EXAMPLES

1.2.3. To find the point of intersection of the two lines $2x + 3y = 6$ and $x - y = 1$, we solve the system:

$$\begin{aligned} 2x+3y&=6\\ x-\ y&=1. \end{aligned}$$

Subtracting twice the second row from the first row yields the equation: $5y = 4$. Therefore: $y = 4/5$. Substituting in the second equation, we get: $x = 1 + y = 1 + 4/5 = 9/5$.

4.5.6. Solve the systems of linear equations:

$$\text{(i)}\quad \begin{aligned} x-\ y+2z&=1\\ 2x-2y+6z&=0\\ x+2y+3z&=1 \end{aligned} \qquad \text{and} \qquad \text{(ii)}\quad \begin{aligned} x_1+\ x_2+2x_3+\ x_4&=0\\ 2x_1+2x_2+3x_3+4x_4&=2\\ x_1+\ x_2+\ x_3-\ x_4&=2. \end{aligned}$$

(i)
$$\begin{matrix} x & y & z \end{matrix}$$
$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 1\\ 2 & -2 & 6 & 0\\ 1 & 2 & 3 & 1 \end{array}\right) \overset{R2\leftarrow R2-2R1}{\longrightarrow} \left(\begin{array}{ccc|c} \underline{1} & -1 & 2 & 1\\ 0 & 0 & 2 & -2\\ 1 & 2 & 3 & 1 \end{array}\right) \overset{R3\leftarrow R3-R1}{\longrightarrow} \left(\begin{array}{ccc|c} \underline{1} & -1 & 2 & 1\\ 0 & 0 & 2 & -2\\ 0 & 3 & 1 & 0 \end{array}\right)$$
$$\overset{R2\leftrightarrow R3}{\longrightarrow} \left(\begin{array}{ccc|c} \underline{1} & -1 & 2 & 1\\ 0 & \underline{3} & 1 & 0\\ 0 & 0 & \underline{2} & -2 \end{array}\right) \overset{R3\leftarrow \frac{1}{2}R3}{\longrightarrow} \left(\begin{array}{ccc|c} \underline{1} & -1 & 2 & 1\\ 0 & \underline{3} & 1 & 0\\ 0 & 0 & \underline{1} & -1 \end{array}\right)$$

Solution: $z = -1$

$3y + z = 0 \Rightarrow y = 1/3$

$x - y + 2z = 1 \Rightarrow x = 1 + y - 2z = 10/3$

(ii)
$$\begin{matrix} x_1 & x_2 & x_3 & x_4 \end{matrix}$$
$$\left(\begin{array}{cccc|c} \underline{1} & 1 & 2 & 1 & 0\\ 2 & 2 & 3 & 4 & 2\\ 1 & 1 & 1 & -1 & 2 \end{array}\right) \begin{matrix} R2\leftarrow R2-2R1\\ \longrightarrow\\ R3\leftarrow R3-R1 \end{matrix} \left(\begin{array}{cccc|c} \underline{1} & 1 & 2 & 1 & 0\\ 0 & 0 & \underline{-1} & 2 & 2\\ 0 & 0 & -1 & -2 & 2 \end{array}\right) \overset{R3\leftarrow R3-R2}{\longrightarrow} \left(\begin{array}{cccc|c} \underline{1} & 1 & 2 & 1 & 0\\ 0 & 0 & \underline{-1} & 2 & 2\\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Solution: $x_2,\ x_3$ free

$-x_3 + 2x_4 = 1 \Rightarrow x_3 = 2x_4 - 1$

$x_1 + x_2 + 2x_3 + x_4 = 0 \Rightarrow x_1 = -x_2 - 2x_3 - x_4 = 2 - x_2 - 5x_4$

In the preceding example, replace the number 2 on the right-hand side of the third equation by 3 and check that the corresponding system is inconsistant.

