
Research Article

A parallel between two classes of pricing problems in transportation and marketing

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ABSTRACT In this study, we establish a parallel between two classes of pricing problems that have attracted the attention of researchers in marketing, theoretical computer science and operations research, each community addressing issues from its own vantage point. More precisely, we contrast the problems of pricing a network or a product line, in order to achieve maximum revenue, given that customers maximize their individual utility. Throughout the article, we focus on problems that can be formulated as mixed-integer programs.

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INTRODUCTION

The issue of setting the right price for a product lies at the core of the economics or

marketing processes. In this study, we focus on mathematical models that are fairly elementary, yet pose challenges, both theoretical and



computational. More specifically, we contrast two streams of research that have lived in parallel in the marketing and operations research literature for sometime. The first, exemplified by the seminal paper of Dobson and Kalish (1988), consists in devising profit-maximizing prices for a generic product sold on a market of utility-maximizing customers. The second, related to the paper Labbé *et al* (1998), deals with the setting of revenue-maximizing tolls on a subset of links of a transportation network. While the first leads to ‘classical’ mathematical programming formulations, the second one is set within the framework of bilevel programming, a branch of optimization that deals with mathematical programs whose constraint set is described by an auxiliary problem, and is closely related to Stackelberg games in economics.

The aim of this study is to provide an overview of results, either theoretical (worst-case complexity), methodological (applications) or numerical (exact or heuristic algorithms), associated with both the original models and variants thereof. Focusing on models that can be formulated as mixed-integer programs, we highlight the relationships between these models, as well as their treatment by the various communities of researchers. This article is organized as follows: the next section is devoted to product pricing models, the subsequent section to toll setting and the penultimate section to the relationships between the respective models.

PRODUCT PRICING

In the area of pricing that we are concerned with, three essential paradigms have emerged: buyer welfare, seller welfare and share-of-choices. Although all these consider utility-maximizing customers, they involve different objective functions and/or constraints. As they frequently appear jointly in the literature, we first provide a presentation of all three, followed by an overview of the main contributions. Next, various extensions of the seller welfare

problem are presented. Throughout, we do *not* consider models where customer choice is subject to randomness and could be cast within the framework of probabilistic discrete choice theory. The interested reader is referred to Krieger and Green (2002), Shioda *et al* (2007) or Maddah and Bish (2008) for further details with regard to this topic.

Three paradigms

Given products and purchasers, the buyer and seller welfare problems consist in determining which subset of products should be introduced into the market, so as to maximize the sum of the purchasers’ utilities and the seller’s revenue, respectively. The share-of-choices problem differs in that its aim is to devise a ‘profile’ for each product so as to maximize the number of satisfied purchasers.

We now introduce a common notation. Let \mathcal{K} be a set of purchasers and \mathcal{I} a set of products. The purchasers’ preferences for the various products are described by a utility matrix whose elements are u_i^k : $k \in \mathcal{K}$, $i \in \mathcal{I}$. In the buyer and seller welfare problems, a subset of products $\mathcal{S} \subseteq \mathcal{I}$ is first introduced into the market, the cardinality of \mathcal{S} being restricted by an upper bound Y . Each purchaser then selects the product with the largest utility, so far as this utility is positive, otherwise he refrains from buying. Let y_i denote the binary variable that specifies whether a product i is introduced into the market or not, while x_i^k specifies whether product i is selected by purchaser k . The *buyer welfare problem* can then be formulated as the mixed-integer program:

$$\text{BWP: } \max_{x,y} \sum_{k \in \mathcal{K}} \sum_{i \in \mathcal{I}} u_i^k x_i^k$$

subject to:

$$\sum_{j \in \mathcal{I}} u_j^k x_j^k \geq u_i^k y_i \quad \forall k \in \mathcal{K}, \forall i \in \mathcal{I} \quad (1)$$

$$\sum_{i \in \mathcal{I}} x_i^k \leq 1 \quad \forall k \in \mathcal{K} \quad (2)$$

$$x_i^k \leq y_i \quad \forall k \in \mathcal{K}, \forall i \in \mathcal{I} \quad (3)$$

$$\sum_{i \in \mathcal{I}} y_i \leq Y \tag{4}$$

$$x_i^k, y_i \in \{0, 1\} \quad \forall k \in \mathcal{K}, \forall i \in \mathcal{I}. \tag{5}$$

In the above, global utility is maximized, while ensuring by constraints (1) that individual utility-maximizing products are purchased. Constraints (2) force each purchaser to select at most one product, constraints (3) impose that the products selected by the purchasers are available, while constraint (4) provides an upper bound on the number of products that can be introduced into the market.

If one integrates additional parameters v_i^k representing the seller's income resulting from the purchase by customer $k \in \mathcal{K}$ of product $i \in \mathcal{I}$, then we can formulate the *seller welfare problem* as:

$$\begin{aligned} \text{SWP: } \max_{x,y} \sum_{k \in \mathcal{K}} \sum_{i \in \mathcal{I}} v_i^k x_i^k \\ \text{subject to constraints (1) - (5)}. \end{aligned} \tag{6}$$

In the *share-of-choices problem* framework, product description is more detailed. More specifically, a set \mathcal{A} of attributes, which can occur in various levels belonging to the set \mathcal{L}_a , is associated with each product. A profile for product $i \in \mathcal{I}$ is then characterized by the assignment of a level to each attribute of a given product, and is represented by a vector $P = (l_1, l_2, \dots, l_{|\mathcal{A}|})$ of its attribute levels. We let \mathcal{P}_i denote the set of available profiles for product i . On the demand side, each purchaser k assigns a preference value w_{al}^k to level l of attribute a , which are normalized to lie between -1 and 1 . Next, purchaser k selects the product i whose profile $p \in \mathcal{P}_i$ has the largest utility $w^k(p) = w_{1l_1}^k + w_{2l_2}^k + \dots + w_{|\mathcal{A}|l_{|\mathcal{A}|}}^k$, provided that the latter is positive. Otherwise he refrains from buying and is not 'satisfied'. On the supply side, a firm determines profiles $p \in \mathcal{P}_i$ for each product $i \in \mathcal{I}$ so as to maximize the number of satisfied customers.

