

A Polyhedral Study of the Network Pricing Problem with Connected Toll Arcs

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We consider the problem of setting revenue-maximizing tolls on a subset of arcs of a transportation network, assuming that the users of the network are assigned to shortest paths with respect to the sum of tolls and initial costs. Our main results are concerned with a polyhedral study of the problem, i.e., the design of valid inequalities and facets for this pricing problem and some of its variants. © 2009 Wiley Periodicals, Inc. NETWORKS, Vol. 00(00), 000–000 2010

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1. INTRODUCTION

In this article, we consider the pricing problem introduced by Labbé et al. [16], in which a revenue maximizing authority sets tolls on a subset of the arcs of a multicommodity transportation network. Commodities correspond to users traveling from an origin to a destination, and are assigned to shortest paths with respect to the sum of initial costs and tolls. The problem can be formulated as a bilevel program involving a leader (the authority) who anticipates the reaction of the follower (the network users) to its decisions. For

a recent review of this field, the reader is referred to Dempe [4], Marcotte and Savard [17], or Colson et al. [3].

The problem under consideration involves a multicommodity network defined by a node set \mathcal{N} , an arc set $\mathcal{A} \cup \mathcal{B}$, and a set of origin-destination pairs $\{(o^k, d^k) : k \in \mathcal{K}\}$, called commodities, each one endowed with a demand η^k . Let \mathcal{A} be a subset of arcs a on which tolls t_a can be added to the original fixed cost vector c , and \mathcal{B} the complementary subset of toll free arcs, for which the cost vector c is also given. Assuming that, for a given toll policy $t = (t_a)_{a \in \mathcal{A}}$, network users travel on shortest paths with respect to the toll and fixed cost structure, the Network Pricing Problem consists in devising a revenue maximizing toll policy. On the introduction of vectors $x^k = (x_a^k)_{a \in \mathcal{A} \cup \mathcal{B}}$ that represent the flows on commodities (equivalently, origin-destination pairs) $k \in \mathcal{K}$, the Network Pricing Problem can be formulated as the bilevel program (Labbé et al. [16]):

$$(TP) \max_{t,x} \sum_{k \in \mathcal{K}} \sum_{a \in \mathcal{A}} \eta^k t_a x_a^k \quad (1)$$

subject to:

$$t_a \geq 0 \quad \forall a \in \mathcal{A} \quad (2)$$

$$x \in \arg \min_x \sum_{k \in \mathcal{K}} \left(\sum_{a \in \mathcal{A}} (c_a + t_a) x_a^k + \sum_{a \in \mathcal{B}} c_a x_a^k \right) \quad (3)$$

subject to:

$$\sum_{a \in i^- \cap \mathcal{A}} x_a^k + \sum_{a \in i^- \cap \mathcal{B}} x_a^k - \sum_{a \in i^+ \cap \mathcal{A}} x_a^k - \sum_{a \in i^+ \cap \mathcal{B}} x_a^k$$

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$$= \begin{cases} -1 & \text{if } i = o^k \\ 1 & \text{if } i = d^k \\ 0 & \text{otherwise} \end{cases} \quad \forall k \in \mathcal{K}, \forall i \in \mathcal{N} \quad (4)$$

$$0 \leq x_a^k \leq 1 \quad \forall k \in \mathcal{K}, \forall a \in \mathcal{A}, \quad (5)$$

where i^- (resp. i^+) denotes the set of arcs having node i as their head (resp. tail) and, slightly abusing notation, the vector x denotes both the optimal solution and the argument of the lower level problem.

The Network Pricing Problem is strongly \mathcal{NP} -hard (see Roch et al. [18], Grigoriev et al. [10], Guruswami et al. [11]). Together with some of its variants, it has been analyzed by Brotcorne et al. [2], Bouhtou et al. [1], van Hoesel et al. [20], Grigoriev et al. [10], Heilporn et al. [13], and Dewez et al. [6]. It is closely related to the problem known as Product Line Design in the economic literature. See Green and Krieger [9], Dobson and Kalish [7, 8], Kohli and Krishnamurti [14], Kohli and Sukumar [15], or Shioda et al. [19] for further details.

The aim of this article is to study the polyhedral structure of a pricing problem involving connected toll arcs, as would occur for a highway. Hence, models, complexity results, valid inequalities, and proofs of facets are at the core of our research. The article is organized as follows. In Section 2, we present three versions of the Network Pricing Problem with Connected Toll Arcs, the first two serving as an introduction to the main version. For the sake of conciseness, an \mathcal{NP} -hardness proof is provided only for the latter in Section 3, whereas complexity results of the other variants are presented as corollaries. Next, two families of valid inequalities, which exploit the underlying network structure and strengthen important constraints of the initial model, are proposed in Section 4, and their theoretical efficiency is considered in Section 5. We prove that the valid inequalities, as well as several initial constraints of the model, define facets of the convex hull of feasible solutions in the single commodity case. Furthermore, the valid inequalities provide a complete description of the convex hull of feasible solutions for one variant of the problem.

2. NETWORK PRICING WITH CONNECTED TOLL ARCS

Throughout the article, we assume that toll arcs are connected and form a highway, and consider three variants of the problem. In the first variant (Basic NPP), tolls are additive. The second variant (General Complete Toll NPP), which allows for arbitrary and nonadditive tolls at the entry and exit points of the highway, involves a complete toll subgraph, i.e., each toll arc corresponds to a toll subpath linking an entry-exit couple. The third variant (Constrained Complete Toll NPP), which will be the main version of the problem considered in the article, involves a complete toll subgraph together with specific constraints encountered in real-life situations. The three variants are described later.

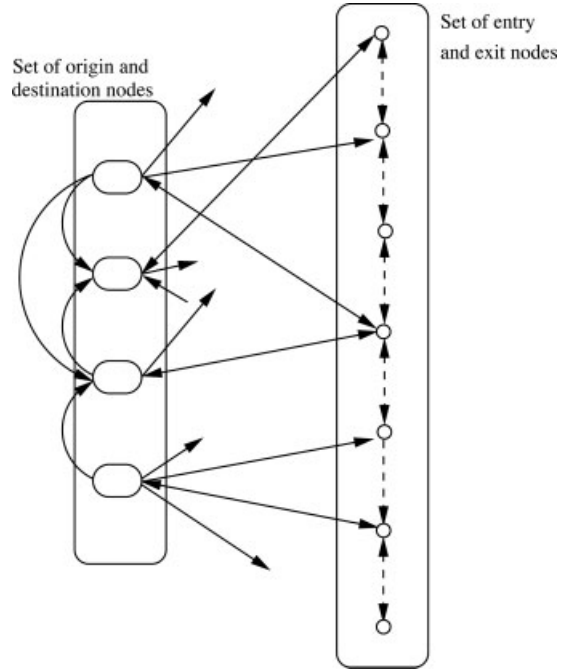


FIG. 1. Basic NPP.

In the first variant, which is directly derived from the Network Pricing Problem presented in Section 1, users may either travel along the shortest toll free path from their origin to their destination, or take the highway, using shortest toll free paths to and from the highway. We assume that users who have left the highway are not allowed to reenter, which implies that paths are uniquely determined by their respective entry and exit nodes. This “**Basic NPP**” is illustrated in Figure 1, in which toll arcs are dashed. Toll free arcs are inserted between origin and destination nodes, as well as from the origin and destination nodes to the highway. These arcs represent shortest toll free paths between the corresponding nodes. Each arc is endowed with a fixed cost, which provides a measure of the distance, time, or gas consumed on this arc. The fixed cost set on a toll free arc is set to the cost of a shortest toll free path between its nodes.

The above mathematical formulation (TP) applies to this situation. However, additional constraints must be appended to (TP) in order to forbid reentry. To this end, let us introduce the set $\tilde{\mathcal{N}} \subseteq \mathcal{N}$ of all possible origin and destination nodes, i.e., $\tilde{\mathcal{N}} = \{o^k, d^k : k \in \mathcal{K}\}$. Assuming that each shortest toll free path is represented by a single arc, the Basic NPP takes the form of (TP), with the additional constraints

$$\sum_{a \in i^- \cap \mathcal{B}} x_a^k + \sum_{a \in i^+ \cap \mathcal{B}} x_a^k = 0 \quad \forall k \in \mathcal{K}, \forall i \in \tilde{\mathcal{N}} : i \neq o^k, d^k. \quad (6)$$

Figure 2 depicts “**General Complete Toll NPP**,” which dispenses with the additivity condition, and requires some notation. For each arc $a \in \mathcal{A}$, let $t(a), h(a) \in \mathcal{N}$ be its tail and head nodes, respectively. For each commodity $k \in \mathcal{K}$ and for each toll arc $a \in \mathcal{A}$, let c_a^k denote the fixed cost on the corresponding path $o^k \rightarrow t(a) \rightarrow h(a) \rightarrow d^k$, where $t(a)$,

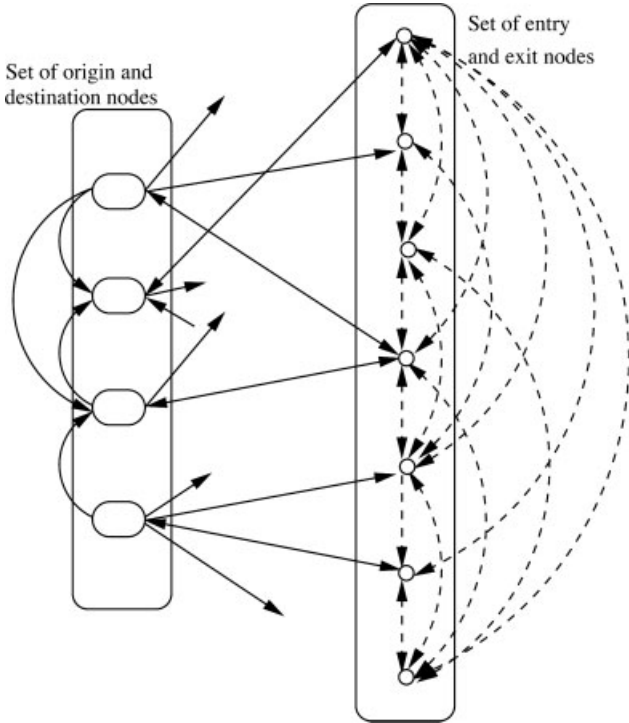


FIG. 2. Complete toll NPP.

and $h(a) \in \mathcal{N}$ are the entry and exit nodes on the highway. The fixed cost on the toll free path $o^k \rightarrow d^k$ is denoted by c_{od}^k , whereas the corresponding flow variable is x_{od}^k . For each commodity $k \in \mathcal{K}$ and for each toll arc $a \in \mathcal{A}$, variable x_a^k represents the flow on the corresponding path $o^k \rightarrow t(a) \rightarrow h(a) \rightarrow d^k$, whereas variable t_a is the toll on the arc a (i.e., toll subpath a). Furthermore, we consider that nodes are labeled by the indices 1 to m , leading to $|\mathcal{A}| = n = m(m - 1)$ toll arcs.

