i, j) \in A} x_{ij}^k - \sum_{j: (i, j) \in B} y_{ij}^k = b_i^k \quad \forall k \in K, i \in N, \\ & \lambda_j^k - \lambda_i^k \leq c_{ij} + t_{ij} \quad \forall k \in K, (i, j) \in A, \\ & \lambda_j^k - \lambda_i^k \leq c_{ij} \quad \forall k \in K, (i, j) \in B, \\ & \sum_{(i, j) \in A} (c_{ij} + t_{ij}) x_{ij}^k + \sum_{(i, j) \in B} c_{ij} y_{ij}^k = \lambda_{d(k)}^k - \lambda_{o(k)}^k \quad \forall k \in K. & (2) \end{aligned}$$

The bilinear terms in (1) and (2) that impose equality between the primal and dual objectives lead to a nonlinear and nonconvex problem. As shown in [7], it is optimal to send each commodity flow along a single path and to restrict the flow proportions x_{ij}^k to binary numbers. Now, let $\tau_{ij}^k = t_{ij} x_{ij}^k$. From this definition, $x_{ij}^k = 0$ implies that $\tau_{ij}^k = 0$ even when t_{ij} is positive. On the other hand, $\tau_{ij}^k = t_{ij}$ when $x_{ij}^k = 1$. The constraints below capture these implications mathematically:

$$\tau_{ij}^k \leq M_{ij}^k x_{ij}^k \quad \forall k \in K, (i, j) \in A, \quad (3)$$

$$t_{ij} - \tau_{ij}^k \leq N_{ij} (1 - x_{ij}^k) \quad \forall k \in K, (i, j) \in A, \quad (4)$$

$$\tau_{ij}^k \leq t_{ij} \quad \forall k \in K, (i, j) \in A, \quad (5)$$

where the constants M_{ij}^k and N_{ij} must satisfy $\tau_{ij}^k \leq M_{ij}^k$ and $t_{ij} \leq N_{ij}$ for some optimal τ_{ij}^k and t_{ij} . This linearization yields the mixed 0-1 linear program:

$$\begin{aligned} \text{TOP-ARCS : } & \max_{t, \tau, x, y \geq 0, \lambda} \sum_{k \in K} \sum_{(i, j) \in A} n^k \tau_{ij}^k \\ \text{s.t.} & \sum_{j: (j, i) \in A} x_{ji}^k + \sum_{j: (j, i) \in B} y_{ji}^k - \sum_{j: (i, j) \in A} x_{ij}^k - \sum_{j: (i, j) \in B} y_{ij}^k = b_i^k \quad \forall k \in K, i \in N, \\ & \lambda_j^k - \lambda_i^k \leq c_{ij} + t_{ij} \quad \forall k \in K, (i, j) \in A, \\ & \lambda_j^k - \lambda_i^k \leq c_{ij} \quad \forall k \in K, (i, j) \in B, \\ & \sum_{(i, j) \in A} (c_{ij} x_{ij}^k + \tau_{ij}^k) + \sum_{(i, j) \in B} c_{ij} y_{ij}^k = \lambda_{d(k)}^k - \lambda_{o(k)}^k \quad \forall k \in K, \\ & \tau_{ij}^k \leq M_{ij}^k x_{ij}^k \quad \forall k \in K, (i, j) \in A, \\ & t_{ij} - \tau_{ij}^k \leq N_{ij} (1 - x_{ij}^k) \quad \forall k \in K, (i, j) \in A, \\ & \tau_{ij}^k \leq t_{ij} \quad \forall k \in K, (i, j) \in A, \\ & x_{ij}^k \in \{0, 1\} \quad \forall k \in K, (i, j) \in A. \end{aligned}$$

Note that the proportions x_{ij}^k on the toll arcs are required to be Boolean in order to enforce the compatibility between τ_{ij}^k and t_{ij} , while this is not the case for the toll-free proportions y_{ij}^k .