7. Pursue the reduction process on the preceding examples.

(i)
$$\left(\begin{array}{ccc|c} \underline{1} & -1 & 2 & 1\\ 0 & \underline{3} & 1 & 0\\ 0 & 0 & \underline{1} & -1 \end{array}\right) \overset{R2\leftarrow R2-R3}{\longrightarrow} \left(\begin{array}{ccc|c} \underline{1} & -1 & 2 & 1\\ 0 & \underline{3} & 0 & 1\\ 0 & 0 & \underline{1} & -1 \end{array}\right) \overset{R1\leftarrow R1-2R3}{\longrightarrow} \left(\begin{array}{ccc|c} \underline{1} & -1 & 0 & 3\\ 0 & \underline{3} & 0 & 1\\ 0 & 0 & \underline{1} & -1 \end{array}\right)$$
$$\overset{R2\leftarrow \frac{1}{3}R2}{\longrightarrow} \left(\begin{array}{ccc|c} \underline{1} & -1 & 0 & 3\\ 0 & \underline{1} & 0 & \frac{1}{3}\\ 0 & 0 & \underline{1} & -1 \end{array}\right) \overset{R1\leftarrow R1+R2}{\longrightarrow} \left(\begin{array}{ccc|c} \underline{1} & 0 & 0 & \frac{10}{3}\\ 0 & \underline{1} & 0 & \frac{1}{3}\\ 0 & 0 & \underline{1} & -1 \end{array}\right)$$

Solution: $x = 10/3$, $y = 1/3$, $z = -1$.

(ii)
$$\left(\begin{array}{cccc|c} \underline{1} & 1 & 2 & 1 & 0\\ 0 & 0 & \underline{-1} & 2 & 2\\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \overset{R1\leftarrow R1+2R2}{\longrightarrow} \left(\begin{array}{cccc|c} \underline{1} & 1 & 0 & 5 & 2\\ 0 & 0 & \underline{-1} & 2 & 2\\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Solution: x_2 and x_4 free, $x_3 = 2x_4 - 1$, $x_1 = 2 - x_2 - 5x_4$.

8.9.

$$\text{Let: } A=\begin{pmatrix}1&2&3\\0&4&-1\end{pmatrix}\quad B=\begin{pmatrix}1&2\\0&1\end{pmatrix}\quad C=\begin{pmatrix}1&1\\-1&0\end{pmatrix}\quad D=\begin{pmatrix}1&-1\\2&-2\end{pmatrix}\quad E=\begin{pmatrix}1&1\\1&1\end{pmatrix}.$$

$$\text{Then: } AA^t=\begin{pmatrix}14&5\\5&17\end{pmatrix}\quad A^tA=\begin{pmatrix}1&2&3\\2&20&2\\3&2&10\end{pmatrix}\quad BC=\begin{pmatrix}-1&1\\-1&0\end{pmatrix}\neq CB=\begin{pmatrix}1&3\\1&-2\end{pmatrix}.$$

Remark. The matrices AA^t and A^tA are *always* symmetric.

Check that $DE=0$, although $D\neq 0$ and $E\neq 0$. However: $ED=\begin{pmatrix}3&-3\\3&-3\end{pmatrix}\neq 0$.

Remark. $(A+B)^2=A^2+AB+BA+B^2\neq A^2+2AB+B^2$ unless $AB=BA$!

10. Let $A=\begin{pmatrix}1&2&-1\\0&2&4\end{pmatrix}$ and $B=\begin{pmatrix}2\\0\end{pmatrix}$. The matrix equation $AX=B$ is equivalent to the system:

$$\begin{aligned}x_1+2x_2-\ x_3&=2\\2x_2+4x_3&=0\end{aligned}\qquad \text{where } X=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}.$$

11. Let $A=\begin{pmatrix}1&2&3\\-1&1&0\end{pmatrix}$. Solve the equation $AX=0$.

$$\begin{matrix}x_1&x_2&x_3\end{matrix}\\ \left(\begin{array}{ccc|c}\underline{1}&2&3&0\\-1&1&0&0\end{array}\right)\overset{R2\leftarrow R2+R1}{\longrightarrow}\left(\begin{array}{ccc|c}\underline{1}&2&3&0\\0&\underline{3}&3&0\end{array}\right)$$

Solution: x_3 is free

$3x_2+x_3=0\Rightarrow x_2=-x_3$

$x_1+2x_2+3x_3=0\Rightarrow x_1=-2x_2-3x_3=-x_3$

12. Let $A=\begin{pmatrix}1&2\\3&4\end{pmatrix}$ and $B=\frac{1}{2}\begin{pmatrix}-4&2\\3&-1\end{pmatrix}$. Since $BA=I$, B is a left inverse of A. Thus it should also be a right inverse (check it). Also: $A^2=\begin{pmatrix}7&10\\15&22\end{pmatrix}$ and $(A^2)^{-1}=(A^{-1})^2=B^2=\frac{1}{4}\begin{pmatrix}22&-10\\-15&7\end{pmatrix}$.