For the sake of realism, Triangle and Monotonicity inequalities are introduced, resulting in the problem “**Constrained Complete Toll NPP**.” The former constraints prevent leaving the highway upstream and reentering downstream, whereas Monotonicity constraints imply that the toll on a path cannot be less than the toll of any subpath. They are expressed mathematically as

$$t_a \leq t_b + t_c \quad \forall a, b, c \in \mathcal{A} : \\ t(a) = t(b), h(b) = t(c), h(c) = h(a) \quad (7)$$

and

$$t_a \geq t_b \quad \forall a, b \in \mathcal{A} : t(a) = t(b) < h(a) = h(b) + 1 \\ \text{or } t(a) = t(b) - 1 < h(a) = h(b) \\ \text{or } t(a) = t(b) > h(a) = h(b) - 1 \\ \text{or } t(a) = t(b) + 1 > h(a) = h(b), \quad (8)$$

respectively. Subnetworks on which these inequalities apply are illustrated in Figure 3.

This yields the formulation (Dewez [5] and Weisgerber [21]):

$$(HP1) \max_{t,x} \sum_{k \in \mathcal{K}} \sum_{a \in \mathcal{A}} \eta^k t_a x_a^k \quad (9)$$

subject to constraints (7), (8) and:

$$t_a \geq 0 \quad \forall a \in \mathcal{A} \quad (10)$$

$$x \in \arg \min_x \sum_{k \in \mathcal{K}} \left(\sum_{a \in \mathcal{A}} (c_a^k + t_a) x_a^k + c_{od}^k x_{od}^k \right) \quad (11)$$

subject to:

$$\sum_{a \in \mathcal{A}} x_a^k + x_{od}^k = 1 \quad \forall k \in \mathcal{K} \quad (12)$$

$$x_a^k \in \{0, 1\} \quad \forall k \in \mathcal{K}, \forall a \in \mathcal{A}. \quad (13)$$

Note that the characterization of lower level solutions as origin-destination paths carrying either no flow or the total origin-destination flow allows us to obtain an integer programming formulation of (HP1) that involves binary variables. Now, in view of the unimodularity of the constraint matrix associated with the shortest path problem at the lower level, one could drop the integrality requirements for the flow variables x . It follows that the lower level problem can be replaced by its primal and dual constraints and primal-dual optimality conditions, yielding a single-level program involving complementarity (i.e., disjunctive) constraints. Furthermore, to obtain a linear model, variables

$$p_a^k = \begin{cases} t_a & \text{if commodity } k \text{ uses arc } a \in \mathcal{A}, \\ 0 & \text{otherwise} \end{cases}$$

are introduced, corresponding to the actual unit profit associated with arc $a \in \mathcal{A}$ and commodity $k \in \mathcal{K}$. This yields (Dewez [5] and Weisgerber [21]):

$$(HP2) \max_p \sum_{k \in \mathcal{K}} \sum_{a \in \mathcal{A}} \eta^k p_a^k \quad (14)$$

subject to constraints (7), (8) and:

$$\sum_{a \in \mathcal{A}} x_a^k + x_{od}^k = 1 \quad \forall k \in \mathcal{K} \quad (15)$$

$$\lambda^k \leq c_a^k + t_a \quad \forall k \in \mathcal{K}, \forall a \in \mathcal{A} \quad (16)$$

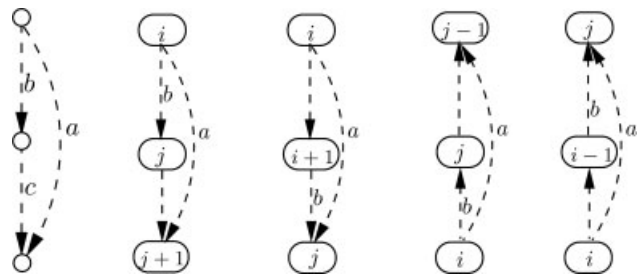


FIG. 3. Subnetworks on which triangle and monotonicity inequalities apply.

$$\lambda^k \leq c_{od}^k \quad \forall k \in \mathcal{K} \quad (17)$$

$$\sum_{a \in \mathcal{A}} (c_a^k x_a^k + p_a^k) + c_{od}^k x_{od}^k = \lambda^k \quad \forall k \in \mathcal{K} \quad (18)$$

$$p_a^k \leq M_a^k x_a^k \quad \forall k \in \mathcal{K}, \forall a \in \mathcal{A} \quad (19)$$

$$t_a - p_a^k \leq N_a(1 - x_a^k) \quad \forall k \in \mathcal{K}, \forall a \in \mathcal{A} \quad (20)$$

$$p_a^k \leq t_a \quad \forall k \in \mathcal{K}, \forall a \in \mathcal{A} \quad (21)$$

$$p_a^k \geq 0 \quad \forall k \in \mathcal{K}, \forall a \in \mathcal{A} \quad (22)$$

$$x_a^k \geq 0 \quad \forall k \in \mathcal{K} \quad (23)$$

$$x_a^k \in \{0, 1\} \quad \forall a \in \mathcal{A}, \forall k \in \mathcal{K}, \quad (24)$$

where M_a^k and N_a are suitably large constants. We set $M_a^k = \max\{0, c_{od}^k - c_a^k\}$ and $N_a = N = \max_{k \in \mathcal{K}, a \in \mathcal{A}} M_a^k$ for all $k \in \mathcal{K}$ and for all $a \in \mathcal{A}$.

Although the lower level optimality conditions in (HP2) involve arc-flow variables, it can be alternatively expressed in terms of path flows, without resorting to dual variables. The optimality conditions (16), (17), and (18) of (HP2) are then replaced by the equivalent

$$\sum_{a \in \mathcal{A}} (c_a^k x_a^k + p_a^k) + c_{od}^k x_{od}^k \leq c_b^k + t_b \quad \forall k \in \mathcal{K}, \forall b \in \mathcal{A} \quad (25)$$

$$\sum_{a \in \mathcal{A}} (c_a^k x_a^k + p_a^k) + c_{od}^k x_{od}^k \leq c_{od}^k \quad \forall k \in \mathcal{K}. \quad (26)$$

However, constraints (26) are obviously redundant in view of constraints (15), (19) and the definition of constants M_a^k : $k \in \mathcal{K}, a \in \mathcal{A}$. Next, based on constraints (15), variables x_{od}^k can be removed, yielding the more compact model:

$$(HP3) \max_p \sum_{k \in \mathcal{K}} \sum_{a \in \mathcal{A}} \eta^k p_a^k \quad (27)$$

subject to constraints (7), (8) and:

$$\sum_{a \in \mathcal{A}} x_a^k \leq 1 \quad \forall k \in \mathcal{K} \quad (28)$$

$$\sum_{b \in \mathcal{A}} (c_b^k x_b^k + p_b^k) + c_{od}^k \left(1 - \sum_{b \in \mathcal{A}} x_b^k\right) \leq c_a^k + t_a \quad \forall k \in \mathcal{K}, \forall a \in \mathcal{A} \quad (29)$$

$$p_a^k \leq M_a^k x_a^k \quad \forall k \in \mathcal{K}, \forall a \in \mathcal{A} \quad (30)$$

$$t_a - p_a^k \leq N_a(1 - x_a^k) \quad \forall k \in \mathcal{K}, \forall a \in \mathcal{A} \quad (31)$$

$$p_a^k \leq t_a \quad \forall k \in \mathcal{K}, \forall a \in \mathcal{A} \quad (32)$$

$$p_a^k \geq 0 \quad \forall k \in \mathcal{K}, \forall a \in \mathcal{A} \quad (33)$$

$$x_a^k \in \{0, 1\} \quad \forall k \in \mathcal{K}, \forall a \in \mathcal{A}. \quad (34)$$

In the sequel, we consider two variants of this program. In the General Complete Toll NPP, tolls are independent, whereas the Constrained Complete Toll NPP imposes Triangle and Monotonicity constraints (7) and (8). The corresponding models are labeled (HP3*) and (HP3), respectively.

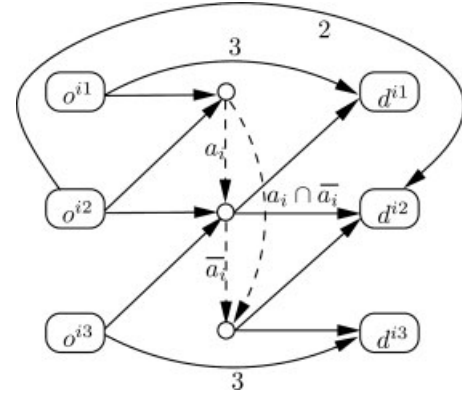


FIG. 4. Subnetwork corresponding to variable x_i (single directional constrained complete toll NPP).

In the next section, we show that the Constrained Complete Toll NPP is strongly \mathcal{NP} -hard.

3. COMPLEXITY ANALYSIS

It has been shown by Dewez [5] that the Network Pricing Problem with Connected Toll Arcs is polynomially solvable when either a single commodity or a single toll arc is involved. In contrast, we prove that the Constrained Complete Toll NPP is strongly \mathcal{NP} -hard, whether toll arcs are single or bidirectional. The proofs presented here are inspired from the ones presented by Roch et al. [18] for the Network Pricing Problem, or Grigoriev et al. [10] for the Cross River Network Pricing Problem, which are also based on a reduction from the strongly \mathcal{NP} -hard problem 3-SAT. Let x_1, \dots, x_n be n Boolean variables, and let $F = \bigwedge_{i=1}^m (l_{i1} \vee l_{i2} \vee l_{i3})$ be a conjunctive normal form of m clauses with literals l_{ij} : $j \in \{1, 2, 3\}$ that represent a variable x_i or its negation. Given such a conjunctive normal form, 3-SAT consists in finding an assignment of TRUE or FALSE values to the variables such that F is TRUE.

Proposition 1. *The single directional constrained complete toll NPP is strongly \mathcal{NP} -hard.*

Proof. Any conjunctive normal form $F = \bigwedge_{i=1}^m (l_{i1} \vee l_{i2} \vee l_{i3})$, where l_{ij} : $j \in \{1, 2, 3\}$ represents a variable x_i : $i \in \{1, \dots, n\}$ or its negation, can be polynomially converted to an instance of the Constrained Complete Toll NPP, in its decision form.