In the mathematical programming formulation of the problem, binary variables z^k specify whether or not the purchaser k is satisfied,

while variables q_{ip} are set to 1 if the profile $p \in \mathcal{P}_i$ is assigned to product i . With the help of the assignment variables x_i^k : $k \in \mathcal{K}$, $i \in \mathcal{I}$ introduced earlier, the share-of-choices problem can be formulated as:

$$\text{SCP: } \max_{q,x,z} \sum_{k \in \mathcal{K}} z^k$$

subject to:

$$\sum_{p \in \mathcal{P}_i} q_{ip} = 1 \quad \forall i \in \mathcal{I} \tag{7}$$

$$\begin{aligned} w^k(p) q_{ip} x_i^k &\geq 0 \\ \forall k \in \mathcal{K}, \forall i \in \mathcal{I}, \forall p \in \mathcal{P}_i \end{aligned} \tag{8}$$

$$\begin{aligned} \sum_{j \in \mathcal{I}, \bar{p} \in \mathcal{P}_j} w^k(\bar{p}) q_{j\bar{p}} x_j^k &\geq w^k(p) z^k q_{ip} \\ \forall k \in \mathcal{K}, \forall i \in \mathcal{I}, \forall p \in \mathcal{P}_i \end{aligned} \tag{9}$$

$$\sum_{i \in \mathcal{I}} x_i^k \leq 1 \quad \forall k \in \mathcal{K} \tag{10}$$

$$x_i^k \leq z^k \quad \forall k \in \mathcal{K}, \forall i \in \mathcal{I} \tag{11}$$

$$\begin{aligned} q_{ip}, x_i^k, z^k &\in \{0, 1\} \\ \forall k \in \mathcal{K}, \forall i \in \mathcal{I}, \forall p \in \mathcal{P}_i. \end{aligned} \tag{12}$$

Constraints (7) ensure that a profile is assigned to each product. Constraints (8) impose that a product can be selected by a purchaser only if the corresponding profile has positive utility. Note that, although this requirement was implicitly ensured in the formulations of the buyer and seller welfare problems, it is not the case for the share-of-choices problem. As before, constraints (9) imply that a purchaser selects the product maximizing its own utility, provided that the latter is positive. Finally, constraints (10) ensure that a purchaser buys at most one product while, according to (11), a customer who buys a product is characterized as 'satisfied'.

Several variants of the above problems have been investigated in the literature. We start with Green and Krieger (1985), who consider the buyer welfare problem under the



assumption that the cardinality of the subset \mathcal{S} is fixed, that is, $\sum_{i \in \mathcal{I}} y_i = |\mathcal{S}|$. In view of the intractability of enumerating all feasible solutions, the authors propose for its solution heuristics based on a greedy approach, as well as Lagrangean relaxation. Whereas these methods proved effective for the buyer welfare problem, they did not perform well for the seller welfare problem. According to these authors, neither Lagrangean relaxation nor exact methods can address with some success instances of realistic sizes. Moreover, a greedy heuristic approach yielded very poor results.

For the single-product share-of-choices problem, Kohli and Krishnamurti (1987) showed that a fast dynamic programming heuristic outperformed a Lagrangean relaxation heuristic in terms of both computational time and quality of solutions. On the theoretical side, \mathcal{NP} -hardness of this problem was proved by Kohli and Krishnamurti (1989). Based upon a graph representation of the problem, these authors developed two heuristic procedures based, respectively, on dynamic programming and shortest path computations. Although both heuristics possess arbitrarily bad worst-case bounds, they exhibited good practical performance, the dynamic programming procedure performing better than that based on shortest paths.

Kohli and Sukumar (1990) presented dynamic programming heuristics for the buyer welfare, seller welfare and share-of-choices problems, assuming multi-product sets for the latter, and a multi-attribute structure similar to that of the share-of-choices problem for buyer and seller welfare (levels have to be determined for each attribute of the products). For multi-attribute buyer welfare, seller welfare and share-of-choices problems, Nair *et al* (1995) implemented a beam search heuristic, that is, breadth-first search with no backtracking, with breadth limited to a given number of promising nodes in the enumeration tree, obtaining results superior to those of Kohli and Sukumar. Keeping with meta-

heuristics, Alexouda and Paparrizos (2001) implemented genetic algorithms for solving the multi-attribute seller welfare problem. The method outperformed the beam search heuristic of Nair *et al*, both in terms of solution quality and computing times. Finally, observing that constraint (1) is only active when there exists $j \in \mathcal{I}$ such that $x_j^k = 1$ and $u_j^k < u_i^k$ (then one must have $y_i = 0$), McBride and Zufryden (1988) replaced constraint (1) of the seller welfare problem by the equivalent:

$$y_i + x_j^k \leq 1 \\ \forall k \in \mathcal{K}, \forall i, j \in \mathcal{I} : i \neq j, u_i^k > u_j^k.$$

This reformulation, which could be solved by a generic mathematical solver, yielded good results on small- to medium-size instances.

Simultaneous profit and bundle pricing problems

In a seminal paper, Dobson and Kalish (1988) considered an extension of the seller welfare problem, in which price variables π_i , $i \in \mathcal{I}$ are explicitly considered. The *profit problem* consists in determining a subset of products $\mathcal{S} \subseteq \mathcal{I}$ and the corresponding product prices yielding maximum profit for the seller. Note that, in contrast with the model of Green and Krieger (1985), the number of products in subset \mathcal{S} is endogenous. The population of purchasers is partitioned into segments, such that $k \in \mathcal{K}$ now denotes a purchaser segment, rather than a single purchaser as previously. Each segment is characterized by its total demand η^k : $k \in \mathcal{K}$. Further, a reservation price r_i^k provides a measure of the value of product i for segment k of the customer population. The utility u_i^k associated to segment $k \in \mathcal{K}$ and product $i \in \mathcal{I}$ is then defined as the difference between the reservation price r_i^k and the product price π_i . Also, the cost of introducing a product i into the market induces a fixed cost f_i for the seller. Finally, in order to manage the case in which a segment refrains of buying because all corresponding utilities are negative, a ‘null’ product

is introduced. The mixed-integer programming formulation of this pricing problem is as follows:

$$\text{PP: } \max_{p, x, \gamma} \sum_{k \in \mathcal{K}, i \in \mathcal{I}} \eta^k \pi_i x_i^k - \sum_{i \in \mathcal{I}} f_i \gamma_i$$

subject to:

$$\sum_{j \in \mathcal{I}} (r_j^k - \pi_j) x_j^k \geq (r_i^k - \pi_i) \gamma_i \quad \forall k \in \mathcal{K}, \forall i \in \mathcal{I} \quad (13)$$

$$\sum_{i \in \mathcal{I}} x_i^k = 1 \quad \forall k \in \mathcal{K} \quad (14)$$

$$x_i^k \leq \gamma_i \quad \forall k \in \mathcal{K}, \forall i \in \mathcal{I} \quad (15)$$

$$\sum_{i \in \mathcal{I}} \gamma_i \leq Y \quad (16)$$

$$\pi_0 = 0 \quad (17)$$

$$\pi_i \geq 0 \quad \forall i \in \mathcal{I} \quad (18)$$

$$x_i^k, \gamma_i \in \{0, 1\} \quad \forall k \in \mathcal{K}, \forall i \in \mathcal{I}. \quad (19)$$