In **8.9.** check that $\operatorname{tr}(BC)=-1+0=\operatorname{tr}(CB)=1+(-2)=-1$.

13. Let us find the inverse of $A=\begin{pmatrix}0&1&2\\1&0&1\\1&1&1\end{pmatrix}$:

$$\left(\begin{array}{ccc|ccc}0&1&2&1&0&0\\1&0&1&0&1&0\\1&1&1&0&0&1\end{array}\right)\overset{R1\leftrightarrow R2}{\longrightarrow}\left(\begin{array}{ccc|ccc}\underline{1}&0&1&0&1&0\\0&\underline{1}&2&1&0&0\\1&1&1&0&0&1\end{array}\right)\overset{R3\leftarrow R3-R1}{\longrightarrow}\left(\begin{array}{ccc|ccc}\underline{1}&0&1&0&1&0\\0&\underline{1}&2&1&0&0\\0&1&0&0&-1&1\end{array}\right)$$

$$\overset{R3\leftarrow R3-R2}{\longrightarrow}\left(\begin{array}{ccc|ccc}\underline{1}&0&1&0&1&0\\0&\underline{1}&2&1&0&0\\0&0&\underline{-2}&-1&-1&1\end{array}\right)\overset{R2\leftarrow R2+R3}{\longrightarrow}\left(\begin{array}{ccc|ccc}\underline{1}&0&1&0&1&0\\0&\underline{1}&0&1&-1&1\\0&0&\underline{-2}&-1&-1&1\end{array}\right)$$

$$\overset{R3\leftarrow -\frac{1}{2}R3}{\longrightarrow}\left(\begin{array}{ccc|ccc}\underline{1}&0&1&0&1&0\\0&\underline{1}&0&1&-1&1\\0&0&\underline{1}&\frac{1}{2}&\frac{1}{2}&-\frac{1}{2}\end{array}\right)\overset{R1\leftarrow R1-R3}{\longrightarrow}\begin{matrix}&&&I&&&&&A^{-1}\end{matrix}\\ \left(\begin{array}{ccc|ccc}\underline{1}&0&0&-\frac{1}{2}&\frac{1}{2}&\frac{1}{2}\\0&\underline{1}&0&0&-1&1\\0&0&\underline{1}&\frac{1}{2}&\frac{1}{2}&-\frac{1}{2}\end{array}\right)$$

$$
\begin{aligned}
A^{-1} &= E_6E_5E_4E_3E_2E_1 \\
&= \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
A &= E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1}E_6^{-1} \\
&= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}
$$

Now consider the equation $AX = B$ with $B = (\,1 \quad 0 \quad 2\,)^t$. Its solution is:

$$
X = A^{-1}B = \frac{1}{2}\begin{pmatrix} -1 & 1 & 1 \\ 0 & -2 & 2 \\ 1 & 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -3 \\ -4 \\ 3 \end{pmatrix} = \begin{pmatrix} -\frac{3}{2} \\ -2 \\ \frac{3}{2} \end{pmatrix}.
$$

14. Let $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$

permutations	number of switches	parity	signed el. products
123	0	+	aei
132	1	−	$-afh$
213	1	−	$-bdi$
231	2	+	bfg
312	2	+	$-cdh$
321	1	−	$-ceg$

$|A| = aei - afh - bdi + bfg + cdh - ceg$

If $A = \begin{pmatrix} 2 & 2 & 3 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{pmatrix}$ then $|A| = 2 \cdot 4 \cdot (-2) = -16$ (the matrix is invertible).