For each variable x_i : $i \in \{1, \dots, n\}$, a subnetwork is constructed as shown in Figure 4. Each of these subnetworks contains three commodities (o^{ij}, d^{ij}): $j \in \{1, 2, 3\}$ with unit demand, and two toll arcs a_i, \bar{a}_i of zero fixed cost corresponding to the true and false assignments for variable x_i , respectively. Furthermore, in any subnetwork, a toll arc is added from the tail node of arc a_i to the head node of arc \bar{a}_i , which corresponds to $a_i \cap \bar{a}_i$. Toll free arcs of cost zero connect o^{i1} (resp. o^{i3}) to the tail node of arc a_i (resp. \bar{a}_i), the head node of arc a_i (resp. \bar{a}_i) to d^{i1} (resp. d^{i3}), o^{i2} to both tail nodes of toll arcs, and both head nodes of toll arcs to

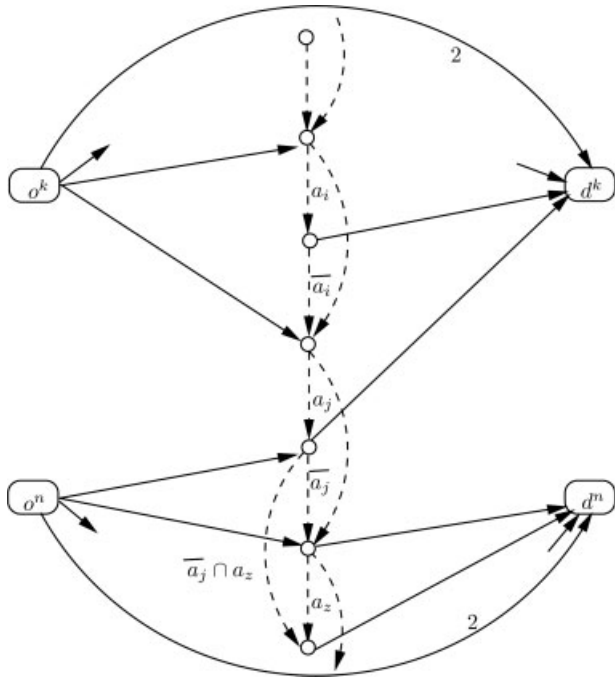


FIG. 5. Subnetwork for $F = (\dots \vee x_i \vee x_j) \wedge (\bar{x}_j \vee x_z \vee \dots) \wedge \dots$ (single directional constrained complete toll NPP).

d^{i2} . Toll free arcs (o^{i1}, d^{i1}) of cost 3, (o^{i3}, d^{i3}) of cost 3, and (o^{i2}, d^{i2}) of cost 2 are also added. Hence an upper bound on the revenue for each subnetwork is 7, obtained by setting to 2 the toll on either a_i or \bar{a}_i and the other ones to 3. In all other cases, the revenue cannot exceed 6.

Note that the toll of 3 on the arc $a_i \cap \bar{a}_i$ ensures that this arc is never taken. Indeed, suppose that commodity $i2$ chooses the arc $a_i \cap \bar{a}_i$ (there is no path using this arc for commodities $i1$ and $i3$). As the revenue on $i2$ is bounded by a toll free arc of cost 2, the toll on the arc $a_i \cap \bar{a}_i$ must be smaller than or equal to 2. But then, due to the Monotonicity constraints added to the problem, tolls on the other two arcs of the subnetworks cannot exceed 2 and the maximal revenue of 7 cannot be reached, which is a contradiction. Also note that the toll free arcs that do not appear from some origins to tail nodes of toll arcs (resp. from head nodes of toll arcs to some destinations) are supposed to be so expensive that they can never be used. They are not depicted in the network graph.

Then the subnetworks are linked together such that the single directional highway corresponds to the set of all connected subnetworks.

Furthermore, for each clause k , a clause-commodity (o^k, d^k) with unit demand is constructed as depicted in Figure 5.

If variable x_i (resp. \bar{x}_i) is a literal of clause k , toll free arcs of cost 0 are added from o^k to the tail node of a_i (resp. \bar{a}_i) and from the head node of a_i (resp. \bar{a}_i) to d^k , which means that the toll arc a_i belongs to the clause-commodity k . Furthermore, if two or three connected (in the highway graph) toll arcs belong to the same clause-commodity, toll arcs are added between the corresponding entry and exit nodes of the network. In the subnetwork example depicted in Figure 5, a single toll arc

is added between the tail node of \bar{a}_j and the head node of a_z . Those additional toll arcs correspond to the intersection of corresponding variables, i.e., in the example, the toll arc corresponds to $\bar{a}_j \cap a_z$. Note that there is no arc linking tail and head nodes of toll arcs belonging to different clause-commodities because there does not exist any path which could include them both. Tolls on the new arcs $\bar{a}_j \cap a_z$ are set to 2 if \bar{a}_j and a_z are both set to 2, and to 3 otherwise. Finally, an additional toll free arc (o^k, d^k) of cost 2 is added, which defines an upper bound of 2 on the revenue from each clause-commodity.

Now let us show that a truth assignment for F exists if and only if the revenue for the Constrained Complete Toll NPP is equal to $2m + 7n$, where m is the number of clauses and n is the number of variables.

Suppose there exists a truth assignment, which means that at least one literal is true in each clause. We set the corresponding tolls to 2, and the other tolls (in the corresponding subnetworks) to 3. Then the total revenue from all clause-commodities is $2m$. For all remaining subnetworks, if any (this situation only occurs if a variable x_i does not appear in any clause), the toll arcs are set arbitrarily to 2 and 3 for a variable and its negation, respectively. Hence the revenue from all subnetworks is $7n$, i.e., the total revenue is equal to $2m + 7n$.

Conversely, suppose that there exist tolls such that the total revenue is $2m + 7n$. The maximal possible revenue from all subnetworks is $7n$, only achievable by setting one toll per subnetwork to 2 and the other ones to 3. On the other hand, the maximal possible revenue from all clause-commodities is $2m$. We set to true the literals corresponding to arcs with toll 2, and to false the others. This corresponds to a well-defined assignment for F , as there is exactly one toll of 2 in each subnetwork. Furthermore, each clause-commodity contributes to the total revenue with a toll of 2, which means that at least one literal per clause is true, and there exists a true assignment for F . ■

Corollary 1. *The bidirectional constrained complete toll NPP is strongly \mathcal{NP} -hard.*

Proof. As toll arcs can now appear in both directions of the highway, subnetworks for variables $x_i : i \in \{1, \dots, n\}$ are constructed in a slightly different way. Indeed, the toll arcs a_i and \bar{a}_i corresponding to the true and false assignments for variable x_i are now in opposite directions, as shown in Figure 6 (left).

Next we construct the clause-commodities similarly to the single directional case. Indeed, if two or three toll arcs of a clause-commodity k are connected and in the same direction, toll arcs are added between the corresponding entry and exit nodes of the network. Such a network is depicted in Figure 6 (right). As the users are assumed to choose only one direction of the highway, no toll arc is added between the tail node of a toll arc in one direction of the highway network to the head node of another toll arc in the reverse direction of the

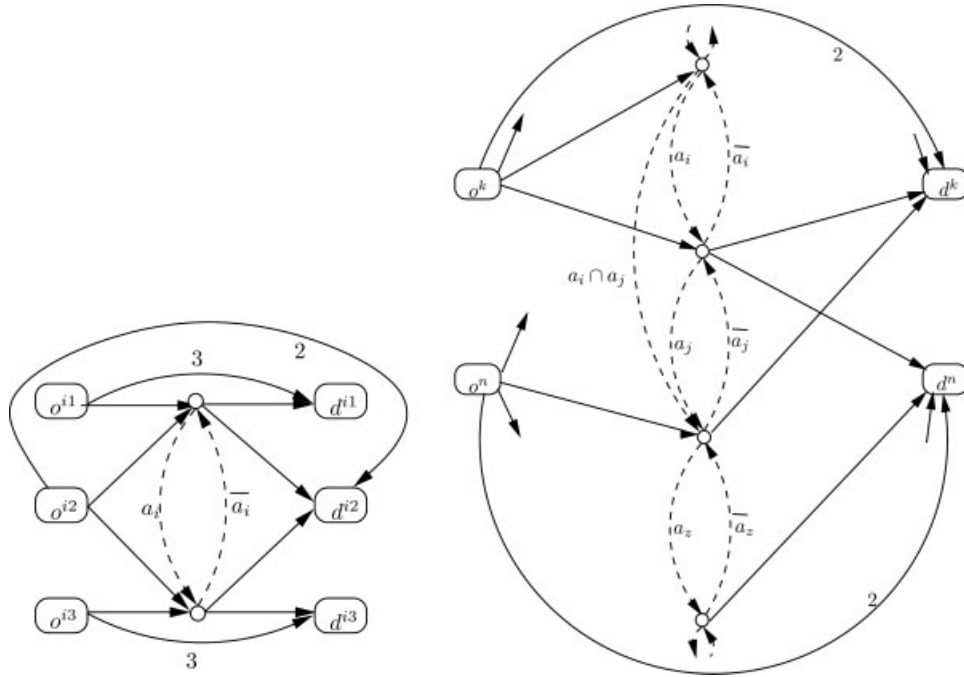


FIG. 6. Left: Subnetwork for variable x_i . Right: Subnetwork for $F = (\dots \vee x_i \vee x_j) \wedge (\bar{x}_j \vee x_z \vee \dots) \wedge \dots$ (bidirectional constrained complete toll NPP).

highway network. Then the same argument as before can be applied. ■

Note that the problem instances constructed in both preceding proofs preclude access either to some destinations from the highway or to the highway from some origins. This means that the corresponding paths are so expensive that they could never be taken. Hence we analyze a third case, in which the bidirectional highway network contains feasible access from all origins to all entry nodes of the highway (resp. from all exit nodes of the highway to all destinations).