2.1. Bounds on the ‘Big-M’ constants

In order that the linear relaxation of TOP-ARCS be as tight as possible, the constants M_{ij}^k and N_{ij} should be set to the smallest values that yet ensure the validity of the formulation. Before addressing this issue, we introduce some key quantities:

- p_{ij} = the cost of a shortest path from i to j using only arcs in B , i.e., a toll-free path
- $\bar{\lambda}_j^k$ = the cost of a shortest path from origin $o(k)$ to j using only arcs in B
- $\underline{\lambda}_j^k$ = the cost of a shortest path from origin $o(k)$ to j using arcs in $A \cup B$, for $t = 0$
- \bar{s}_j^k = the cost of a shortest path from j to destination $d(k)$ using only arcs in B
- \underline{s}_j^k = the cost of a shortest path from j to destination $d(k)$ using arcs in $A \cup B$, for $t = 0$.

We set the above quantities to ∞ if the corresponding path does not exist, and set $\infty - \infty = \infty$.

Proposition 1. For every toll arc $(i, j) \in A$ and commodity k ,

$$\max\{0, \min\{p_{ij} - c_{ij}, \bar{\lambda}_j^k - \underline{\lambda}_i^k - c_{ij}, \bar{\lambda}_{d(k)}^k - (\underline{\lambda}_i^k + c_{ij} + \underline{s}_j^k), \bar{s}_i^k - \underline{s}_j^k - c_{ij}\}\}$$

is a valid constant M_{ij}^k for TOP-ARCS.

Proof. The fifth constraint of TOP-ARCS specifies that $\tau_{ij}^k \leq M_{ij}^k$ whenever $x_{ij}^k = 1$, i.e., any upper bound on τ_{ij}^k that translates into a valid M_{ij}^k must be an upper bound on τ_{ij}^k . Now, for fixed toll vector t , a toll arc (i, j) may lie on a shortest path associated with commodity k only if the following inequalities hold:

1. The length of arc (i, j) does not exceed the shortest toll-free distance from i to j , that is, $c_{ij} + \tau_{ij}^k \leq p_{ij}$. A first valid upper bound on τ_{ij}^k is thus $p_{ij} - c_{ij}$.
2. The shortest distance from $o(k)$ to node j is larger than the shortest distance of a path going through toll arc (i, j) , that is, $\underline{\lambda}_i^k + c_{ij} + \tau_{ij}^k \leq \bar{\lambda}_j^k$. This yields the valid bound $\bar{\lambda}_j^k - \underline{\lambda}_i^k - c_{ij}$.
3. The length of the shortest toll-free path from $o(k)$ to $d(k)$ must exceed the shortest distance from $o(k)$ to $d(k)$ going through toll arc (i, j) , that is, $\underline{\lambda}_i^k + c_{ij} + \tau_{ij}^k + \underline{s}_j^k \leq \bar{\lambda}_{d(k)}^k$. This yields the bound $\bar{\lambda}_{d(k)}^k - (\underline{\lambda}_i^k + c_{ij} + \underline{s}_j^k)$.
4. The cost of the toll-free path from node i to the destination $d(k)$ exceeds the shortest distance from i to $d(k)$ going through arc (i, j) , that is, $\underline{s}_j^k + c_{ij} + \tau_{ij}^k \leq \bar{s}_i^k$. This yields the bound $\bar{s}_i^k - \underline{s}_j^k - c_{ij}$.

Now, the fact that 0 is a lower bound and that the minimum of valid bounds is also valid concludes the proof. In her Ph.D. thesis, Dewez [2] showed that none of the bounds dominates any other. \square

Proposition 2. $\max_{k \in K} \{M_{ij}^k\}$ is a valid bound for N_{ij} .

Proof. If $x_{ij}^k = 1$ for some k , then $\tau_{ij}^k = t_{ij}$ and the bounds derived for the former also apply to the latter. Otherwise, the toll t_{ij} should be such that arc (i, j) does *not* belong to the shortest path used by any commodity. If there is a tie, the flow is assigned to a path that does not contain arc (i, j) . This is exactly what is expressed mathematically by the inequality $\lambda_j^k - \lambda_i^k \leq c_{ij} + t_{ij}$. Now, the inequality

$$t_{ij} \geq \min\{p_{ij} - c_{ij}, \bar{\lambda}_j^k - \underline{\lambda}_i^k - c_{ij}, \bar{\lambda}_{d(k)}^k - (\underline{\lambda}_i^k + c_{ij} + \underline{s}_j^k), \bar{s}_i^k - \underline{s}_j^k - c_{ij}\},$$

obtained by comparing paths including or not arc (i, j) , implies that there must exist, for each origin–destination pair, a shortest path that does *not* go through arc (i, j) , and it follows that the above minimum is a valid upper bound for the ‘big-M’ constant N_{ij} . \square

2.2. Valid inequalities

In this section, we derive valid inequalities for TOP-ARCS from the complementarity conditions. Order k inequalities refer to inequalities that involve k commodities simultaneously.