For the numerical solution of this problem, the authors proposed a ‘reverse greedy heuristic’ that exploits the underlying structure of the problem. More precisely, the optimal prices corresponding to a given assignment of products to customers can be computed in polynomial time by solving the shortest path problems. The procedure stops when no further improvement can be achieved. The selection criterion for choosing the segment to reassign at each iteration is the seller’s profit, that is, among all segments that prevent the seller from increasing its prices, one selects the one that would lead to the largest improvement of the objective function. In a related paper, Dobson and Kalish (1993) adapted their results to the buyer welfare and profit problems. They formally show that the buyer welfare and the plant location problems are equivalent; hence the former is \mathcal{NP} -hard. They also propose heuristics for this problem: greedy heuristic (start with an empty subset \mathcal{S} of

products and append products one at a time in \mathcal{S} , whenever this increases profit), greedy interchange (greedy assignment of purchasers to products, followed by pairwise product interchanges until no improvement is possible), reverse greedy (see Dobson and Kalish, 1988) and reverse greedy interchange. These authors also showed that the profit problem is \mathcal{NP} -hard through a reduction involving Vertex Cover. For this problem, they considered the reverse greedy heuristic introduced in Dobson and Kalish (1988), as well as a greedy heuristic.

These algorithms have been adapted by Shioda *et al* (forthcoming) to the *full profit problem* variant, where all products are available ($Y = \infty$, $\mathcal{S} = \mathcal{I}$ and $\gamma_i = 1$ for all i in \mathcal{I}) and fixed costs of putting a product on the market are set to zero. These authors also derived a novel linear mixed-integer formulation for which they derived valid inequalities.

Hanson and Martin (1990) considered the profit problem with zero fixed costs in the context of ‘global elements’, such as data-processing software, that possess up to n components. One can then form $2^n - 1$ subsets or ‘products’. For this variant of the *bundle pricing problem*, they assumed that product prices are sub-additive, that is, whenever product $i \in \mathcal{I}$ is the union of other products, then its price is lower than the sum of the prices of these other products:

$$\pi_i \leq \sum_{j \in \mathcal{S}} \pi_j \quad \forall i \in \mathcal{I}, \forall \mathcal{S} \subseteq \mathcal{I} : i = \cup_{j \in \mathcal{S}} j \quad (20)$$

Based on a mixed-integer formulation of the problem, they devised an exact solution method. Whenever the number of components n is large, the authors restricted the number of subsets, observing that there often exists a ‘key component’ that belongs to all subsets supplied on the market, and to which additional components could be appended. In Guruswani *et al* (2005), all products of the bundle pricing problem are considered, and sub-additivity is not assumed. The problem is shown to be



APX -hard¹ through a reduction to Vertex Cover, and a logarithmic approximation algorithm is proposed. Approximation or polynomial time algorithms, together with other theoretical results, were also provided for specific cases.

Finally, Nichols and Venkataramanan (2005) considered a formulation of the *conjoint buyer welfare and profit problem* similar to the one suggested by Dobson and Kalish (1988, 1993), in which the objective function consists of a weighted sum of seller's income and purchasers' utilities (that is, a weighted sum of both objective functions from the buyer welfare and profit problems). Three heuristics were proposed for its solution. The first one is a 'pure' genetic algorithm used for comparison purposes. Next, a genetic procedure was used to generate product prices, while utility-maximizing assignments were obtained by applying a partial enumeration (branch-and-bound) procedure. A third heuristic randomly selects the products to be introduced into the market. Next, for a given set of flows, corresponding prices are obtained by solving an inverse problem. The results show that, on large instances, the relaxation methods perform better than a pure genetic algorithm, thus

prompting the development of approaches where only a subset of 'hard' variables is genetically treated.

We close this section with three figures (Figures 1, 2 and 3) that illustrate a taxonomy of research pertaining to the buyer welfare, seller welfare and share-of-choices problems, respectively.

NETWORK PRICING

In this section, we address the issue of pricing the arcs of a general transportation network. A generic *network pricing problem* is introduced in the section 'Problem definition' together with the main contributions from the literature. Several variants involving specific network structures are presented in the section 'Highway pricing'.

Problem definition

Let us consider a transportation network composed of a set of nodes (cities), a set of arcs (routes) linking pairs of nodes, and commodities corresponding to sets of commuters having identical origins and destinations. In addition to a fixed travel cost associated with every arc of the network, tolls are imposed

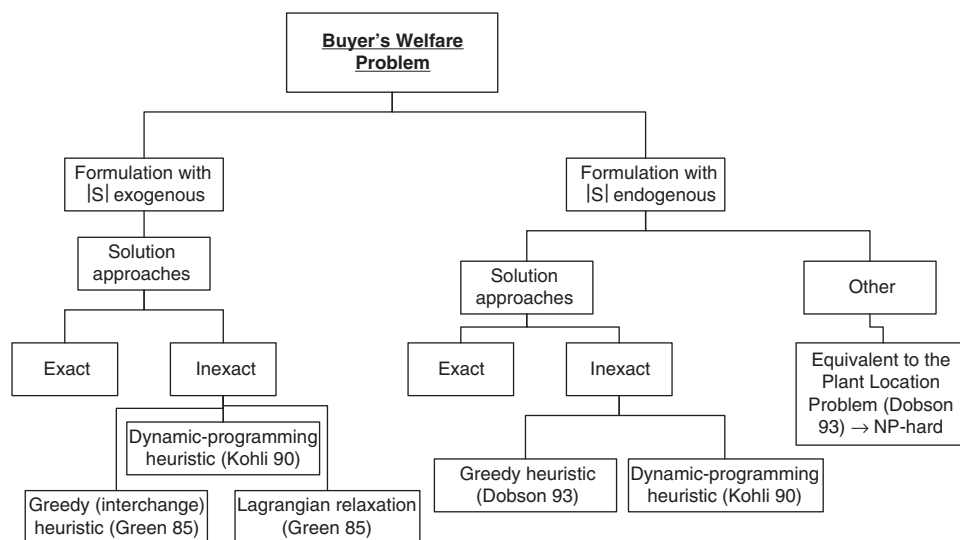


Figure 1: Taxonomy for the buyer welfare problem.