Proposition 2. *The bidirectional Constrained Complete Toll NPP is strongly \mathcal{NP} -hard, even if access to all entry points (resp. from all exit points) is feasible from all origins (resp. to all destinations).*

Proof. The additional condition implies that no path is so expensive that it could never be taken, which is slightly different from the situation described earlier. Subnetworks are constructed as before, except that additional toll free arcs (those that were too expensive) are added such that there is one toll free arc from any origin to any tail node of a toll arc, and from any head node of a toll arc to any destination. For each commodity k , the cost on those arcs $(o^k, t(a_i))$ and $(h(a_i), d^k)$ are set such that the sum of the fixed cost of these two arcs is equal to the cost of the toll free arc (o^k, d^k) . Such a subnetwork is depicted in Figure 7. Hence the new arcs can only be chosen if tolls are set to zero on the corresponding arcs, which does not yield a maximal revenue for the leader. Tolls are then set as before and the same argument can be applied. ■

Corollary 2. *The single or bidirectional general complete toll NPP is strongly \mathcal{NP} -hard, even if access to all entry points (resp. from all exit points) is feasible from all origins (resp. to all destinations).*

Corollary 3. *The single or bidirectional Basic NPP is strongly \mathcal{NP} -hard, even if access to all entry points (resp. from all exit points) is feasible from all origins (resp. to all destinations).*

Proof. Similar proofs to the ones presented for Propositions 1 and 2 can be applied, with the proviso that the networks contain no toll arc of type $\bar{a}_i \cap a_i$ or $\bar{a}_i \cap a_j$ ($i, j \in \{1, \dots, n\}$). ■

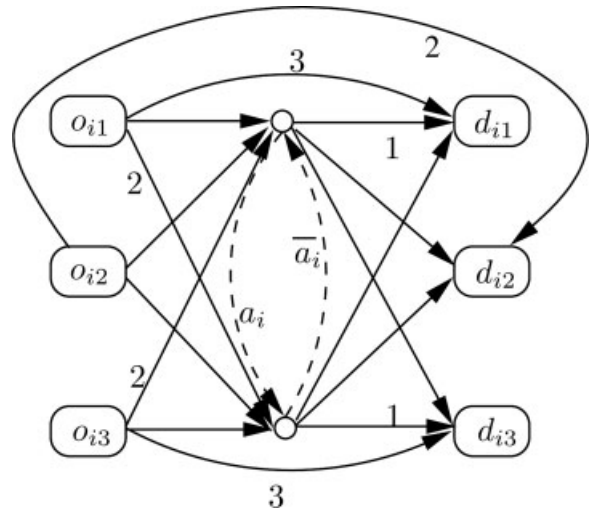


FIG. 7. Subnetwork for variable x_i (universal access constrained complete toll NPP).

4. VALID INEQUALITIES

In this section, we introduce two families of valid inequalities for the Constrained or General Complete Toll NPP. The first one strengthens the ‘‘Shortest Path’’ constraints (29) of (HP3) and (HP3*). The second family of inequalities is valid for the Constrained Complete Toll NPP, as those strengthen the Monotonicity constraints (8) of (HP3).

Proposition 3 (Strengthened shortest path inequalities). *Inequalities*

$$\sum_{b \in \mathcal{A}} (p_b^k + c_b^k x_b^k) + c_{od}^k \left(1 - \sum_{b \in \mathcal{A}} x_b^k \right) \leq t_a + c_a^k + \sum_{b \in \mathcal{A} \setminus (S \cup \{a\})} (p_b^k + (c_b^k - c_a^k) x_b^k) \quad (35)$$

where $k \in \mathcal{K}$, $a \in \mathcal{A}$, and S is any subset of \mathcal{A} (possibly the empty set) are valid for the Constrained Complete Toll NPP and General Complete Toll NPP.

Proof. Let $k \in \mathcal{K}$, $a \in \mathcal{A}$, and $S \subseteq \mathcal{A}$. If $x_b^k = 0$ for all $b \in \mathcal{A}$, then $p_b^k = 0$ for all $b \in \mathcal{A}$ by (30) and the inequality becomes $c_{od}^k \leq t_a + c_a^k$, which is valid by (29). If $x_a^k = 1$, then $x_b^k = 0 = p_b^k$ for all $b \in \mathcal{A} \setminus \{a\}$ by (28) and (29). The inequality becomes $p_a^k \leq t_a$, which is valid by (32). If $x_b^k = 1$ for $b \in \mathcal{A} \setminus (S \cup \{a\})$, the inequality becomes $0 \leq t_a$, which is valid by (32) and (33). Finally, if $x_b^k = 1$ for $b \in S$, then the inequality becomes $p_b^k + c_b^k \leq t_a + c_a^k$. As $p_b^k \leq t_b$ by (32), it is valid because $x_b^k = 1$, which means that the path corresponding to $b \in \mathcal{A}$ is shorter than the path corresponding to a . ■

For a given commodity $k \in \mathcal{K}$ and a given toll arc $a \in \mathcal{A}$, either the cost of the optimal path is equal to $t_a + c_a^k$, i.e., the cost of path $o^k \rightarrow t(a) \rightarrow h(a) \rightarrow d^k$, or the inequality gives the cost decrease corresponding to the choice of another path $o^k \rightarrow t(b) \rightarrow h(b) \rightarrow d^k$ as optimal path for this commodity. Note that similar valid inequalities, involving the toll free arcs $(o^k, d^k) : k \in \mathcal{K}$ instead of the toll arcs $a \in \mathcal{A}$, can be constructed as:

$$\sum_{b \in \mathcal{A}} (p_b^k + c_b^k x_b^k) + c_{od}^k \left(1 - \sum_{b \in \mathcal{A}} x_b^k \right) \leq c_{od}^k + \sum_{b \in \mathcal{A} \setminus S} (p_b^k + (c_b^k - c_{od}^k) x_b^k) \quad \forall k \in \mathcal{K}, \forall S \subseteq \mathcal{A}. \quad (36)$$

However, these inequalities can be equivalently written as:

$$\sum_{b \in S} p_b^k \leq \sum_{b \in S} (c_{od}^k - c_b^k) x_b^k \quad \forall k \in \mathcal{K}, \forall S \subseteq \mathcal{A}, \quad (37)$$

which are redundant by (30).

Any choice for the set S being valid, the number of inequalities (35) is exponential. Thus, an efficient separation procedure is required. Let $(\bar{t}, \bar{p}, \bar{x})$ be a current fractional

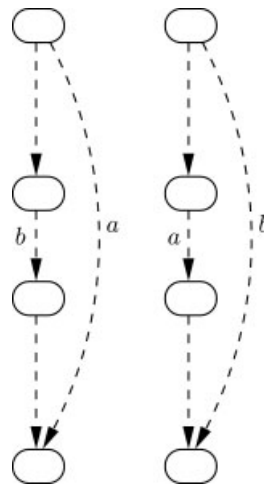


FIG. 8. Examples of $b \in \underline{C}_a$ and $b \in \bar{C}_a$.

solution of (HP3). The separation problem consists in determining, for each commodity $k \in \mathcal{K}$, a toll arc $a \in \mathcal{A}$ and a subset S of \mathcal{A} such that the corresponding inequality (35) is the most violated, i.e., minimizing the right-hand side of this inequality. Let us restrict our attention to inequalities such that $c_b^k \leq c_a^k$ for all $b \in \mathcal{A} \setminus (S \cup \{a\})$. We will see later that this is sufficient, as only these last inequalities are facet defining for the convex hull of feasible solutions of (HP3).

For each commodity $k \in \mathcal{K}$, the separation procedure goes as follows. First, we sort the toll arcs such that $c_1^k \leq \dots \leq c_n^k$, where n is the number of toll arcs in the network. As our goal is to construct a right-hand side of (35) as small as possible, let us consider, for a given toll arc a , the toll arcs b in $\mathcal{A} \setminus (S \cup \{a\})$ for which

$$\bar{p}_b^k + (c_b^k - c_a^k) \bar{x}_b^k \leq 0, \quad \text{i.e.,} \quad \frac{\bar{p}_b^k + c_b^k \bar{x}_b^k}{\bar{x}_b^k} \leq c_a^k. \quad (38)$$

Each toll arc $b \in \mathcal{A}$ is assigned to a node of a singly linked list such that the corresponding values $(\bar{p}_b^k + c_b^k \bar{x}_b^k) / \bar{x}_b^k$ are sorted in increasing order, i.e., each node of this list contains a toll arc as data and points to the node which contains the toll arc $b \in \mathcal{A}$ with the next largest term $(\bar{p}_b^k + c_b^k \bar{x}_b^k) / \bar{x}_b^k$.

Then, for each toll arc a going from 1 to n , we construct the set $\mathcal{A} \setminus S$ sequentially to obtain the smallest value for the right-hand side of inequalities (35) corresponding to commodity k . Note that the smallest right-hand side values of (35) do not differ very much from a given toll arc a to the next one $a + 1$ (in terms of increasing order of the corresponding fixed costs). Indeed, if $b \in \mathcal{A} \setminus S$ for a given toll arc a , then $b \in \mathcal{A} \setminus S$ for the next toll arc $a + 1$ by (38). The complexity of the separation procedure is dominated by that of the sorting operation, and can therefore be performed in $O(n \log n)$ time for each commodity $k \in \mathcal{K}$.

Now, for every $a \in \mathcal{A}$, let us define the set

$$\underline{C}_a = \{b \in \mathcal{A} : t(a) \leq t(b) < h(b) \leq h(a) \text{ or } t(a) \geq t(b) > h(b) \geq h(a)\}, \quad (39)$$

$$\begin{aligned} \bar{C}_a = \{b \in \mathcal{A} : t(b) \leq t(a) < h(a) \leq h(b) \\ \text{or } t(b) \geq t(a) > h(a) \geq h(b)\}, \quad (40) \end{aligned}$$

whose corresponding subnetworks are depicted in Figure 8. If Triangle and Monotonicity inequalities are included in the problem, the toll variables must satisfy $t_a \geq t_b$ for all b in \underline{C}_a . Note that an arc b is in \bar{C}_a if and only if a is in \underline{C}_b .

We also define an elementary arc as a toll arc $a \in \mathcal{A}$ such that $h(a) = t(a) + 1$ or $h(a) = t(a) - 1$. The set of all elementary arcs is denoted $\mathcal{A}_\mathcal{E}$, i.e.,

$$\mathcal{A}_\mathcal{E} = \{a \in \mathcal{A} : h(a) = t(a) + 1 \text{ or } h(a) = t(a) - 1\}. \quad (41)$$

With these notations, we can present a second family of valid inequalities for model (HP3) that strengthen the Monotonicity constraints (8).

Proposition 4 (Strengthened monotonicity inequalities). *Inequalities*

$$t_a \geq t_b + (c_b^k - c_{b^*}^k)x_b^k \quad \forall k \in \mathcal{K}, \forall a \in \mathcal{A}, \forall b \in \underline{C}_a \cap \mathcal{A}_\mathcal{E} \quad (42)$$

where b^* is the arc with the smallest fixed cost in $\underline{C}_a \cap \mathcal{A}_\mathcal{E}$ are valid for the Constrained Complete Toll NPP.

Proof. Let $k \in \mathcal{K}$, $a \in \mathcal{A}$ and suppose that $t(a) < h(a)$. If $x_b^k = 0$ for every toll arc $b \in \underline{C}_a \cap \mathcal{A}_\mathcal{E}$, then (42) becomes $t_a \geq t_b$ which is valid by the Monotonicity inequalities (8). Otherwise, i.e., if $x_b^k = 1$ for a toll arc $b \in \underline{C}_a \cap \mathcal{A}_\mathcal{E}$, then (42) becomes $t_a \geq t_b + (c_b^k - c_{b^*}^k)$. As inequalities $t_a \geq t_{b^*}$ and $t_b + c_b^k \leq t_{b^*} + c_{b^*}^k$ hold, the first due to (8) and the second because the cost of the path containing toll arc b is smaller than the cost of any other path, it follows that (42) holds. ■

For each commodity $k \in \mathcal{K}$, the separation procedure for inequalities (42) is $O(n^{3/2})$. For $a \in \mathcal{A}$ fixed, one may find the toll arc $b \in \underline{C}_a \cap \mathcal{A}_\mathcal{E}$ with minimal fixed cost in $O(n^{1/2})$ time. Indeed, as a highway network with m entry and exit nodes contains $n = m(m - 1)$ toll arcs, among which $2(m - 1)$ are elementary toll arcs, the number of elementary toll arcs is $O(n^{1/2})$. Then, according to the current solution $(\mathbf{t}, \bar{\mathbf{p}}, \bar{\mathbf{x}})$ of (HP3), one looks for the largest right-hand side value of the corresponding inequalities (42) in $O(n)$ time, and then checks whether this value is larger than t_a .