2.2.1. Order 1 inequalities

In the MIP formulation TOP-ARCS, the constraint that forces the equality of the primal and dual objectives is equivalent to the pair of complementarity constraints

$$(c_{ij} - \lambda_j^k + \lambda_i^k)y_{ij}^k = 0, \quad (6)$$

$$(c_{ij} + t_{ij}^k - \lambda_j^k + \lambda_i^k)x_{ij}^k = 0. \quad (7)$$

Proposition 3 (Toll-free inequalities). *For every $(i, j) \in B$ and $k \in K$, we have*

$$\underline{\lambda}_j^k \leq \lambda_j^k,$$

$$\bar{\lambda}_j^k \geq \lambda_j^k,$$

$$(c_{ij} + \bar{\lambda}_i^k - \underline{\lambda}_j^k)y_{ij}^k + \lambda_i^k - \lambda_j^k \leq \bar{\lambda}_i^k - \underline{\lambda}_j^k.$$

Proof. The first two inequalities follow trivially from the definition of $\underline{\lambda}_j^k$ and $\bar{\lambda}_j^k$. From (6), together with dual feasibility (second constraint of TOP-ARCS), we infer that

$$-L_{ij}^k(1 - y_{ij}^k) \leq \lambda_j^k - \lambda_i^k - c_{ij} \leq 0$$

whenever L_{ij}^k satisfies $-L_{ij}^k \leq \lambda_j^k - \lambda_i^k - c_{ij}$. For $y_{ij}^k = 0$, this inequality holds if L_{ij}^k is set to $c_{ij} + \bar{\lambda}_i^k - \underline{\lambda}_j^k$. Setting L_{ij}^k to this bound yields the result. \square

Proposition 4 (Toll inequalities). *For every $(i, j) \in A$ and $k \in K$, we have*

$$\underline{\lambda}_j^k \leq \lambda_j^k,$$

$$\bar{\lambda}_j^k \geq \lambda_j^k,$$

$$(c_{ij} + \bar{\lambda}_i^k - \underline{\lambda}_j^k)x_{ij}^k + t_{ij}^k + \lambda_i^k - \lambda_j^k \leq \bar{\lambda}_i^k - \underline{\lambda}_j^k.$$

Proof. The proof makes use of the second complementary condition (7). As either $x_{ij}^k = 0$ or $\lambda_j^k - \lambda_i^k - c_{ij} - t_{ij} = 0$, the constraint

$$-R_{ij}^k(1 - x_{ij}^k) \leq \lambda_j^k - \lambda_i^k - c_{ij} - t_{ij} \leq 0$$

is valid for TOP-ARCS whenever the constant R_{ij}^k satisfies $-R_{ij}^k \leq \lambda_j^k - \lambda_i^k - c_{ij} - t_{ij}$ for $x_{ij}^k = 0$. Setting $R_{ij}^k = c_{ij} + t_{ij} + \bar{\lambda}_i^k - \underline{\lambda}_j^k$ yields the result. \square

2.2.2. Order k inequalities

Proposition 5. *Let s and t be two distinct nodes such that there is no path from t to s . Then, for every $k_1 \neq k_2$, $a \neq a'$, $b \neq b'$, the inequalities*

$$x_{sa'}^{k_1} + x_{sa}^{k_2} + x_{b't}^{k_1} + x_{bt}^{k_2} \leq 3 \quad (8)$$

are valid for TOP-ARCS.