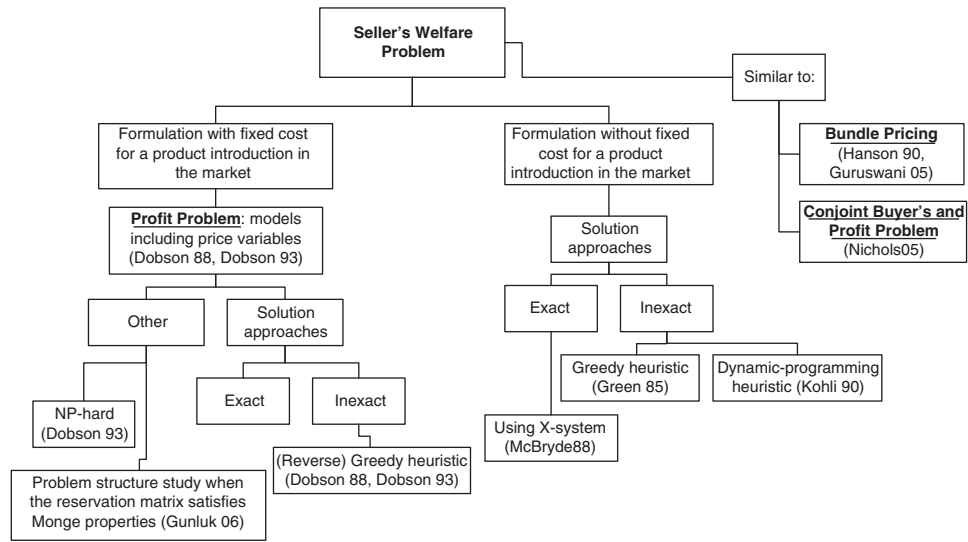


Figure 2: Main contributions to the seller welfare problem.

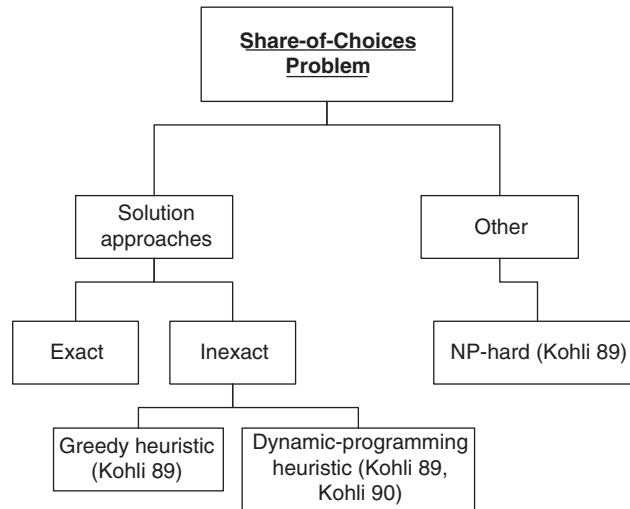


Figure 3: Main contributions to the share-of-choices problem.

by the authority on a specified subset of arcs of the network. The *network pricing problem* consists in devising the toll levels in order to maximize the authority's revenues. Reacting to tolls, each commodity travels on the cheapest path² from its origin to its destination, with respect to a cost set to the sum of tolls and fixed costs.

As tolls are set before flows are assigned, the problem belongs to a class of hierarchical, sequential and non-cooperative optimization

problems known as bilevel programs, in which a leader (the authority) integrates within its optimization process the reaction of a follower (the commodities) to its decisions. Bilevel programming and the related mathematical programs with equilibrium constraints have been the topic of several studies, for which the reader is referred to the surveys of Demepe (2002), Marcotte and Savard (2005), Colson et al (2007) and Luo et al (1996).



Bilevel network pricing was first introduced by Labbé *et al* (1998). They considered a multi-commodity network defined by a node set \mathcal{N} , an arc set $\mathcal{I} \cup \mathcal{J}$ and a set of commodities $\{(o^k, d^k): k \in \mathcal{K}\}$, each one endowed with a demand η^k . Fixed costs c_i are supplied for all arcs $i \in \mathcal{I} \cup \mathcal{J}$, and provide a measure of delay, distance or gas consumption. Let \mathcal{I} be the subset of arcs i upon which tolls t_i can be added to the original fixed-cost vector c , and \mathcal{J} the complementary subset of toll free arcs. Upon the introduction of variables x_i^k that specify the unit flows on commodities $k \in \mathcal{K}$ and arcs $i \in \mathcal{I} \cup \mathcal{J}$, the network pricing problem can be formulated as the bilevel program (Labbé *et al*, 1998):

$$\text{NP: } \max_{t, x} \sum_{k \in \mathcal{K}} \sum_{i \in \mathcal{I}} \eta^k t_i x_i^k$$

subject to:

$$t_i \geq 0 \quad \forall i \in \mathcal{I} \quad (21)$$

$$x \in \arg \min_x \sum_{k \in \mathcal{K}} \left(\sum_{i \in \mathcal{I}} (c_i + t_i) x_i^k + \sum_{i \in \mathcal{J}} c_i x_i^k \right) \quad (22)$$

subject to:

$$\sum_{i \in n^- \cap (\mathcal{I} \cup \mathcal{J})} x_i^k - \sum_{i \in n^+ \cap (\mathcal{I} \cup \mathcal{J})} x_i^k = \begin{cases} -1 & \text{if } n = o^k \\ 1 & \text{if } n = d^k \\ 0 & \text{otherwise} \end{cases} \quad (23)$$

$$\forall k \in \mathcal{K}, \forall n \in \mathcal{N}$$

$$0 \leq x_i^k \leq 1 \quad \forall k \in \mathcal{K}, \forall i \in \mathcal{I}, \quad (24)$$

where n^- (respectively n^+) denotes the set of arcs having node n as their head (respectively tail). Note that, because the constraint matrix associated with a shortest path problem is totally unimodular, the lower level solutions can be restricted to origin-destination paths carrying either no flow or the total origin-destination flow.

Alternatively to the above arc flow formulation, one may express the optimality of the

lower-level problem in terms of path flows. To this aim, let us introduce the set \mathcal{P}^k of feasible paths associated with commodity $k \in \mathcal{K}$. This yields a binary program that can be linearized by introducing variables $p_i^k: k \in \mathcal{K}, i \in \mathcal{I}$ that represent the actual unit revenue raised from each arc-commodity, that is, $p_i^k = t_i x_i^k$ for all $k \in \mathcal{K}$ and $i \in \mathcal{I}$. Based on these remarks, one can derive a Mixed Integer Linear Program for the problem (see Labbé *et al*, 1998; Heilporn *et al*, 2006):

$$\text{NP2: } \max \sum_{k \in \mathcal{K}} \sum_{i \in \mathcal{I}} \eta^k p_i^k$$

subject to:

$$\sum_{i \in n^- \cap (\mathcal{I} \cup \mathcal{J})} x_i^k - \sum_{i \in n^+ \cap (\mathcal{I} \cup \mathcal{J})} x_i^k = \begin{cases} -1 & \text{if } n = o^k \\ 1 & \text{if } n = d^k \\ 0 & \text{otherwise} \end{cases} \quad (25)$$