15. Evaluation of a determinant by row reduction to a triangular form:

$$
\begin{vmatrix} 0 & 2 & 3 \\ 4 & 0 & 4 \\ 2 & 3 & 1 \end{vmatrix} \overset{R2\leftarrow\frac{1}{4}R2}{=} 4\begin{vmatrix} 0 & 2 & 3 \\ 1 & 0 & 1 \\ 2 & 3 & 1 \end{vmatrix} \overset{R2\leftrightarrow R1}{=} -4\begin{vmatrix} \underline{1} & 0 & 1 \\ 0 & \underline{2} & 3 \\ 2 & 3 & 1 \end{vmatrix}
$$

$$
\overset{R3\leftarrow R3-2R1}{=} -4\begin{vmatrix} \underline{1} & 0 & 1 \\ 0 & \underline{2} & 3 \\ 0 & 3 & -1 \end{vmatrix} \overset{R2\leftarrow\frac{1}{2}R2}{=} -8\begin{vmatrix} \underline{1} & 0 & 1 \\ 0 & \underline{1} & \frac{3}{2} \\ 0 & 3 & -1 \end{vmatrix} \overset{R3\leftarrow R3-3R2}{=} -8\begin{vmatrix} \underline{1} & 0 & 1 \\ 0 & \underline{1} & 3/2 \\ 0 & 0 & -\underline{\frac{11}{2}} \end{vmatrix}
$$

$= -8 \times 1 \times 1 \times (-\dfrac{11}{2}) = 44 \neq 0$ and the matrix is invertible (see **17.**)

16.17. $\begin{vmatrix} a-3 & -2 \\ -2 & a-2 \end{vmatrix} = (a-3)(a-2) - 4 = a^2 - 5a + 2 = 0 \iff a = (5 \pm \sqrt{17})/2$. If $a \neq (5 \pm \sqrt{17})/2$, then the matrix (call it A) is invertible and $|A^{-1}| = 1/|A| = 1/(a^2 - 5a + 2)$.

18. Let $A = \begin{pmatrix} 1^+ & 2^- & 3^+ \\ 2^- & 0^+ & -1^- \\ 0^+ & -1^- & 4^+ \end{pmatrix}$. The cofactor expansion of $\det(A)$ along row 2 yields:

$$|A| = -2\begin{vmatrix} 2 & 3 \\ -1 & 4 \end{vmatrix} + 0 - (-1)\begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix} = -22 - 1 = -23.$$

The expansion along the second column yields:

$$|A| = -2\begin{vmatrix} 2 & -1 \\ 0 & 4 \end{vmatrix} + 0 - (-1)\begin{vmatrix} 1 & 3 \\ 2 & -1 \end{vmatrix} = -16 - 7 = -23.$$

An interesting strategy is to mix row reduction and cofactor expansion. For example:

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 0 & -1 \\ 0 & -1 & 4 \end{vmatrix} \overset{R2\leftarrow R2-2R1}{=} \begin{vmatrix} 1 & 2 & 3 \\ 0 & -4 & -7 \\ 0 & -1 & 4 \end{vmatrix} = 1\begin{vmatrix} -4 & -7 \\ -1 & 4 \end{vmatrix} \text{ (expansion along column 1) } = -23.$$

19. Let $A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. Then: $|A| \overset{R3\leftarrow R3-R2}{=} \begin{vmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = -1\begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} = 2 \neq 0.$

$$\text{co}(A) = \begin{pmatrix} +\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} \\ -\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} & +\begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} \\ +\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix} & +\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -1 \end{pmatrix}$$

$$\text{adj}(A) = \begin{pmatrix} -1 & 1 & 1 \\ 0 & -2 & 2 \\ 1 & 1 & -1 \end{pmatrix} \text{ and } A^{-1} = \frac{1}{2}\begin{pmatrix} -1 & 1 & 1 \\ 0 & -2 & 2 \\ 1 & 1 & -1 \end{pmatrix}$$

20.

Let us solve, using Cramer's rule:
$$\begin{array}{r} x_2+2x_3=3 \\ x_1 \quad + \ x_3=3 \\ x_1+x_2+ \ x_3=2 \end{array}$$