Proof. We may restrict our attention to solutions where, between two given nodes s and t , at most one subpath, actually the shortest one, is used. If the left-hand side of (8) is larger than 3 then, given flow integrality, two distinct subpaths must be used. Now the assumption implies that nodes s and t must be visited in that order by both commodities, which contradicts the previous sentence. \square

Whenever two commodities share the same destination (resp. origin) and both go through a given node, there must exist a solution where they go through the same exiting arc (resp. entering arc). This can be exploited in the following fashion. Let K_d (resp. K_o) be the set of commodities having the same destination d (resp. origin o). For a given node i , let $K_d(i)$ (resp. $K_o(i)$) be a disjoint subset of K_d (resp. K_o) with cardinality equal to the number of exiting arcs from node i (resp. entering arcs into i). Each element of $K_d(i)$ (resp. $K_o(j)$) is mapped to an exiting (resp. entering) arc (i, j) and noted $k(j)$ (resp. $k(i)$).

Proposition 6. Let i^+ (resp. i^-) denote the forward star (resp. backward star) at node i . For any i and subsets $K_d(i)$ and $K_o(i)$, we have

$$\sum_{j \in i^+ \cap A} x_{ij}^{k(j)} + \sum_{j \in i^+ \cap B} y_{ij}^{k(j)} \leq 1,$$

$$\sum_{j \in i^- \cap A} x_{ji}^{k(i)} + \sum_{j \in i^- \cap B} y_{ji}^{k(i)} \leq 1.$$

3. Path formulation

Path-based formulations have been proposed in [3,10]. The approach is advantageous when the number of potential paths is small. Also, many paths can typically be excluded a priori because they would never become shortest in an optimal solution. This remarks lies behind the reduction proposed in [10], yielding a network with a specific structure, the shortest paths graph model (SPGM). In this section, we propose two new arc elimination conditions, as well as bounds and valid inequalities for the formulation of Didi et al. [3].

3.1. The shortest paths graph model

The graph corresponding to a toll problem can be simplified by replacing the toll-free paths between toll arcs by a single arc endowed with the shortest length between those arcs. Similar reductions apply to origin and destination nodes (see [10] for details).

The SPGM has been further streamlined by van Hoesel et al. To this end, let u_{ij} (resp. l_{ij}) denote the cost of a toll-free shortest path from node i to node j in the SPGM (resp. with null tolls). If the arc (i, j) does not exist in the SPGM, its cost u_{ij} is set to ∞ . The number u_{ij} (resp. l_{ij}) constitutes an upper bound (resp. lower bound) on the length of a path from i to j .

1. If $l_{jd(k)} = u_{jd(k)}$, then every arc leaving node j can be removed from G^k , the sub-network specific to commodity k with the exception of $(j, d(k))$.
2. If $l_{o(k)i} = u_{o(k)i}$, then every arc entering node i can be removed G^k , with the exception of $(o(k), i)$.
3. Let (i_1, j_1) be a toll arc and i_2 be the head node of a toll arc. If $u_{j_1 d(k)} \leq u_{j_1 i_2} + l_{i_2 d(k)}$ and $i_2 \neq d(k)$, then (j_1, i_2) can be removed from G^k .
4. Let (i_1, j_1) be a toll arc and j_2 the tail node of a toll arc. If $u_{o(k) i_1} \leq l_{o(k) j_2} + u_{j_2 i_1}$ and $j_2 \neq o(k)$, then arc (j_2, i_1) can be removed from G^k .
5. If $u_{o(k) d(k)} \leq l_{o(k) i} + l_{ij} + l_{jd(k)}$, then (i, j) can be deleted from G^k .
6. Let (i_1, j_1) be a toll arc and i_2 the head node of a toll arc. If $u_{o(k) d(k)} \leq l_{o(k) j_1} + u_{j_1 i_2} + l_{i_2 d(k)}$, $j_1 \neq o(k)$ and $i_2 \neq d(k)$, then (j_1, i_2) can be deleted from G^k .

The next two results are new.

Proposition 7. Let i be the head node of a toll arc and $i \neq d(k)$. If $(o(k), i)$ is a toll-free arc and

$$u_{o(k) d(k)} \leq u_{o(k) i} + l_{id(k)},$$

then the arc $(o(k), i)$ can be deleted from G^k .

Proof. A lower bound on the cost of a path from $o(k)$ to $d(k)$ using the arc $(o(k), i)$ is $u_{o(k) i} + l_{id(k)}$. Hence there is no room for a positive toll, and the arc $(o(k), i)$ will never be used in a shortest path. \square

A similar reasoning yields the following dual result:

Proposition 8. Let j be the tail node of a toll arc and $j \neq o(k)$. If $(j, d(k))$ is a toll-free arc and

$$u_{o(k) d(k)} \leq l_{o(k) j} + u_{jd(k)},$$

then the arc $(o(k), i)$ can be deleted from G^k .