$$\forall k \in \mathcal{K}, \forall n \in \mathcal{N}$$

$$\sum_{i \in \mathcal{I}} (c_i x_i^k + p_i^k) + \sum_{i \in \mathcal{J}} c_i x_i^k \leq \sum_{i \in \mathcal{I} \cap p} (c_i + t_i) + \sum_{i \in \mathcal{J} \cap p} c_i \quad (26)$$

$$\forall k \in \mathcal{K}, \forall p \in \mathcal{P}^k$$

$$p_i^k \leq M_i^k x_i^k \quad \forall k \in \mathcal{K}, \forall i \in \mathcal{I} \quad (27)$$

$$t_i - p_i^k \leq N_i (1 - x_i^k) \quad \forall k \in \mathcal{K}, \forall i \in \mathcal{I} \quad (28)$$

$$p_i^k \leq t_i \quad \forall k \in \mathcal{K}, \forall i \in \mathcal{I} \quad (29)$$

$$p_i^k \geq 0 \quad \forall k \in \mathcal{K}, \forall i \in \mathcal{I} \quad (30)$$

$$x_i^k \in \{0, 1\} \quad \forall k \in \mathcal{K}, \forall i \in \mathcal{I} \quad (31)$$

$$x_i^k \geq 0 \quad \forall k \in \mathcal{K}, \forall i \in \mathcal{J}, \quad (32)$$

where M_i^k and N_i are 'big-M' constants that must theoretically be set to an upper bound on the largest possible revenue associated with arc i and commodity k . In this formulation, (25)

describe the polyhedron of feasible flows (the flow conservation equations), constraints (26) impose that the cost of an optimal path for a commodity $k \in \mathcal{K}$ is smaller than or equal to the cost of any other path for the associated commodity, and (27), (28) and (29) come from the linearization of the model and ensure that $p_i^k = t_i x_i^k$ for all $k \in \mathcal{K}, i \in \mathcal{I}$.

The network pricing problem has been investigated by several researchers. From the theoretical standpoint, Roch *et al* (2005) and Grigoriev *et al* (2005) have proved its \mathcal{NP} -hardness, under various restrictive conditions. However, some particular cases are polynomially solvable, such as the network pricing problem with a single toll arc (see Brotcorne *et al*, 2000). In fact, Van Hoesel *et al* (2003) showed that, when the number of toll arcs is bounded, the optimal solution can be obtained by solving a polynomial number of linear programs. The latter authors also considered other polynomially solvable variants.

In contrast with the ‘arc formulation’ NP2, Bouhtou *et al* (2007) and Didi-Biha *et al* (2006) have proposed formulations involving path flow variables. These are based on a graph reduction whose size is, in practice, much smaller than that of the original graph.

Unfortunately, no off-the-shelf software can address the above formulations for large-scale instances, mainly because of the poor quality of the upper bound obtained by relaxing the integrality requirements in mixed-integer formulations. To overcome this difficulty, several avenues have been investigated. By computing upper bounds on the toll arcs, Dewez *et al* (2008) obtained tight values for the constants M_i^k, N_i : $k \in \mathcal{K}, i \in \mathcal{I}$ in formulation NP2, while simultaneously introducing valid inequalities for both the arc and path formulations. The authors showed that the proposed bounds allow to halve the duality gap at the root node of the branch-and-bound tree, whereas the valid cuts allow a further reduction of the number of explored nodes, as well as a reduction of computing time.

Other improvements can be achieved by focusing on the efficient resolution of the inverse problem, which consists in finding revenue-maximizing tolls compatible with a given flow assignment. As the latter possesses the structure of a side-constrained flow problem, it is amenable to column generation (see Cirinei, 2007). Numerical tests indicated that the method significantly speeds up the solution process. Coupled with an efficient generation of the lower level solutions and a clever use of data structures, the algorithm also improves sharply the upper bounds on the revenue.

On larger instances, Brotcorne *et al* (2001) presented two heuristics: a quick and greedy method that sets tolls sequentially over the arcs, and a primal-dual approach based on penalizing the complementarity constraints that occur when the lower-level problem is replaced by its primal-dual optimality conditions. A similar approach was applied by Brotcorne *et al* (2000) in the framework of a single-commodity transportation problem. From a different perspective, Cirinei (2007) implemented a tabu metaheuristic that exploits the network structure of the lower-level problem. He obtained better results than Brotcorne *et al* (2001). Let us also mention a paper of Roch *et al* (2005), in which an approximation algorithm is proposed for the single-commodity network pricing problem together with worst case lower bounds.

Brotcorne *et al* (2008) addressed an extension of the network pricing problem in which the leader must simultaneously determine which toll arcs belong to the network, and is reminiscent of the profit problem introduced by Dobson and Kalish (1988). They proposed for its solution a Lagrangean-based heuristic. In the next subsection, we provide an overview of variants of the basic network pricing problem.

Highway pricing

The problem of maximizing the revenue raised from a toll highway can be cast in the framework of network pricing. For instance, Dewez (2004), Heilporn (2008) and Heilporn



et al (2009, forthcoming) have considered network topologies that reflect the features of an actual toll highway. More specifically, they consider structured networks composed of a toll path (the highway), toll free arcs linking the origin and destination nodes, and toll free arcs to and from the highway. Commodities (commuters) either travel on the shortest toll free path, or take the highway, using shortest toll free paths to and from it. Throughout, we assume that commodities who have left the highway are not allowed to re-enter, which implies that paths are uniquely determined by their respective entry and exit nodes. In the case where tolls are set with respect to entry–exit pairs, the situation can be represented by a complete toll subgraph where every single feasible toll subpath (that is, a subpath on the highway) from any origin to any destination in the network is represented by exactly one toll arc. This highway structure is illustrated in Figure 4, in which toll arcs are dashed.

Precisely, for each toll arc $i \in \mathcal{I}$, let $\text{tail}(i)$, $\text{head}(i) \in \mathcal{N}$ denote its tail and head nodes, respectively. For each commodity $k \in \mathcal{K}$ and for

each toll arc $i \in \mathcal{I}$, let c_i^k denote the fixed cost on the corresponding path $o^k \rightarrow \text{tail}(i) \rightarrow \text{head}(i) \rightarrow d^k$, where $\text{tail}(i)$, $\text{head}(i) \in \mathcal{N}$ are the entry and exit nodes on the highway, respectively. The fixed cost on the toll free path $o^k \rightarrow d^k$ is denoted by c_{od}^k . For each commodity $k \in \mathcal{K}$ and for each toll arc $i \in \mathcal{I}$, variable x_i^k represents the unit flow on the corresponding path $o^k \rightarrow \text{tail}(i) \rightarrow \text{head}(i) \rightarrow d^k$, while variable t_i denotes the toll on the toll arc i . This yields the bilevel program (Labbé *et al*, 1998; Dewez, 2004):

$$\text{HP: } \max_{t,x} \sum_{k \in \mathcal{K}} \sum_{i \in \mathcal{I}} \eta^k t_i x_i^k$$

subject to:

$$t_i \geq 0 \quad \forall i \in \mathcal{I} \quad (33)$$

$$x \in \arg \min_x \sum_{k \in \mathcal{K}} \left(\sum_{i \in \mathcal{I}} (c_i^k + t_i) x_i^k + c_{od}^k (1 - \sum_{i \in \mathcal{I}} x_i^k) \right) \quad (34)$$

subject to:

$$\sum_{i \in \mathcal{I}} x_i^k \leq 1 \quad \forall k \in \mathcal{K} \quad (35)$$

$$0 \leq x_i^k \leq 1 \quad \forall k \in \mathcal{K}, \forall i \in \mathcal{I}, \quad (36)$$