The matrix of the system is $A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ and its determinant is $\det(A) \overset{R3\leftarrow R3-R2}{=} \begin{vmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix}$ $= -\begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} = 2$. Thus we have:

$$x_1 = \frac{1}{\det(A)}\begin{vmatrix} \mathbf{3} & 1 & 2 \\ \mathbf{3} & 0 & 1 \\ \mathbf{2} & 1 & 1 \end{vmatrix} \overset{R3\leftarrow R3-R1}{=} \frac{1}{2}\begin{vmatrix} 3 & 1^- & 2 \\ 3 & 0^+ & 1 \\ -1 & 0^- & -1 \end{vmatrix} = -\frac{1}{2}\begin{vmatrix} 3 & 1 \\ -1 & -1 \end{vmatrix} = 1$$

$$x_2 = \frac{1}{\det(A)}\begin{vmatrix} 0 & \mathbf{3} & 2 \\ 1 & \mathbf{3} & 1 \\ 1 & \mathbf{2} & 1 \end{vmatrix} \overset{R3\leftarrow R3-R2}{=} \frac{1}{2}\begin{vmatrix} 0^+ & 3 & 2 \\ 1^- & 3 & 1 \\ 0^+ & -1 & 0 \end{vmatrix} = -\frac{1}{2}\begin{vmatrix} 3 & 2 \\ -1 & 0 \end{vmatrix} = -1$$

$$x_3 = \frac{1}{\det(A)}\begin{vmatrix} 0 & 1 & \mathbf{3} \\ 1 & 0 & \mathbf{3} \\ 1 & 1 & \mathbf{2} \end{vmatrix} \overset{R3\leftarrow R3-R2}{=} \frac{1}{2}\begin{vmatrix} 0^+ & 1 & 3 \\ 1^- & 0 & 3 \\ 0^+ & 1 & -1 \end{vmatrix} = -\frac{1}{2}\begin{vmatrix} 1 & 3 \\ 1 & -1 \end{vmatrix} = 2.$$

21. $P = (1, 2, -1) \quad Q = (2, 0, 3) \qquad \mathbf{v} = \overrightarrow{PQ} = (2 - 1, 0 - 2, 3 - (-1)) = (1, -2, 4) \qquad -\mathbf{v} = (-1, 2, -4)$

22.

$A = (1, 2, 3) \qquad B = (2, -1, 0) \qquad C = (3, 0, 4)$

$\mathbf{u} = \overrightarrow{AB} = (1, -3, -3) \quad \mathbf{v} = \overrightarrow{BC} = (1, 1, 4) \quad \mathbf{u} + \mathbf{v} = \overrightarrow{AB} + \overrightarrow{BC} = (1 + 1, -3 + 1, -3 + 4) = (2, -2, 1) = \overrightarrow{AC}$

$-6\mathbf{u} = -6(1, -3, -3) = (-6, 18, 18)$

$\|\mathbf{u}\| = \sqrt{1^2 + (-3)^2 + (-3)^2} = \sqrt{19} \qquad \|\mathbf{v}\|^2 = \sqrt{1^2 + 1^2 + 4^2} = \sqrt{18}$

$\mathbf{u} \cdot \mathbf{v} = (1, -3, -3) \cdot (1, 1, 4) = 1 \cdot 1 + (-3) \cdot 1 + (-3) \cdot 4 = -14$

$$\cos(\mathbf{u}, \mathbf{v}) = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \, \|\mathbf{v}\|} = \frac{-14}{\sqrt{19}\sqrt{18}} \approx -.757$$

and the angle between the vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ is approximately 139°.

The dot product of the vectors $(1, 2)$ and $(4, -2)$ is zero. Hence they are orthogonal ($(1, 2) \perp (4, -2)$).

23. Let $\mathbf{u} = (1, 2, -2)$ and $\mathbf{v} = (3, 0, 4)$. Using cofactor expansion along the first row, we have:

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -2 \\ 3 & 0 & 4 \end{vmatrix} = \mathbf{i}(8) - \mathbf{j}(4 - (-6)) + \mathbf{k}(0 - 6) = (8, -10, -6).$$

$$|\sin(\mathbf{u}, \mathbf{v})| = \frac{\|\mathbf{u} \times \mathbf{v}\|}{\|\mathbf{u}\| \, \|\mathbf{v}\|} = \frac{\sqrt{64 + 100 + 36}}{\sqrt{1 + 4 + 4}\sqrt{9 + 0 + 16}} = \frac{2\sqrt{2}}{3} \qquad \cos(\mathbf{u}, \mathbf{v}) = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \, \|\mathbf{v}\|} = -\frac{1}{3}$$

and check that $\sin^2(\mathbf{u}, \mathbf{v}) + \cos^2(\mathbf{u}, \mathbf{v}) = 8/9 + 1/9 = 1$

24. Let $O' = (-1, 2, 0)$ be the new origin. The point $P = (11, 2, 3)$ in the old coordinate system xOy becomes $P = (11 - (-1), 2 - 2, 3 - 0) = (12, 0, 3)$ in the new system $x'O'y'$.