Fig. 1, where the graph on the left corresponds to the ‘standard’ SPGM, illustrates the reduction process. Arc $(2, 3)$ can be eliminated since the shortest path from node 2 to node $d(k)$ is the arc $(2, d(k))$. Rule 2 eliminates arcs $(6,3)$ and $(4,5)$, while rules 4 and 5 eliminate arcs $(4,1)$ and $(3,4)$, respectively. Finally, Proposition 7 allows the deletion of arc $(o(k), 1)$.

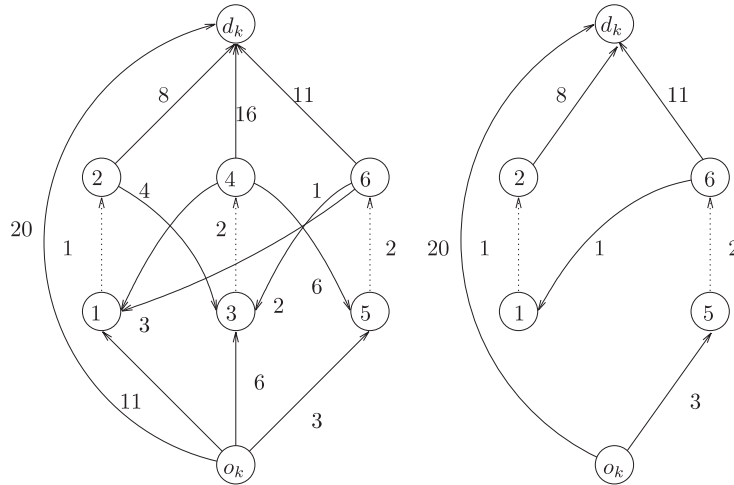


Fig. 1. SPGM reduction.

3.2. A path formulation: bounds and inequalities

Consider the path-based formulation in [3] in which the decision variable $\mathcal{X}_{\mathcal{P}}$ is binary and indicates whether path P is used. Let L^k be the total cost of the path assigned to commodity k and T^k denote the portion of L^k consisting of tolls charged along the path. For the path $\mathcal{P}_k^0 \in P_k$ actually used by commodity k , we have

$$T^k = \sum_{a \in \mathcal{P}_k^0 \cap A} t_a,$$

$$L^k = \sum_{a \in \mathcal{P}_k^0} c_a + \sum_{a \in \mathcal{P}_k^0 \cap A} t_a$$

and, since exactly one $\mathcal{X}_{\mathcal{P}}$ takes the value 1:

$$L^k = T^k + \sum_{\mathcal{P} \in P_k} \mathcal{X}_{\mathcal{P}} \sum_{a \in \mathcal{P}} c_a.$$

This leads to the MIP formulation

$$\begin{aligned} \text{PATH : } & \max_{T^k, \mathcal{X}, L^k} \sum_{k \in K} n^k T^k \\ \text{s.t. } & \sum_{a \in \mathcal{P} \cap A} t_a + \sum_{a \in \mathcal{P}} c_a - M_k^{\mathcal{P}} (1 - \mathcal{X}_{\mathcal{P}}) \leq L^k \leq \sum_{a \in \mathcal{P} \cap A} t_a + \sum_{a \in \mathcal{P}} c_a \quad \forall \mathcal{P} \in P_k, \quad \forall k \in K, \\ & T^k \leq \sum_{a \in \mathcal{P} \cap A} t_a + M_k (1 - \mathcal{X}_{\mathcal{P}}) \quad \forall \mathcal{P} \in P_k, \quad \forall k \in K, \\ & L^k = T^k + \sum_{\mathcal{P} \in P_k} \mathcal{X}_{\mathcal{P}} \sum_{a \in \mathcal{P}} c_a \quad \forall k \in K, \\ & \sum_{\mathcal{P} \in P_k} \mathcal{X}_{\mathcal{P}} = 1 \quad \forall k \in K, \\ & \mathcal{X}_{\mathcal{P}} \in \{0, 1\} \quad \forall \mathcal{P} \in P_k, \quad \forall k \in K, \end{aligned}$$

where the first constraint ensures that L^k is indeed the length of the shortest path travelled by commodity k , and the next two force T^k to match the actual unit revenue for commodity k .