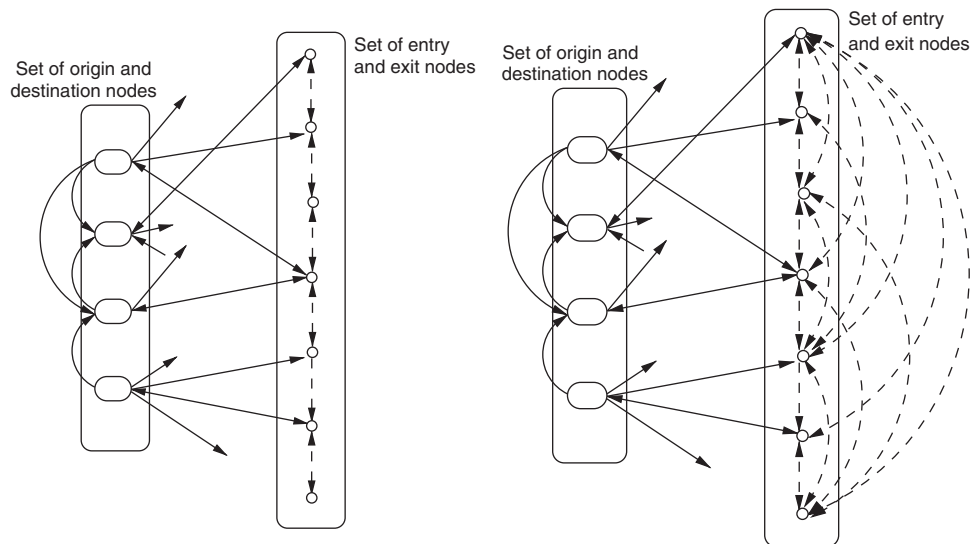


Figure 4: Transformation of the highway (on the left) into a complete toll subgraph (on the right).

Note: Only the shortest toll free paths need to be represented in addition to the highway, as commodities are assigned to the shortest paths.

Constraints (35) ensure flow conservation. Indeed, if $x_i^k = 0$ for all toll arcs $i \in \mathcal{I}$, the corresponding commodity is assigned to the shortest toll free path from its origin to its destination.

Similar to the network pricing problem, Heilporn *et al* (forthcoming) expressed the optimality of the lower-level problem in terms of path flows. Variables $p_i^k: k \in \mathcal{K}, i \in \mathcal{I}$ are also introduced so that $p_i^k = t_i x_i^k$ for all $k \in \mathcal{K}$ and $i \in \mathcal{I}$. This yields the linear mixed-integer program (see Dewez, 2004; Heilporn, 2008):

$$\text{HP2 : } \max_P \sum_{k \in \mathcal{K}} \sum_{i \in \mathcal{I}} \eta^k p_i^k$$

subject to:

$$\sum_{i \in \mathcal{I}} x_i^k \leq 1 \quad \forall k \in \mathcal{K} \quad (37)$$

$$\sum_{j \in \mathcal{I}} (p_j^k + c_j^k x_j^k) + c_{od}^k (1 - \sum_{j \in \mathcal{I}} x_j^k) \leq c_i^k + t_i \quad \forall k \in \mathcal{K}, \forall i \in \mathcal{I} \quad (38)$$

$$p_i^k \leq M_i^k x_i^k \quad \forall k \in \mathcal{K}, \forall i \in \mathcal{I} \quad (39)$$

$$t_i - p_i^k \leq N_i (1 - x_i^k) \quad \forall k \in \mathcal{K}, \forall i \in \mathcal{I} \quad (40)$$

$$p_i^k \leq t_i \quad \forall k \in \mathcal{K}, \forall i \in \mathcal{I} \quad (41)$$

$$p_i^k \geq 0 \quad \forall i \in \mathcal{I} \quad (42)$$

$$x_i^k \in \{0, 1\} \quad \forall k \in \mathcal{K}, \forall i \in \mathcal{I}, \quad (43)$$

where $M_i^k: k \in \mathcal{K}, i \in \mathcal{I}$ and $N_i: i \in \mathcal{I}$ are suitably large constants. By the flow constraints (37), each commodity $k \in \mathcal{K}$ chooses at most one toll path i . By constraints (38), the cost of the optimal path for a commodity $k \in \mathcal{K}$ is smaller than the cost of any other path for this commodity. Constraints (39)–(41) ensure that the actual revenue is consistent with the commodity revenue, that is, $p_i^k = t_i x_i^k$ for all $k \in \mathcal{K}, i \in \mathcal{I}$.

Next, for the sake of realism, additional constraints on the toll structure are appended to the model. The triangle constraints (44) ensure

that it cannot be profitable to leave and re-enter the highway, whereas the monotonicity constraints (45) force the toll along a path to exceed the toll on any of its subpaths, that is:

$$\begin{aligned} t_i &\leq t_j + t_l \quad \forall i, j, l \in \mathcal{I}: \\ \text{tail}(i) &= \text{tail}(j), \text{head}(j) \\ &= \text{tail}(l), \text{head}(l) = \text{head}(i) \end{aligned} \quad (44)$$

$$\begin{aligned} t_i \geq t_j \quad \forall i, j \in \mathcal{I} : & \text{tail}(i) = \text{tail}(j) < \text{head}(i) = \text{head}(j) + 1 \\ & \text{or } \text{tail}(i) = \text{tail}(j) - 1 < \text{head}(i) = \text{head}(j) \\ & \text{or } \text{tail}(i) = \text{tail}(j) > \text{head}(i) = \text{head}(j) - 1 \\ & \text{or } \text{tail}(i) = \text{tail}(j) + 1 > \text{head}(i) = \text{head}(j) \end{aligned} \quad (45)$$

In the absence of monotonicity constraints, the single-commodity problem is trivially solved. Indeed, the toll arc yielding the largest potential revenue for the leader can be found in $O(n)$ -time, and the optimal solution consists in setting its toll to the maximum value compatible with the toll free path. Other tolls are set to arbitrarily large values. For multi-commodity problems, Dewez (2004) proposed a solution approach based on the enumeration of the lower level solutions. Unfortunately, the time required to solve the problem to optimality grows exponentially with the number of commodities and the number of nodes in the network. Alternatively, the author proposed several flow selection heuristics. Once flows are determined, optimal tolls can be recovered through the solution of an inverse problem, which consists in determining revenue maximizing tolls compatible with a given flow assignment, and has polynomial complexity.