25. $\|\mathbf{u} + \mathbf{v}\|^2 = (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) = \mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{u} = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\mathbf{u} \cdot \mathbf{v}$.

26. Let $\mathbf{u} = (2, -1, 3)$ and $\mathbf{v} = (4, -1, 2)$. We have:

$$\mathbf{w}_1 = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v} = \frac{15}{21}(4, -1, 2) = \left(\frac{20}{7}, -\frac{5}{7}, \frac{10}{7}\right)$$

$$\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = \left(-\frac{6}{7}, -\frac{2}{7}, \frac{11}{7}\right).$$

27.28. • The plane $x+3y-z=4$ has the normal vector $\mathbf{n}=(1,3,-1)$. The two *unit* normal vectors are $\mathbf{n}/\|\mathbf{n}\| = (1,3,-1)/\sqrt{5}$ and $(-1,-3,1)/\sqrt{5}$. This plane has the point-normal form (there are others):

$$1(x-4)+3(y-0)-1(z-0)=0.$$

• Let $A=(1,0,1)$, $B=(1,2,3)$ and $C=(-1,1,0)$, and $ax+by+cz+d=0$ be the equation of the plane through A, B and C. We have:

$$\begin{aligned} a+\quad\ \ c+d&=0\\ a+2b+3c+d&=0\\ -a+b\quad\ \ +d&=0. \end{aligned}$$

Let us solve this homogeneous system:

$$\begin{array}{cccc} a & b & c & d \\ \end{array}$$
$$\left(\begin{array}{cccc|c} \underline{1} & 0 & 1 & 1 & 0\\ 1 & 2 & 3 & 1 & 0\\ -1 & 1 & 0 & 1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} \underline{1} & 0 & 1 & 1 & 0\\ 0 & \underline{2} & 2 & 0 & 0\\ 0 & 1 & 1 & 2 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} \underline{1} & 0 & 1 & 1 & 0\\ 0 & \underline{1} & 1 & 0 & 0\\ 0 & 0 & 0 & \underline{2} & 0 \end{array}\right).$$

Its solution is: $d=0$, c free, $b=-c$, $a=-c$. A particular solution is found by assigning the value 1 to c, and the equation of the plane is:

$$-x-y+z=0.$$

Alternatively, one could have found a vector normal to the plane: $\mathbf{n}=\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ 0 & 2 & 2\\ -2 & 1 & -1 \end{vmatrix}=(-4,-4,4)$,

and a point-normal form (using point A) is:

$$-4(x-1)-4(y-0)+4(z-1)=0 \qquad \text{i.e.} \qquad x+y-z=0.$$

• Let $P=(1,0,2)$ and $\mathbf{v}=(1,2,-1)$. The line going through P and having the same direction as $\mathbf{v}$ has the parametric equation:

$$x-1=t \qquad y-0=2t \qquad z-2=-t.$$

Its symmetric equation is: $x-1=y/2=(z-2)/(-1)$.

• To find the intersection of 2, 3 or more planes in 3D, we solve the corresponding system of equations which can have 0, 1 or infinitely many solutions, depending on the relative positions of the planes.

(i)

$$\begin{array}{rcrcrl} x & + & y & + & z & =1\\ & & y & - & z & =0\\ 2x & + & y & & & =2 \end{array} \qquad \begin{array}{cccc} x & y & z & \\ \end{array}$$
$$\left(\begin{array}{ccc|c} \underline{1} & 1 & 1 & 1\\ 0 & 1 & -1 & 0\\ 2 & 1 & 0 & 2 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} \underline{1} & 1 & 1 & 1\\ 0 & \underline{1} & -1 & 0\\ 0 & 0 & \underline{1} & 0 \end{array}\right)$$

The intersection of the 3 planes occurs at the point $(x,y,z)=(1,0,0)$.