Proposition 9. In the PATH formulation, $\bar{\lambda}_k - \underline{\lambda}_k$ is a valid bound for M_k .

Proof. If, for some path \mathcal{P}^1 , $\mathcal{X}_{\mathcal{P}^1} = 0$, the first inequality of PATH reduces to

$$T^k \leq \sum_{a \in \mathcal{P}^1 \cap A} t_a + M_k.$$

By definition of \mathcal{P}_k^0 , we have that

$$\sum_{a \in \mathcal{P}_k^0 \cap A} t_a = T^k.$$

Combining these two formulas yields

$$M_k \geq \sum_{a \in \mathcal{P}_k^0 \cap A} t_a - \sum_{a \in \mathcal{P}^1 \cap A} t_a.$$

The latter condition will obviously hold if

$$M_k \geq \sum_{a \in \mathcal{P}_k^0 \cap A} t_a$$

and the result follows from the fact that $\bar{\lambda}_k - \underline{\lambda}_k$ is an upper bound on the revenue raised from commodity k on path \mathcal{P}_k^0 . \square

Proposition 10. Let N_a be any upper bound for the toll on arc a . Then $\sum_{a \in \mathcal{P} \cap A} N_a + \sum_{a \in \mathcal{P}} c_a - \underline{\lambda}_k$ is a valid bound for $M_k^{\mathcal{P}}$ in the formulation PATH.

Proof. If $\mathcal{X}_{\mathcal{P}^1} = 0$, the second constraint of PATH yields

$$\sum_{a \in \mathcal{P}^1 \cap A} t_a + \sum_{a \in \mathcal{P}^1} c_a - M_k^{\mathcal{P}^1} \leq L^k,$$

implying that

$$\sum_{a \in \mathcal{P}^1 \cap A} t_a + \sum_{a \in \mathcal{P}^1} c_a - L^k$$

is a valid bound for $M_k^{\mathcal{P}^1}$. Since N_a is an upper bound on t_a and L^k is larger than $\underline{\lambda}_k$, we reach the desired conclusion. \square

Note that the upper bounds on arc tolls computed in Section 2.1 are still valid for the PATH formulation.

Proposition 11. The following inequalities are valid for PATH:

$$T^k \geq \sum_{a \in \mathcal{P} \cap A} t_a - \sum_{a \in \mathcal{P} \cap A} N_a(1 - \mathcal{X}_{\mathcal{P}}) \quad \forall \mathcal{P} \in P_k, \quad \forall k \in K.$$

Proof. If $\mathcal{X}_{\mathcal{P}} = 1$, T^k is exactly $\sum_{a \in \mathcal{P} \cap A} t_a$ and the inequality is tight. If $\mathcal{X}_{\mathcal{P}} = 0$, the right-hand side of the inequality is negative and the inequality holds trivially. \square

Proposition 12. The following inequalities are valid for PATH:

$$T^k \geq \sum_{\mathcal{P} \in P_k} \sum_{a \in \mathcal{P} \cap A} t_a - \sum_{\mathcal{P} \in P_k} \sum_{a \in \mathcal{P} \cap A} N_a(1 - \mathcal{X}_{\mathcal{P}}) \quad \forall k \in K.$$

Proof. Let \mathcal{P}^1 be the shortest path associated with commodity k . It follows that $\mathcal{X}_{\mathcal{P}} = 1$ if and only if $\mathcal{P} = \mathcal{P}^1$, and the inequality reduces to

$$T^k \geq \sum_{\mathcal{P} \in P_k} \sum_{a \in \mathcal{P} \cap A} t_a - \sum_{\mathcal{P} \in P_k \setminus \mathcal{P}^1} \sum_{a \in \mathcal{P} \cap A} N_a$$

which leads to

$$T^k \geq \sum_{a \in \mathcal{P}^1 \cap A} t_a + \sum_{\mathcal{P} \in P_k \setminus \mathcal{P}^1} \sum_{a \in \mathcal{P} \cap A} t_a - \sum_{\mathcal{P} \in P_k \setminus \mathcal{P}^1} \sum_{a \in \mathcal{P} \cap A} N_a.$$