In the next section, we assess the valid inequalities by focusing on single commodity problems.

5. SINGLE COMMODITY PROBLEMS

The single commodity case is polynomially solvable [5]. Indeed, the toll arc yielding the largest revenue for the leader, i.e., with the largest upper bound $M_a^k : a \in \mathcal{A}$, can be found in $O(n)$ time. The toll on this arc is set to its upper bound, whereas tolls on the other arcs are set to sufficiently large values. Note that, in this case, constraints (31) can be removed from the model (HP3).

Although it is known that the single commodity problem is polynomially solvable, yet its analysis provides some insight. Indeed, a better comprehension of this case allows us to explore the core of the problem, and one should expect a generalization of the results to the multicommodity case. Hence this section focuses on single commodity problems, and aims to highlight several facets of the convex hull of (HP3) feasible solutions, i.e., for the Constrained Complete Toll NPP. Further, given the new family of valid inequalities (35), a complete description of the convex hull of solutions for the General Complete Toll NPP with a single commodity is pointed out. Throughout this section, the commodity index k will be omitted.

5.1. Single Commodity Constrained Complete Toll NPP

Let us denote by \mathcal{P}^C the convex hull of feasible solutions for the Constrained Complete Toll NPP, i.e.,

$$\mathcal{P}^C = \text{conv}\{(\mathbf{t}; \mathbf{p}; \mathbf{x}) \in \mathbb{R}_+^n \times \mathbb{R}_+^n \times \{0, 1\}^n : (7), (8), (28) - (30), (32) - (34)\},$$

where n is the number of toll arcs.

We may impose the condition $c_a \leq c_{od}$ for all a in \mathcal{A} . Indeed, if a toll arc $a \in \mathcal{A}$ is such that $c_a > c_{od}$, the corresponding path is never used and the arc $a \in \mathcal{A}$ can be removed. For the Constrained Complete Toll NPP, an improved preprocessing phase can be applied, based on the Monotonicity constraints.

Proposition 5. *Let $a, b \in \mathcal{A}$ be two toll arcs of the network such that $b \in \underline{C}_a$. If $c_b < c_a$, then one may set x_a to zero, and the corresponding path is never used.*

Proof. The cost of the first path using $b \in \mathcal{A}$ is $c_b + t_b$, and the cost of the second path using $a \in \mathcal{A}$ is $c_a + t_a$. As the Monotonicity inequalities impose $t_a \geq t_b$, the cost of the second path is always larger than the cost of the first path. ■

In what follows, the toll arcs which are never used are removed and the set \mathcal{A} is modified accordingly. We introduce a scalar M such that $M > c_{od} - c_a$ for all $a \in \mathcal{A}$. Furthermore, let $M_a = \max\{0, c_{od} - c_a\}$ as defined in (HP3), and denote by \mathbf{e}_a a unit vector in the direction a . We also denote by $\mathbf{1}$ a vector with all coordinates equal to 1.

Throughout the analysis, we assume that the arcs are totally ordered (labels 1 to n) in a manner that is compatible with the partial order induced by the Monotonicity and Triangle inequalities. More specifically, the coordinates of toll arcs are such that if $b \in \underline{C}_a$ for some couple $a, b \in \mathcal{A}$, i.e., the corresponding toll variables satisfy $t_a \geq t_b$, then a has a smaller index (also denoted “a”) than b .

Proposition 6. *The polyhedron \mathcal{P}^C has full dimension, i.e., $\text{Dim}(\mathcal{P}^C) = 3n$.*

Proof. Assume by contradiction that the points \mathcal{P}^C lie on a generic hyperplane $\alpha\mathbf{t} + \beta\mathbf{p} + \gamma\mathbf{x} = \delta$. The points

$(M\mathbf{1}; \mathbf{0}; \mathbf{0})$ and $(M\mathbf{1} + \sum_{b \in \mathcal{A}: b \leq a} \mathbf{e}_b; \mathbf{0}; \mathbf{0})$ belong to \mathcal{P}^C for all $a \in \mathcal{A}$. It follows that

$$M \sum_{b \in \mathcal{A}} \alpha_b = \delta \quad \text{and} \quad M \sum_{b \in \mathcal{A}} \alpha_b + \sum_{b \in \mathcal{A}: b \leq a} \alpha_b = \delta,$$

thus $\alpha_a = 0$ for all $a \in \mathcal{A}$ and $\delta = 0$. Furthermore, the points $(M(\mathbf{1} - \sum_{b \in \underline{C}_a} \mathbf{e}_b) + \sum_{b \in \mathcal{A}: b < a} \mathbf{e}_b; \mathbf{0}; \mathbf{e}_a)$ and $(M\mathbf{1} + \sum_{b \in \mathcal{A}: b < a} \mathbf{e}_b + (M_a - M) \sum_{b \in \underline{C}_a} \mathbf{e}_b; M_a \mathbf{e}_a; \mathbf{e}_a)$ are in \mathcal{P}^C for all $a \in \mathcal{A}$, thus $\gamma_a = 0 = \beta_a$ for all $a \in \mathcal{A}$ and the result follows. ■

One can also prove that several (HP3) inequalities define facets of \mathcal{P}^C . Let $(\mathbf{t}; \mathbf{p}; \mathbf{x})$ be points of \mathcal{P}^C . In order to prove the result, we define \mathcal{H} as the hyperplane induced by a given inequality, and \mathcal{G} as a generic hyperplane defined by $\alpha \mathbf{t} + \beta \mathbf{p} + \gamma \mathbf{x} = \delta$. Then we select points of $\mathcal{P}^C \cap \mathcal{H}$ and deduce that $\mathcal{G} = \mathcal{H}$.

Proposition 7. *The inequalities*

$$\sum_{a \in \mathcal{A}} x_a \leq 1 \quad (43)$$

$$p_a \leq M_a x_a \quad \forall a \in \mathcal{A} \quad (44)$$

$$p_a \geq 0 \quad \forall a \in \mathcal{A} \quad (45)$$

with $M_a = \max\{0, c_{od} - c_a\}$, are facet defining for \mathcal{P}^C .

Proof.

i. Let $\mathcal{H} = \{(\mathbf{t}; \mathbf{p}; \mathbf{x}) : \sum_{a \in \mathcal{A}} x_a = 1\}$.

For all $a \in \mathcal{A}$, we consider the points $(M(\mathbf{1} - \sum_{b \in \underline{C}_a} \mathbf{e}_b); \mathbf{0}; \mathbf{e}_a)$ and $(M(\mathbf{1} - \sum_{b \in \underline{C}_a} \mathbf{e}_b) + \sum_{b \in \mathcal{A}: b < a} \mathbf{e}_b; \mathbf{0}; \mathbf{e}_a)$ of $\mathcal{P}^C \cap \mathcal{H}$. This yields

$$M \sum_{b \in \mathcal{A} \setminus \underline{C}_a} \alpha_b + \gamma_a = \delta \quad \text{and}$$

$$M \sum_{b \in \mathcal{A} \setminus \underline{C}_a} \alpha_b + \sum_{b \in \mathcal{A}: b < a} \alpha_b + \gamma_a = \delta.$$

Hence $\alpha_a = 0$ and $\gamma_a = \delta$ for all $a \in \mathcal{A}$. Next, the points $(M\mathbf{1} + (M_a - M) \sum_{b \in \underline{C}_a} \mathbf{e}_b; M_a \mathbf{e}_a; \mathbf{e}_a)$ are also in $\mathcal{P}^C \cap \mathcal{H}$ for all $a \in \mathcal{A}$. Hence $\beta_a = 0$ for all $a \in \mathcal{A}$.

ii. Let $\mathcal{H} = \{(\mathbf{t}; \mathbf{p}; \mathbf{x}) : p_{\tilde{a}} = M_{\tilde{a}} x_{\tilde{a}}, \tilde{a} \in \mathcal{A}\}$.

The points $(M\mathbf{1}; \mathbf{0}; \mathbf{0})$ and $(M\mathbf{1} + \sum_{b \in \mathcal{A}: b \leq a} \mathbf{e}_b; \mathbf{0}; \mathbf{0})$ are in $\mathcal{P}^C \cap \mathcal{H}$ for all $a \in \mathcal{A}$, which implies that

$$M \sum_{b \in \mathcal{A}} \alpha_b = \delta \quad \text{and} \quad M \sum_{b \in \mathcal{A}} \alpha_b + \sum_{b \in \mathcal{A}: b \leq a} \alpha_b = \delta,$$

thus $\alpha_a = 0$ for all $a \in \mathcal{A}$ and $\delta = 0$. For all $a \in \mathcal{A} \setminus \{\tilde{a}\}$, points $(M(\mathbf{1} - \sum_{b \in \underline{C}_a} \mathbf{e}_b); \mathbf{0}; \mathbf{e}_a)$ also belong to $\mathcal{P}^C \cap \mathcal{H}$. Hence $\gamma_a = 0$ for all $a \in \mathcal{A} \setminus \{\tilde{a}\}$. As $(M\mathbf{1} + (M_a - M) \sum_{b \in \underline{C}_a} \mathbf{e}_b; M_a \mathbf{e}_a; \mathbf{e}_a)$ are in $\mathcal{P}^C \cap \mathcal{H}$ for all $a \in \mathcal{A}$, it follows that $\beta_a = 0$ for all $a \in \mathcal{A} \setminus \{\tilde{a}\}$ and $\gamma_{\tilde{a}} = -M_{\tilde{a}} \beta_{\tilde{a}}$. The result follows.

iii. Let $\mathcal{H} = \{(\mathbf{t}; \mathbf{p}; \mathbf{x}) : p_{\tilde{a}} = 0, \tilde{a} \in \mathcal{A}\}$.