More recently, Heilporn *et al* (forthcoming) proved the \mathcal{NP} -hardness of the highway pricing problem, and derived strong valid inequalities. Focusing on two-commodity problems, Heilporn *et al* (2009) showed that classes of valid inequalities, as well as classes of constraints in the original formulations, define facets of the convex hull of feasible solutions. In the absence of triangle and monotonicity



constraints, a complete description of the convex hull of feasible solutions was also obtained for single-commodity problems (see Heilporn *et al*, forthcoming). Numerical experiments showed that appending the valid inequalities to HP2 yielded significant decreases of the optimality gap, computing time and number of nodes in the resulting branch-and-cut algorithm.

Let us mention that Grigoriev *et al* (2005) considered a model where commuters select at most one toll arc, which is equivalent to a highway pricing problem involving neither triangle nor monotonicity constraints. As the resulting topology is reminiscent of bridges crossing a river, they refer to it as the *cross river pricing problem*. The authors proved that this instance is \mathcal{NP} -hard. They also showed that uniform pricing yields an n -approximation scheme, where n is the number of toll arcs and

that, under further assumptions, uniform pricing yields an $O(\log n)$ -approximation algorithm.

Contributions to the network pricing problem are summarized in Figure 5.

RELATIONSHIPS BETWEEN PRODUCT AND NETWORK PRICING

At this point in the study, it should be clear that product and network pricing, although they have drawn the attention of different scientific communities, are closely related. Indeed, a common thread to welfare, profit and toll problems lies in the explicit consideration of rational, utility-maximizing customers. However, there are some differences. For instance, in most models presented in the section ‘Product pricing’, the set of products is

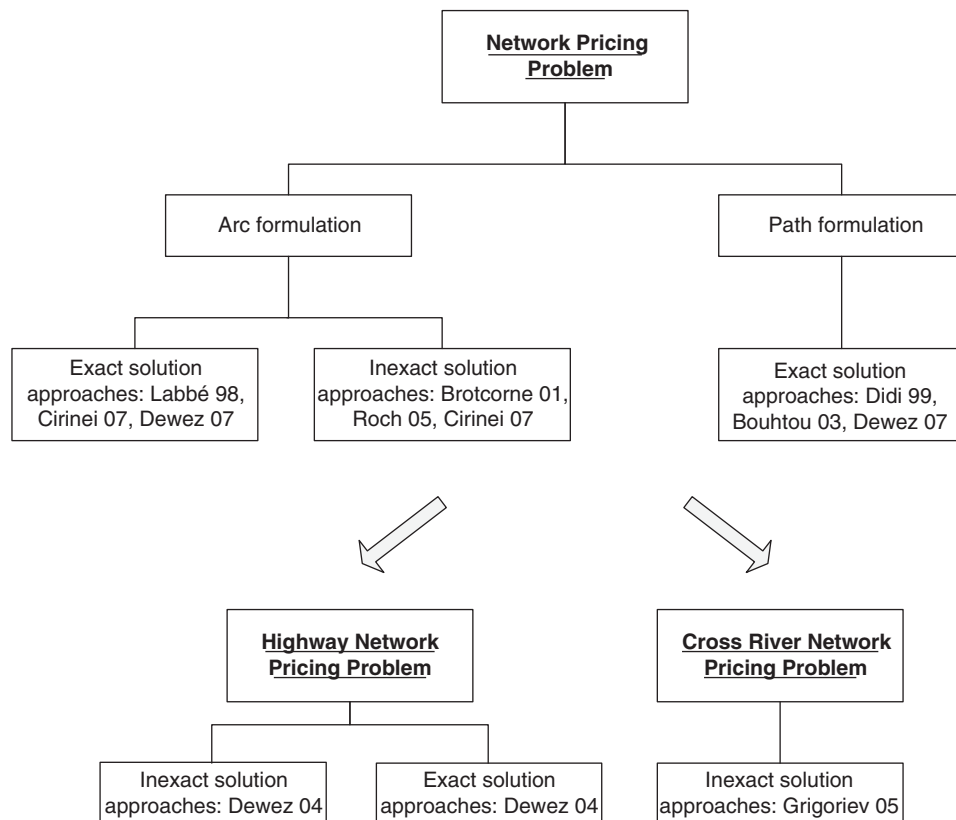


Figure 5: Main contributions to the network pricing problem.

exogenous, whereas this is not the case for network pricing where products correspond to paths, whose number is exponentially large and cannot be enumerated. This has an impact both on model formulation and on numerical resolution. On the other hand, there is a clear parallel between the seller welfare problem and the highway pricing problem, both models having to do with revenue maximization in the context of utility-maximizing purchasers or cost-minimizing commuters, respectively. Note that the full profit problem, where all products are offered, is equivalent to the highway pricing problem. Indeed, let us rewrite the highway pricing problem in its nonlinear form, using the fact that $p_i^k = t_i x_i^k$ in model HP2. One obtains:

$$\text{HP2 - NL: } \max_{t,x} \sum_{k \in \mathcal{K}} \sum_{i \in \mathcal{I}} \eta^k t_i x_i^k$$

subject to:

$$\sum_{i \in \mathcal{I}} x_i^k \leq 1 \quad \forall k \in \mathcal{K} \quad (46)$$

$$\sum_{j \in \mathcal{I}} (t_j + c_j^k) x_j^k + c_{od}^k (1 - \sum_{j \in \mathcal{I}} x_j^k) \leq c_i^k + t_i \quad \forall k \in \mathcal{K}, \forall i \in \mathcal{I} \quad (47)$$

$$t_i \geq 0 \quad \forall i \in \mathcal{I} \quad (48)$$

$$x_i^k \in \{0, 1\} \quad \forall k \in \mathcal{K}, \forall i \in \mathcal{I}. \quad (49)$$

Next, a binary variable x_{od}^k is appended to the model, which is equal to 1 if commodity k is assigned to the toll free arc (o^k, d^k) and 0 otherwise. Constraints (46) and (47) become

$$\sum_{i \in \mathcal{I}} x_i^k + x_{od}^k = 1 \quad \forall k \in \mathcal{K} \quad (50)$$

$$\sum_{j \in \mathcal{I}} (t_j + c_j^k) x_j^k + c_{od}^k x_{od}^k \leq c_i^k + t_i \quad \forall k \in \mathcal{K}, \forall i \in \mathcal{I} \quad (51)$$

Note that (51) could also be written as:

$$\sum_{j \in \mathcal{I} \cup \{od\}} (c_{od}^k - c_j^k - t_j) x_j^k \geq c_{od}^k - c_i^k - t_i \quad \forall k \in \mathcal{K}, \forall i \in \mathcal{I} \quad (52)$$

if one considers a zero toll $t_{od} = 0$. Now one can match purchaser segments with commodities, and products with toll arcs. The product prices $\pi_i; i \in \mathcal{I}$ then correspond to the tolls on arcs $t_i; i \in \mathcal{I}$. Also, the assignment of a commodity to a toll free arc corresponds to the purchase of a null product at zero cost by a demand segment, that is, this segment refrains from buying. Finally, the reservation price r_i^k of purchaser segment k for product i corresponds to the toll window $c_{od}^k - c_i^k$. These correspondences are summarized in Table 1.