Now, since N_a is an upper bound on t_a and $T^k = \sum_{a \in \mathcal{P}^1 \cap A} t_a$ by definition of \mathcal{P}^1 , the latter inequality is valid. \square

Proposition 13. The following inequalities are valid for PATH:

$$T^k \leq (1 - \mathcal{X}_{\mathcal{P}})(\bar{\lambda}_k - \underline{\lambda}_k) + \mathcal{X}_{\mathcal{P}} \sum_{a \in \mathcal{P} \cap A} N_a \quad \forall \mathcal{P} \in P_k, \quad \forall k \in K.$$

Proof. If $\mathcal{X}_\emptyset = 1$, the inequality reduces to $T^k \leq \sum_{a \in \mathcal{P} \cap A} N_a$, which trivially holds. If $\mathcal{X}_\emptyset = 0$, the inequality becomes $T^k \leq \bar{\lambda}_k - \underline{\lambda}_k$. The validity of the latter was established in Proposition 9. \square

4. Numerical experiments

Formulations and inequalities have been tested on random grid networks involving 60 nodes and 208 arcs, designed to promote interactions between commodities, and make for problems that are combinatorially challenging (see [1]). In the experiments, c_{ij} is a random number in the interval [2,20], the number of commodities belongs to the set {20, 30, 40}, and the percentage of toll

Table 1
Path formulation: 30 commodities; toll arc densities of 15% and 20%

			Big M	Sharp M	cut1	cut2	cut3	cut12	cut123	# paths	cpu
15%	gap	μ	20.0	16.2	15.7	16.0	16.2	15.5	15.5		
		σ	4.5	4.5	4.6	4.4	4.5	4.5	4.5		
	cpu	μ	11.1	4.2	4.5	4.9	4.4	4.5	4.9	$\mu = 213.0$	$\mu = 3.5$
		σ	21.0	5.9	4.6	8.0	6.1	4.9	5.7	$\sigma = 71.3$	$\sigma = 4.6$
	nodes	μ	3047.7	779.1	680.3	935.7	779.1	557.7	557.7		
		σ	5794.1	1302.3	882.2	1804.2	1302.3	644.4	644.4		
20%	gap	μ	22.0	19.3	18.9	19.1	19.3	18.8	18.8		
		σ	5.6	5.2	5.2	5.2	5.2	5.2	5.2		
	cpu	μ	577.6	64.9	108.1	82.8	67.5	112.5	112.5	$\mu = 612.3$	$\mu = 117.9$
		σ	598.7	70.0	140.3	83.6	69.8	164.3	164.3	$\sigma = 393.7$	$\sigma = 189.4$
	nodes	μ	58 688.3	5363.6	5663.5	7361.0	5363.6	5142.6	5142.6		
		σ	62 403.3	6504.9	5579.1	8994.7	6504.9	5136.5	5136.5		

Table 2
Networks with 20, 30 and 40 commodities; toll arc densities of 10%, 15% and 20%