The points $(M\mathbf{1}; \mathbf{0}; \mathbf{0})$ and $(M\mathbf{1} + \sum_{b \in \mathcal{A}: b \leq a} \mathbf{e}_b; \mathbf{0}; \mathbf{0})$ are in $\mathcal{P}^C \cap \mathcal{H}$ for all $a \in \mathcal{A}$, thus $\alpha_a = 0$ for all $a \in \mathcal{A}$ and $\delta = 0$. As $(M(\mathbf{1} - \sum_{b \in \underline{C}_a} \mathbf{e}_b); \mathbf{0}; \mathbf{e}_a)$ also belong to

$\mathcal{P}^C \cap \mathcal{H}$ for all $a \in \mathcal{A}$, it follows that $\gamma_a = 0$ for all $a \in \mathcal{A}$. Finally, the points $(M\mathbf{1} + (M_a - M) \sum_{b \in \underline{C}_a} \mathbf{e}_b; M_a \mathbf{e}_a; \mathbf{e}_a)$ are in $\mathcal{P}^C \cap \mathcal{H}$ for all $a \in \mathcal{A} \setminus \{\tilde{a}\}$, thus $\beta_a = 0$ for all $a \in \mathcal{A} \setminus \{\tilde{a}\}$ and the result follows. ■

Proposition 8. *The triangle inequalities*

$$t_a \leq t_b + t_c \quad \forall a, b, c \in \mathcal{A} :$$

$$t(a) = t(b), h(b) = t(c), h(c) = h(a) \quad (46)$$

are facet defining for \mathcal{P}^C .

Proof. Let $\mathcal{H} = \{(\mathbf{t}; \mathbf{p}; \mathbf{x}) : t_{\tilde{a}} = t_{\tilde{b}} + t_{\tilde{c}}, \tilde{a}, \tilde{b}, \tilde{c} \in \mathcal{A} \text{ s.t. } t(\tilde{a}) = t(\tilde{b}), h(\tilde{b}) = t(\tilde{c}), h(\tilde{c}) = h(\tilde{a})\}$. Furthermore, assume that the indices are such that $\tilde{b} < \tilde{c}$. The point $(M(\mathbf{1} + \sum_{b \in \mathcal{A}: b \leq \tilde{a}} \mathbf{e}_b); \mathbf{0}; \mathbf{0})$ belongs to $\mathcal{P}^C \cap \mathcal{H}$, which implies

$$M \sum_{b \in \mathcal{A}} \alpha_b + M \sum_{b \in \mathcal{A}: b \leq \tilde{a}} \alpha_b = \delta.$$

For all $a \in \mathcal{A} \setminus \{\tilde{a}\}$, the points $(M(\mathbf{1} + \sum_{b \in \mathcal{A} \setminus \{\tilde{a}\}: b \leq a} \mathbf{e}_b) + S \mathbf{e}_{\tilde{a}}; \mathbf{0}; \mathbf{0})$ are in $\mathcal{P}^C \cap \mathcal{H}$ with $S = M$ for $a < \tilde{b}$, $S = 2M$ for $\tilde{b} \leq a < \tilde{c}$, and $S = 3M$ for $a \geq \tilde{c}$. It follows that

$$M \sum_{b \in \mathcal{A}} \alpha_b + M \sum_{b \in \mathcal{A} \setminus \{\tilde{a}\}: b \leq a} \alpha_b + S \alpha_{\tilde{a}} = \delta.$$

From the first case $a < \tilde{b}$ ($a \neq \tilde{a}$), we know that $\alpha_a = 0$; from the second one we conclude that $\alpha_{\tilde{a}} = -\alpha_{\tilde{b}}$ and that the other indices α_a are equal to zero; from the last one we obtain $\alpha_{\tilde{a}} = -\alpha_{\tilde{c}}$ and $\alpha_a = 0$ for all other indices. Hence $\delta = 0$.

Now, for all $a \in \mathcal{A}$ such that $\tilde{a}, \tilde{b}, \tilde{c} \notin \underline{C}_a$, the points $(M(\mathbf{1} + \sum_{b \in \underline{C}_a} \mathbf{e}_b - \sum_{b \in \underline{C}_a} \mathbf{e}_b); \mathbf{0}; \mathbf{e}_a)$ belong to $\mathcal{P}^C \cap \mathcal{H}$. In these point coordinates, the first sum ensures that the given Triangle inequality is tight, whereas the second sum ensures that $t_a = p_a$ [as imposed by (31), (32) in (HP3)] and that the Monotonicity inequalities hold. For all $a \in \mathcal{A}$ such that either \tilde{a} or \tilde{b} or \tilde{c} in \underline{C}_a , the points $(M(\mathbf{1} - \sum_{b \in \underline{C}_a} \mathbf{e}_b); \mathbf{0}; \mathbf{e}_a)$ are in $\mathcal{P}^C \cap \mathcal{H}$. Hence $\gamma_a = 0$ for all $a \in \mathcal{A}$.

Next, for all $a \in \mathcal{A}$ such that $\tilde{a}, \tilde{b}, \tilde{c} \notin \underline{C}_a$, one considers the points $(M(\mathbf{1} + \sum_{b \in \underline{C}_a \cap \mathcal{A}} \mathbf{e}_b) + (M_a - M) \sum_{b \in \underline{C}_a} \mathbf{e}_b; M_a \mathbf{e}_a; \mathbf{e}_a)$ of $\mathcal{P}^C \cap \mathcal{H}$. For all $a \in \mathcal{A}$ such that either \tilde{a} or \tilde{b} or \tilde{c} is in \underline{C}_a , the points $(M\mathbf{1} + M_a \sum_{b \in \underline{C}_a} \mathbf{e}_b + (M_a - M) \sum_{b \in \underline{C}_a} \mathbf{e}_b; M_a \mathbf{e}_a; \mathbf{e}_a)$ are in $\mathcal{P}^C \cap \mathcal{H}$. This yields $\beta_a = 0$ for all $a \in \mathcal{A}$ and the result follows. ■

Proposition 9. *The Monotonicity inequalities*

$$t_a \geq t_b \quad \forall a \in \mathcal{A}, \forall b \in \mathcal{A} :$$

$$t(a) = t(b) < h(a) = h(b) + 1$$

$$\text{or } t(a) = t(b) - 1 < h(a) = h(b) \quad (47)$$

$$\text{or } t(a) = t(b) > h(a) = h(b) - 1$$

$$\text{or } t(a) = t(b) + 1 > h(a) = h(b)$$

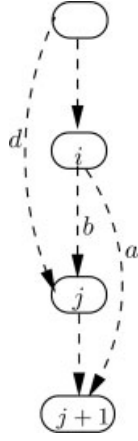


FIG. 9. Subnetwork illustrating the assumption in Proposition 9.

are facet defining for \mathcal{P}^C if and only if $c_d \leq c_{\tilde{a}}$ for all $d \in \mathcal{A}$ such that $\tilde{b} \in \underline{C}_d$ and $\tilde{a} \notin \underline{C}_d$ (see Fig. 9).

Proof. Let $\mathcal{H} = \{(\mathbf{t}; \mathbf{p}; \mathbf{x}) : t_{\tilde{a}} = t_{\tilde{b}} + (c_{\tilde{b}} - c_{\tilde{b}^*})x_{\tilde{b}}, \tilde{a} \in \mathcal{A}, \tilde{b} \in \underline{C}_{\tilde{a}} \cap \mathcal{A}_{\mathcal{E}}\}$ and suppose that the assumption holds. For all $a \in \mathcal{A} \setminus \{\tilde{a}\}$, the points $(M(\mathbf{1} + \sum_{b \in \mathcal{A} \setminus \{\tilde{a}\}; b \leq a} \mathbf{e}_b) + S\mathbf{e}_{\tilde{a}}; \mathbf{0}; \mathbf{0})$ belong to $\mathcal{P}^C \cap \mathcal{H}$ if the constant S is such that $S = 0$ for $a < \tilde{b}$ and $S = M$ for $a \geq \tilde{b}$. As $(M\mathbf{1}; \mathbf{0}; \mathbf{0})$ also belongs to $\mathcal{P}^C \cap \mathcal{H}$, it follows that

$$M \sum_{b \in \mathcal{A}} \alpha_b = \delta \quad \text{and} \quad M \sum_{b \in \mathcal{A}} \alpha_b + M \sum_{b \in \mathcal{A} \setminus \{\tilde{a}\}; b \leq \tilde{a}} \alpha_b + S\alpha_{\tilde{a}} = \delta.$$

From the first case $a < \tilde{b}$ ($a \neq \tilde{a}$), we know that $\alpha_a = 0$; from the second one we conclude that $\alpha_{\tilde{a}} = -\alpha_{\tilde{b}}$ and that all other indices $\alpha_a = 0$. Hence $\delta = 0$.

Now, for all $a \in \mathcal{A}$, the points $(M(\mathbf{1} - \sum_{b \in \underline{C}_a \cup \underline{C}_{\tilde{a}}} \mathbf{e}_b); \mathbf{0}; \mathbf{e}_a)$ are in $\mathcal{P}^C \cap \mathcal{H}$. Note that setting $t_b = 0$ for all $b \in \underline{C}_{\tilde{a}}$ forces the given Monotonicity inequality to be tight if $\tilde{b} \in \underline{C}_a$. This implies that $\gamma_a = 0$ for all $a \in \mathcal{A}$.

Finally, for all $a \in \mathcal{A}$ such that $\tilde{a}, \tilde{b} \notin \underline{C}_a$, one considers the points $(M\mathbf{1} + (M_a - M) \sum_{b \in \underline{C}_a} \mathbf{e}_b; M_a \mathbf{e}_a; \mathbf{e}_a)$ of $\mathcal{P}^C \cap \mathcal{H}$. Otherwise for all $a \in \mathcal{A}$ such that $\tilde{b} \in \underline{C}_a$, the points $(M\mathbf{1} + (M_{\tilde{b}} - M) \sum_{b \in \underline{C}_a \cup \underline{C}_{\tilde{a}}} \mathbf{e}_b; M_{\tilde{b}} \mathbf{e}_a; \mathbf{e}_a)$ belong to $\mathcal{P}^C \cap \mathcal{H}$. Again, the small perturbation in those point coordinates (M_a becomes $M_{\tilde{b}}$) forces the Monotonicity inequality involving $\tilde{a}, \tilde{b} \in \mathcal{A}$ to be tight. This yields $\beta_a = 0$ for all $a \in \mathcal{A}$, and the result follows. ■

Now let us focus on the valid inequalities presented in Section 4, and let us state the conditions under which these define facets of \mathcal{P}^C .

Proposition 10. *The Strengthened Monotonicity inequalities*

$$t_a \geq t_b + (c_b - c_{b^*})x_b \quad \forall a \in \mathcal{A}, \forall b \in \underline{C}_a \cap \mathcal{A}_{\mathcal{E}}, \quad (48)$$

where b^* is the arc with the smallest fixed cost in $\underline{C}_a \cap \mathcal{A}_{\mathcal{E}}$, are facet defining for \mathcal{P}^C if $c_d \leq c_a$ for all $d \in \mathcal{A}$ such that $b \in \underline{C}_d$ and $a \notin \underline{C}_d$.