One obtains:

$$\text{FPP: } \max_{t,x} \sum_{k \in \mathcal{K}} \sum_{i \in \mathcal{I}} \eta^k \pi_i x_i^k$$

subject to:

$$\sum_{i \in \mathcal{I} \cup \{od\}} x_i^k = 1 \quad \forall k \in \mathcal{K} \quad (53)$$

$$\sum_{j \in \mathcal{I} \cup \{od\}} (r_j^k - \pi_j) x_j^k \geq r_i^k - \pi_i \quad \forall k \in \mathcal{K}, \forall i \in \mathcal{I} \quad (54)$$

$$\pi_{od} = 0 \quad (55)$$

Table 1: Notation for the full profit and highway pricing problems

Full profit problem	Highway pricing problem
Purchaser segments $k \in \mathcal{K}$	Commodities $k \in \mathcal{K}$
Products $i \in \mathcal{I}$	Toll arcs $i \in \mathcal{I}$
Reservation prices $r_i^k; k \in \mathcal{K}, i \in \mathcal{I}$	Toll windows $c_{od}^k - c_i^k; k \in \mathcal{K}, i \in \mathcal{I}$
Prices $\pi_i; i \in \mathcal{I}$	Tolls $t_i; i \in \mathcal{I}$
Assignments $x_i^k; k \in \mathcal{K}, i \in \mathcal{I}$	Unit flows $x_i^k; k \in \mathcal{K}, i \in \mathcal{I}$

**Table 2:** Complexity classification

Network pricing problem	\mathcal{NP} -hard (Grigoriev <i>et al</i> , 2005; Roch <i>et al</i> , 2005)
with lower bounded tolls	\mathcal{NP} -hard (Labbé <i>et al</i> , 1998)
with unrestricted tolls	\mathcal{NP} -hard (Roch <i>et al</i> , 2005)
with a single commodity	\mathcal{NP} -hard (Roch <i>et al</i> , 2005)
with a single toll arc	Polynomial (Brotcorne <i>et al</i> , 2001)
with number of toll arcs upper bounded	Polynomial (Van Hoesel <i>et al</i> , 2003)
Cross river pricing problem	\mathcal{NP} -hard (Grigoriev <i>et al</i> , 2005)
Highway pricing problem	\mathcal{NP} -hard (Heilporn <i>et al</i> , forthcoming)
with a single commodity	Polynomial (Dewez, 2004)
Buyer welfare problem	\mathcal{NP} -hard (Dobson and Kalish, 1993)
Profit problem	\mathcal{NP} -hard (Dobson and Kalish, 1993)
Share-of-choices problem	\mathcal{NP} -hard (Kohli and Krishnamurti, 1989)
Bundle pricing problem	\mathcal{APX} -hard (Guruswami <i>et al</i> , 2005)

$$\pi_i \geq 0 \quad \forall i \in \mathcal{I} \quad (56)$$

$$x_i^k \in \{0, 1\} \quad \forall k \in \mathcal{K}, \forall i \in \mathcal{I}, \quad (57)$$

which is exactly the full profit problem described in the section ‘Product pricing’. Hence, while a purchaser segment buys the product that maximizes its utility $r_i^k - \pi_i$, a commodity travels on the toll arc that maximizes the difference $c_{od}^k - c_i^k - t_i$, that is, that minimizes its travel cost $c_i^k + t_i$.

Similar relationships can be established between the bundle pricing problem, where each product to be priced represents a subset of components of a ‘global element’, and the network pricing problem, where each path consists of a subset of arcs of the network. More precisely, while n components of a global element yield $2^n - 1$ products, one also obtains an exponential number of paths by combining the n toll arcs of a transportation network. However, the connection is not as direct as for the previous two problems.

From the complexity viewpoint, both families of problems have been investigated (see Table 2). From the computational perspective, they have motivated different algorithmic approaches. For one, the community of

network pricing has proposed valid inequalities that have led to improved formulations and faster resolution. In contrast, few results of the sort have been proposed for product pricing, with the notable exception of works by McBride and Zufryden (1988) and Shioda *et al* (forthcoming). The situation is different on the heuristic front, which has been the subject of several proposals by both communities of researchers. In this respect, note that the inverse optimization procedure of Labbé *et al* (1998) is a slight generalization of the reverse greedy heuristic initially proposed by Dobson and Kalish (1988).

CONCLUSION

In the current context of deregulations, setting prices right offers a competitive advantage to firms operating in an oligopolistic environment, and is a key component of revenue management, a branch of management science whose objective is to optimize the revenue raised from perishable resources, and that was initiated in the airline industry. The aim of this study was to highlight the close relationship between two families of pricing problems that have been studied, frequently in parallel, in the fields of marketing,

economics, operations research and theoretical computer science. We hope that breakthroughs achieved in any field, whether theoretical, methodological or computational, translate into improved knowledge for design and pricing problems that arise in the fields of industrial marketing or revenue management. As a concrete example, the triangle and monotonicity inequalities that occur naturally in highway pricing are also meaningful for the general product pricing problem. In the latter case, when prices are assigned to product with different product formats, it would make sense, whenever the product quantity X satisfies the relationship $X = Y + Z$, to require the triangle inequality $\pi_X \leq \pi_Y + \pi_Z$, for the sake of market consistency. In the same vein, one would expect that $\pi_X \leq \pi_Y$ if $X \leq Y$, that is, monotonicity inequalities are satisfied.

NOTES

- 1 An \mathcal{APX} -hard problem is an \mathcal{NP} -hard problem that allows a polynomial time approximation algorithm with an approximation ratio bounded by a constant (see Papadimitriou and Yannakakis (1991) or Ausiello et al (1999)).
- 2 The terms ‘shortest path’ and ‘cheapest path’ are used interchangeably throughout the paper.

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