		20 OD pairs (commodities)			30 OD pairs (commodities)			40 OD pairs (commodities)		
		10%	15%	20%	10%	15%	20%	5%	10%	15%
Arc (G) BM	gap	22.8%	16.5%	16.8%	26.2%	20.2%	22.4%	23%	24.5%	23.6%
	cpu	23	349		99	4182		23	158	
	nodes	140	5943		447	47 522		48	1053	
Arc (G) SM	gap	10.2%	9.5%	10.4%	12%	11%	13.1%	7.6%	11.1%	13.2%
	cpu	13	34	221	59	180	720	13	43	1341
	nodes	33	161	2119	173	839	4391	10	61	8235
Arc (G) SM + C	gap	10.1%	9.3%	10%	11.9%	10.7%	12.8%	7.6%	11%	13%
	cpu	11	39	133	38	171	788	9	40	1491
	nodes	26	135	687	77	511	3179	12	53	6257
Arc (SPGM) BM	gap	22.8%	16.5%	16.8%	26.2%	20.2%	22.4%	23%	24.5%	23.6%
	cpu	< 1	12	621	2	106		< 1	3	
	nodes	60	1264	20 193	314	6550		27	305	
Arc (SPGM) SM	gap	10.2%	9.5%	10.3%	12%	11%	12.9%	7.6%	11.1%	13.2%
	cpu	< 1	4	66	1	18	226	< 1	1	73
	nodes	19	199	1818	75	535	2493	7	39	2590
Arc (SPGM) SM + C	gap	10.2%	9.4%	10.2%	12%	10.7%	12.8%	7.6%	11%	13.1%
	cpu	< 1	6	101	2	26	231	< 1	2	156
	nodes	21	110	1613	58	405	2563	7	55	3621
Path (SPGM) BM	gap	22.8%	16.5%	16.8%	26.2%	20.2%	22%	23%	24.5%	23.6%
	cpu	< 1	3	91 + 83	1	16		< 1	1	194
	nodes	77	936	11 266	301	3049		30	185	38 017
Path (SPGM) SM	gap	16%	13.8%	15%	19.2%	18.6%	19.3%	12.9%	16.8%	19.4%
	cpu	< 1	1	21 + 83	< 1	6	223	< 1	< 1	47
	nodes	56	250	1693	172	779	5364	13	55	4001
Path (SPGM) SM + C	gap	14.3%	12%	12.9%	16.6%	18.4%	18.8%	10.3%	15.2%	18.1%
	cpu	< 1	2	27 + 83	< 1	6	257	< 1	< 1	51
	nodes	43	274	1303	155	558	5143	12	56	3151

arcs belongs to the set {5%, 10%, 15%, 20%}. We ensured that there exists at least one toll-free path for each commodity. All MIP formulations were solved using CPLEX 8.1 set at its default values, and run on a Pentium III (500 MHz) under Linux.

Table 1 compares the three classes of valid inequalities developed in Section 3.2, based on the path formulation. The results are averaged (mean μ and standard deviation σ) over a set of 10 problems. The first column header ‘Big M ’ refers to a value of M set arbitrarily high, while ‘Sharp M ’ refers to the values of M defined in Section 3.2. The next five column headers refer to the ‘Sharp M ’ formulation, tightened by valid inequalities. The last two columns provide the average number of paths generated, together with the generation time. The gap is a measure of the distance between the optimal solution Z_{opt} and the linear relaxation Z_{lp} , defined as $\text{gap} = (Z_{\text{lp}} - Z_{\text{opt}})/Z_{\text{opt}}$. Cpu times are expressed in seconds, and ‘nodes’ represents the number of branch-and-bound nodes. One observes that the choice of tight bounds has a significant impact on both the gap and the resolution time. This is particularly true when the number of commodities is small. Cuts reduce the gap slightly, with the first cut yielding the largest individual improvement. In the remaining experiments, all three cuts were introduced, whenever they were introduced at all.

Table 2 shows results for the arc formulation of Section 2, either for the original network (G) or the reduced network (SPGM), and the path formulation. For each formulation, we consider three cases: Big M (BM), Sharp M (SM) and Sharp M + cuts (SM + C). This yields nine sets of experiments for each network size, each set being averaged over 10 randomly generated instances.

Comparing the arc formulation on the original network and the SPGM, we observe that the gaps are roughly equivalent, which supports the intuition that the combinatorial complexity remains similar. However, both the number of nodes in the branch-and-bound tree and the running time are smaller if we apply the arc formulation to an SPGM network incorporating the new reduction method. In contrast, the ‘Big M ’ results are clearly dominated. Indeed, the optimal solution was not found after three hours in 5 out of 6 cases for networks with 20% of toll arcs. The cuts allow to reduce the number of nodes explored, but have almost no impact on the integrality gap.

For ‘Big M ’, the gaps achieved in TOP-ARCS and PATH are identical. For ‘Sharp M ’, the gap is larger for the path formulation, and is reduced once valid inequalities are incorporated. With or without additional cuts, the PATH formulation could easily be tackled by CPLEX.

5. Conclusion

The bilevel model considered in this paper, apart from being theoretically and computationally challenging, provides a rich framework for the analysis of pricing situations in network-based industries. While the analysis of the basic model is by itself relevant, it would be of interest to consider models that integrate real-life features such as congestion, competition, market segmentation, dynamics and randomness. The success of the bilevel approach will be measured by its tackling efficiently some, if not all, of these aspects.

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