Proof. Let $\mathcal{H} = \{(\mathbf{t}; \mathbf{p}; \mathbf{x}) : t_{\tilde{a}} = t_{\tilde{b}} + (c_{\tilde{b}} - c_{\tilde{b}^*})x_{\tilde{b}}, \tilde{a} \in \mathcal{A}, \tilde{b} \in \underline{C}_{\tilde{a}} \cap \mathcal{A}_{\mathcal{E}}\}$ and suppose that the assumption holds. For all $a \in \mathcal{A} \setminus \{\tilde{a}\}$, the points $(M(\mathbf{1} + \sum_{b \in \mathcal{A} \setminus \{\tilde{a}\}; b \leq a} \mathbf{e}_b) + S\mathbf{e}_{\tilde{a}}; \mathbf{0}; \mathbf{0})$ belong to $\mathcal{P}^C \cap \mathcal{H}$ if the constant S is such that $S = 0$ for $a < \tilde{b}$ and $S = M$ for $a \geq \tilde{b}$. As $(M\mathbf{1}; \mathbf{0}; \mathbf{0})$ also belongs to $\mathcal{P}^C \cap \mathcal{H}$, it follows that

$$M \sum_{b \in \mathcal{A}} \alpha_b = \delta \quad \text{and} \quad M \sum_{b \in \mathcal{A}} \alpha_b + M \sum_{b \in \mathcal{A} \setminus \{\tilde{a}\}; b \leq \tilde{a}} \alpha_b + S\alpha_{\tilde{a}} = \delta.$$

From the first case $a < \tilde{b}$ ($a \neq \tilde{a}$), we conclude that $\alpha_a = 0$; from the second one we deduce that $\alpha_{\tilde{a}} = -\alpha_{\tilde{b}}$ and that all other coefficients α_a are null. Hence $\delta = 0$.

Considering the points $(M(\mathbf{1} - \sum_{a \in \underline{C}_{\tilde{a}}} \mathbf{e}_a) + \sum_{a \in \underline{C}_{\tilde{a}} \setminus \{\tilde{b}\}} (c_{\tilde{b}} - c_{\tilde{b}^*})\mathbf{e}_a; \mathbf{0}; \mathbf{e}_{\tilde{b}})$ and $(M(\mathbf{1} - \sum_{b \in \underline{C}_{\tilde{a}}} \mathbf{e}_b) + M_{\tilde{b}}\mathbf{e}_{\tilde{b}} + M_{\tilde{b}^*} \sum_{b \in \underline{C}_{\tilde{a}} \setminus \{\tilde{b}\}} \mathbf{e}_b; M_{\tilde{b}}\mathbf{e}_{\tilde{b}}; \mathbf{e}_{\tilde{b}})$ of $\mathcal{P}^C \cap \mathcal{H}$ yields $\gamma_{\tilde{b}} = -(c_{\tilde{b}} - c_{\tilde{b}^*})\alpha_{\tilde{a}}$ and $\beta_{\tilde{b}} = 0$. Note that the last sums in those point coordinates ensure that the given Strengthened Monotonicity inequality is tight.

Next, for all $a \in \mathcal{A} \setminus \{\tilde{b}\}$, the points $(M(\mathbf{1} - \sum_{b \in \underline{C}_a \cup \underline{C}_{\tilde{a}}} \mathbf{e}_b); \mathbf{0}; \mathbf{e}_a)$ are in $\mathcal{P}^C \cap \mathcal{H}$. Again, setting $t_b = 0$ for all $b \in \underline{C}_{\tilde{a}}$ forces the given Strengthened Monotonicity inequality to be tight if $\tilde{b} \in \underline{C}_a$. This implies $\gamma_a = 0$ for all $a \in \mathcal{A} \setminus \{\tilde{b}\}$.

For all $a \in \mathcal{A} \setminus \{\tilde{b}\}$ such that $\tilde{a}, \tilde{b} \notin \underline{C}_a$, one considers the points $(M\mathbf{1} + (M_a - M) \sum_{b \in \underline{C}_a} \mathbf{e}_b; M_a \mathbf{e}_a; \mathbf{e}_a)$ of $\mathcal{P}^C \cap \mathcal{H}$. Otherwise for all $a \in \mathcal{A} \setminus \{\tilde{b}\}$ such that $\tilde{b} \in \underline{C}_a$, the points $(M\mathbf{1} + (M_{\tilde{b}} - M) \sum_{b \in \underline{C}_a \cup \underline{C}_{\tilde{a}}} \mathbf{e}_b; M_{\tilde{b}} \mathbf{e}_a; \mathbf{e}_a)$ belong to $\mathcal{P}^C \cap \mathcal{H}$. The small perturbation in those points (M_a becomes $M_{\tilde{b}}$) forces the given Strengthened Monotonicity inequality to be tight. Hence $\beta_a = 0$ and the result follows. ■

Proposition 11. *The Strengthened Shortest Path inequalities*

$$\sum_{b \in \mathcal{A}} (p_b + c_b x_b) + c_{od} \left(1 - \sum_{b \in \mathcal{A}} x_b \right) \leq t_a + c_a + \sum_{b \in \mathcal{A} \setminus (S \cup \{a\})} (p_b + (c_b - c_a)x_b) \quad \forall a \in \mathcal{A}, \forall S \subseteq \mathcal{A} \quad (49)$$

are facet defining for \mathcal{P}^C for all sets S such that $c_a \geq c_b$ for all b in $\mathcal{A} \setminus (S \cup \{a\})$.

Proof. Considering a Strengthened Shortest Path inequality in its generic form, let \mathcal{H} be defined as $\mathcal{H} = \{(\mathbf{t}; \mathbf{p}; \mathbf{x}) : t_{\tilde{a}} - \sum_{a \in \mathcal{S} \cup \{\tilde{a}\}} p_a + \sum_{a \in \mathcal{S} \cup \{\tilde{a}\}} (c_{od} - c_a)x_a + \sum_{a \in \mathcal{A} \setminus (\mathcal{S} \cup \{\tilde{a}\})} (c_{od} - c_{\tilde{a}})x_a = c_{od} - c_{\tilde{a}}, \text{ with } \tilde{a} \in \mathcal{A}, \mathcal{S} \subseteq \mathcal{A} \text{ such that } c_{\tilde{a}} \geq c_a \text{ for all } a \in \mathcal{A} \setminus (\mathcal{S} \cup \{\tilde{a}\})\}$.

First, the points $(M\mathbf{1} + (M_{\tilde{a}} - M) \sum_{b \in \underline{C}_{\tilde{a}}} \mathbf{e}_b; \mathbf{0}; \mathbf{0})$ and $(M\mathbf{1} + (M_{\tilde{a}} - M) \sum_{b \in \underline{C}_{\tilde{a}}} \mathbf{e}_b + \sum_{b \in \mathcal{A} \setminus \{\tilde{a}\}: b \leq a} \mathbf{e}_b; \mathbf{0}; \mathbf{0})$ belong to $\mathcal{P}^C \cap \mathcal{H}$ for all $a \in \mathcal{A}$. This implies that

$$M \sum_{b \in \mathcal{A} \setminus \underline{C}_{\tilde{a}}} \alpha_b + M_{\tilde{a}} \sum_{b \in \underline{C}_{\tilde{a}}} \alpha_b = \delta \quad \text{and}$$

$$M \sum_{b \in \mathcal{A} \setminus \underline{C}_{\tilde{a}}} \alpha_b + M_{\tilde{a}} \sum_{b \in \underline{C}_{\tilde{a}}} \alpha_b + \sum_{b \in \mathcal{A} \setminus \{\tilde{a}\}: b \leq a} \alpha_b = \delta,$$

thus $\alpha_b = 0$ for any $b \neq \tilde{a}$, and $\delta = M_{\tilde{a}}\alpha_{\tilde{a}}$. The point $(M\mathbf{1} + (M_{\tilde{a}} - M) \sum_{b \in \underline{C}_{\tilde{a}}} \mathbf{e}_b; M_{\tilde{a}}\mathbf{e}_{\tilde{a}}; \mathbf{e}_{\tilde{a}})$ of $\mathcal{P}^C \cap \mathcal{H}$ yields $M_{\tilde{a}}\alpha_{\tilde{a}} + M_{\tilde{a}}\beta_{\tilde{a}} + \gamma_{\tilde{a}} = M_{\tilde{a}}\alpha_{\tilde{a}}$, and $\gamma_{\tilde{a}} = -M_{\tilde{a}}\beta_{\tilde{a}}$. The point $(M(\mathbf{1} - \sum_{b \in \underline{C}_{\tilde{a}}} \mathbf{e}_b); \mathbf{0}; \mathbf{e}_{\tilde{a}})$ also belongs to $\mathcal{P}^C \cap \mathcal{H}$, i.e., $\gamma_{\tilde{a}} = -\alpha_{\tilde{a}}$.

Furthermore, for all $a \in \mathcal{A} \setminus (\mathcal{S} \cup \{\tilde{a}\})$, the points $(M(\mathbf{1} - \sum_{b \in \underline{C}_{\tilde{a}}} \mathbf{e}_b - \sum_{b \in \underline{C}_a \setminus \underline{C}_{\tilde{a}}} \mathbf{e}_b); \mathbf{0}; \mathbf{e}_a)$ belong to $\mathcal{P}^C \cap \mathcal{H}$, which implies $\gamma_a = M_{\tilde{a}}\alpha_{\tilde{a}}$. From the points $(M(\mathbf{1} - \sum_{b \in \underline{C}_{\tilde{a}}} \mathbf{e}_b) + (c_{\tilde{a}} - c_a - M) \sum_{b \in \underline{C}_a \setminus \underline{C}_{\tilde{a}}} \mathbf{e}_b; (c_{\tilde{a}} - c_a)\mathbf{e}_a; \mathbf{e}_a)$ of $\mathcal{P}^C \cap \mathcal{H}$, it follows that $(c_{\tilde{a}} - c_a)\beta_a + \gamma_a = M_{\tilde{a}}\alpha_{\tilde{a}}$, and $\beta_a = 0$.

For all $a \in \mathcal{S} \setminus \{\tilde{a}\}$, the points $(M\mathbf{1} + (M_{\tilde{a}} - M) \sum_{b \in \underline{C}_{\tilde{a}}} \mathbf{e}_b + (M_a - M) \sum_{b \in \underline{C}_a \setminus \underline{C}_{\tilde{a}}} \mathbf{e}_b; M_a\mathbf{e}_a; \mathbf{e}_a)$ are in $\mathcal{P}^C \cap \mathcal{H}$, yielding $\gamma_a = -M_a\beta_a$. If $c_a \geq c_{\tilde{a}}$, the point $(M(\mathbf{1} - \sum_{b \in \underline{C}_{\tilde{a}}} \mathbf{e}_b) + (c_a - c_{\tilde{a}} - M) \sum_{b \in \underline{C}_{\tilde{a}}} \mathbf{e}_b; \mathbf{0}; \mathbf{e}_a)$ belongs to $\mathcal{P}^C \cap \mathcal{H}$. Hence, $(c_a - c_{\tilde{a}})\alpha_{\tilde{a}} + \gamma_a = M_{\tilde{a}}\alpha_{\tilde{a}}$ and $\gamma_a = M_a\alpha_{\tilde{a}}$. Otherwise, i.e., if $c_a < c_{\tilde{a}}$, the points $(M(\mathbf{1} - \sum_{b \in \underline{C}_{\tilde{a}}} \mathbf{e}_b) + (c_{\tilde{a}} - c_a - M) \sum_{b \in \underline{C}_a \setminus \underline{C}_{\tilde{a}}} \mathbf{e}_b; (c_{\tilde{a}} - c_a)\mathbf{e}_a; \mathbf{e}_a)$ are in $\mathcal{P}^C \cap \mathcal{H}$ and $(c_{\tilde{a}} - c_a)\beta_a + \gamma_a = M_{\tilde{a}}\alpha_{\tilde{a}}$. As $\gamma_a = -M_a\beta_a$, we obtain $\beta_a = -\alpha_{\tilde{a}}$ for all $a \in \mathcal{S} \setminus \{\tilde{a}\}$ and the result follows. ■

In the next subsection, we restrict our attention to the General Complete Toll NPP. In this case, the Strengthened Shortest Path inequalities introduced in Section 4 allow us to present a complete description of the convex hull of feasible solutions for the corresponding single commodity problem.

5.2. Single Commodity General Complete Toll NPP

Consider the single commodity General Complete Toll NPP, and let us define \mathcal{P} as the convex hull of its feasible solutions, i.e.,

$$\mathcal{P} = \text{conv}\{(\mathbf{t}; \mathbf{p}; \mathbf{x}) \in \mathbb{R}_+^n \times \mathbb{R}_+^n \times \{0, 1\}^n : (28) - (30), (32) - (34)\}.$$

As $\mathcal{P}^C \subseteq \mathcal{P}$, we have the following results.

Corollary 4. $\text{Dim}(\mathcal{P}) = 3n$.

Corollary 5. Inequalities

$$\sum_{a \in \mathcal{A}} x_a \leq 1 \quad (50)$$

$$p_a \leq M_a x_a \quad \forall a \in \mathcal{A} \quad (51)$$

$$p_a \geq 0 \quad \forall a \in \mathcal{A} \quad (52)$$

$$\sum_{b \in \mathcal{A}} (p_b + c_b x_b) + c_{od} \left(1 - \sum_{b \in \mathcal{A}} x_b\right) \leq t_a + c_a$$

$$+ \sum_{b \in \mathcal{A} \setminus (\mathcal{S} \cup \{a\})} (p_b + (c_b - c_a)x_b)$$

$$\forall a \in \mathcal{A}, \forall \mathcal{S} \subseteq \mathcal{A} : c_a \geq c_b \quad \forall b \in \mathcal{A} \setminus (\mathcal{S} \cup \{a\}) \quad (53)$$

with $M_a = \max\{0, c_{od} - c_a\}$, are facet defining for \mathcal{P}^C .

Furthermore, this allows us to present a complete description of \mathcal{P} .

Proposition 12. Let $\tilde{\mathcal{P}} = \{(\mathbf{t}; \mathbf{p}; \mathbf{x}) \in \mathbb{R}_+^n \times \mathbb{R}_+^n \times \mathbb{R}_+^n :$

$$\sum_{a \in \mathcal{A}} x_a \leq 1 \quad (54)$$

$$p_a \leq M_a x_a \quad \forall a \in \mathcal{A} \quad (55)$$

$$p_a \geq 0 \quad \forall a \in \mathcal{A} \quad (56)$$

$$\sum_{b \in \mathcal{A}} (p_b + c_b x_b) + c_{od} \left(1 - \sum_{b \in \mathcal{A}} x_b\right) \leq t_a + c_a$$

$$+ \sum_{b \in \mathcal{A} \setminus (\mathcal{S} \cup \{a\})} (p_b + (c_b - c_a)x_b)$$

$$\forall a \in \mathcal{A}, \forall \mathcal{S} \subseteq \mathcal{A} : c_a \geq c_b \quad \forall b \in \mathcal{A} \setminus (\mathcal{S} \cup \{a\}) \quad (57)$$

with $M_a = \max\{0, c_{od} - c_a\}$. Then $\tilde{\mathcal{P}} = \mathcal{P}$.

Proof. Let $\mathcal{A} = \{1, \dots, n\}$ be the toll arcs of the network. We define a fractional point of $\tilde{\mathcal{P}}$ as a point of $\tilde{\mathcal{P}}$ with a fractional component \mathbf{x} , i.e., there exists i in $\{1, \dots, n\}$ such that $0 < x_i < 1$. Similarly, an integer point of $\tilde{\mathcal{P}}$ is defined as a point of $\tilde{\mathcal{P}}$ with an integer component \mathbf{x} , i.e., $x_i \in \{0, 1\}$ for all i in $\{1, \dots, n\}$.

We have proved that inequalities (54)–(57) define facets of \mathcal{P} . Now let us show that any fractional point of $\tilde{\mathcal{P}}$ is not extremal. Indeed, let $(\bar{\mathbf{t}}, \bar{\mathbf{p}}, \bar{\mathbf{x}})$ be a fractional point of $\tilde{\mathcal{P}}$, where $0 \leq \bar{x}_i \leq 1$ for all $i \in \{1, \dots, n\}$ and there exists at least one \bar{x}_i such that $0 < \bar{x}_i < 1$. This point can be presented as a convex combination of integer points of $\tilde{\mathcal{P}}$:

$$(\bar{\mathbf{t}}; \bar{\mathbf{p}}; \bar{\mathbf{x}})$$

$$= \sum_{i \in \{1, \dots, n\}: \bar{x}_i \neq 0} \bar{x}_i \left(\mathbf{t}^i; \frac{1}{\bar{x}_i} \bar{p}_i \mathbf{e}_i; \mathbf{e}_i \right) + \left(1 - \sum_{i=1}^n \bar{x}_i \right) (\mathbf{t}^{n+1}; \mathbf{0}; \mathbf{0}) \quad (58)$$

$$= \left(\sum_{i \in \{1, \dots, n\}: \bar{x}_i \neq 0} \bar{x}_i \mathbf{t}^i + \left(1 - \sum_{i=1}^n \bar{x}_i \right) \mathbf{t}^{n+1}; \bar{\mathbf{p}}; \bar{\mathbf{x}} \right), \quad (59)$$

with $0 \leq \bar{x}_i \leq 1 \forall i \in \{1, \dots, n\}$, $\exists i : 0 < \bar{x}_i < 1$, $\mathbf{t}^i = (t_j^i)_{j=1, \dots, n+1}$ and

$$t_j^i = \max \left\{ 0, \frac{\bar{p}_i}{\bar{x}_i} + c_i - c_j \right\} \quad \forall i, j \in \{1, \dots, n\} : \bar{x}_i \neq 0 \quad (60)$$

$$t_j^{n+1} = c_{od} - c_j + s_j \quad \forall j \in \{1, \dots, n\}, \quad (61)$$

with $s_j \in \mathbb{R}_+$.

These integer points belong to $\tilde{\mathcal{P}}$. Indeed, inequalities (54)–(56) are clearly satisfied. Next, for the integer points corresponding to $i \in \{1, \dots, n\}$, (57) yields either $t_j^i \geq 0$ or $t_j^i \geq \frac{\bar{p}_i}{\bar{x}_i} + c_i - c_j$ depending on whether $i \in \mathcal{A} \setminus (S \cup \{j\})$ or not, which is ensured by (60). For the last integer point $n+1$, the null components \mathbf{x} and \mathbf{p} force tolls $t_j^{n+1} : j \in \mathcal{A}$ to be larger than $c_{od} - c_j$ in (57), which is also ensured by the definition (61).

Next, for each $j \in \mathcal{A}$, we introduce the set $\mathcal{B}(j) = \{i \in \mathcal{A} : \bar{x}_i \neq 0, \frac{\bar{p}_i}{\bar{x}_i} + c_i - c_j > 0\}$ and let s_j be such that

$$s_j = \frac{1}{1 - \sum_{i=1}^n \bar{x}_i} \left[\bar{t}_j - \sum_{i \in \mathcal{B}(j)} (\bar{p}_i + c_i \bar{x}_i - c_j \bar{x}_i) - \left(1 - \sum_{i=1}^n \bar{x}_i \right) (c_{od} - c_j) \right].$$

As the fractional point $(\bar{\mathbf{t}}, \bar{\mathbf{p}}, \bar{\mathbf{x}})$ satisfies (57) with $S \cup \{j\} = \mathcal{B}(j)$, it follows that $s_j \geq 0$. Hence we have

$$\sum_{i \in \{1, \dots, n\} : \bar{x}_i \neq 0} \bar{x}_i t_j^i + \left(1 - \sum_{i=1}^n \bar{x}_i \right) t_j^{n+1} \quad (62)$$

$$= \sum_{i \in \{1, \dots, n\} : \bar{x}_i \neq 0} \bar{x}_i \max \left\{ 0, \frac{1}{\bar{x}_i} \bar{p}_i + c_i - c_j \right\} + \left(1 - \sum_{i=1}^n \bar{x}_i \right) (c_{od} - c_j + s_j) \quad (63)$$

$$= (c_{od} - c_j) \left(1 - \sum_{i=1}^n \bar{x}_i \right) + \sum_{i \in \{1, \dots, n\} : \bar{x}_i \neq 0} \bar{x}_i \max \left\{ 0, \frac{1}{\bar{x}_i} \bar{p}_i + c_i - c_j \right\} + \bar{t}_j - \sum_{i \in \mathcal{B}(j)} (\bar{p}_i + c_i \bar{x}_i - c_j \bar{x}_i) - \left(1 - \sum_{i=1}^n \bar{x}_i \right) (c_{od} - c_j), \quad (64)$$

which is equal to \bar{t}_j by definition of $\mathcal{B}(j)$. ■

6. CONCLUSION

Although the problem of optimal pricing is pervasive in the economics literature, it is fairly recent that its complexity

and theoretical properties have been assessed. In this article, we have provided an in-depth theoretical study of a network variant that possesses features encountered in real highway topologies. Specifically, we obtained complexity results and derived strong valid inequalities that, in a particular case, allow for a complete description of the convex hull of integer solutions associated with the MIP formulation of the bilevel program. Finally, although we did not include them in the article, preliminary numerical tests have shown that the inequalities yield significant decreases in both the optimality gap and number of nodes required in a Branch-and-Cut algorithm (see Heilporn [12]). Based on these results and extensions thereof, it is possible to design improved algorithms for addressing the NPP, which will be the topic of a companion article.

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