(ii)

$$\begin{array}{rcrcrl} x & + & y & - & z & =1\\ x & + & y & + & z & =5 \end{array} \qquad \begin{array}{ccc} x & y & z \\ \end{array}$$
$$\left(\begin{array}{ccc|c} \underline{1} & 1 & -1 & 1\\ 1 & 1 & 1 & 5 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} \underline{1} & 1 & -1 & 1\\ 0 & 0 & \underline{1} & 2 \end{array}\right)$$

The solution is: $z=2$, y free and $x=3-y$. The parametric equation of the line of intersection of the two planes is: $x=3-t$, $y=t$, $z=2$.

(iii) Let us find the point of intersection of the plane $x+2y-z=2$ and the line $x=t$, $y=t+2$, $z=2t-1$. We insert the parametric equations into the plane equation to obtain:

$$\begin{aligned} t+2(t+2)-(2t-1)&=2\\ t&=3 \end{aligned}$$

and the required point is $(-3,-1,-7)$.

29.

$$\left\| \frac{\mathbf{u}}{\|\mathbf{u}\|} \right\| = \frac{1}{\|\mathbf{u}\|}\|\mathbf{u}\| = 1.$$

$$\begin{aligned}\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 &= (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) + (\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) \\ &= 2\mathbf{u}\cdot\mathbf{u} + 2\mathbf{v}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{u} - \mathbf{u}\cdot\mathbf{v} - \mathbf{v}\cdot\mathbf{u} \\ &= 2(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2).\end{aligned}$$

30. Examples of vector spaces: the set of real functions, the set of 2×2 matrices of the form $\begin{pmatrix} a & a+b \\ 0 & a+3b \end{pmatrix}$, a line going through the origin.

Proof that $k\mathbf{0} = \mathbf{0}$:

$$\begin{aligned} & \mathbf{0}+\mathbf{0} = \mathbf{0} \qquad \text{(axiom 4)} \\ \Rightarrow \quad & k(\mathbf{0}+\mathbf{0}) = k\mathbf{0} \\ \Rightarrow \quad & k\mathbf{0}+k\mathbf{0} = k\mathbf{0} \qquad \text{(axiom 7)} \\ \Rightarrow \quad & k\mathbf{0}+k\mathbf{0}+(-k\mathbf{0}) = k\mathbf{0}+(-k\mathbf{0}) \qquad \text{(axiom 5)} \\ \Rightarrow \quad & k\mathbf{0} = \mathbf{0} \qquad \text{(axiom 5)}\end{aligned}$$

31. The first quadrant of R^2: $Q_1 = \{(x,y)|x \geq 0, y \geq 0\}$ is not closed under scalar multiplication and therefore not a vector space.

The union of Q_1 and the third quadrant $Q_3 = \{(x,y)|x \leq 0, y \leq 0\}$ is closed under scalar multiplication but *not* under addition.

32.
- A set containing the zero vector is always dependent.
- The subset $\{(3,-1),(4,5),(-4,34)\}$ of R^2 is dependent since it contains more than two vectors.
- The set $\{(1,2,3),(-2,-4,-6)\}$ is dependent since the second vector is a multiple of the first.
- The set $\{(1,2),(3,4)\}$ is an independent set since $\alpha_1\begin{pmatrix}1\\2\end{pmatrix} + \alpha_2\begin{pmatrix}3\\4\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$ implies that:

$\alpha_1 + 3\alpha_2 = 0$, $2\alpha_1 + 4\alpha_2 = 0$, and the unique solution to this homogeneous system is the trivial solution.

33. Let $z_1 = 1-5i$ and $z_2 = 3+4i$. We have:

- $|z_1|^2 = 1^2 + (-5)^2 = 26$
- $z_1 z_2 = 3 - 20i^2 + 4i - 15i = 23 - 11i$
- $\overline{z_2} = 3 - 4i$
- $\dfrac{z_1}{z_2} = \dfrac{1-5i}{3+4i} = \dfrac{(1-5i)(3-4i)}{9+16} = \dfrac{1}{25}(-17-19i)$
- $iz = 2 - i \quad \Rightarrow \quad z = (2-i)/i = \dfrac{2}{i} - 1 = \dfrac{2i}{-1} - 1 = -1 - 2i$

34. Let $z = 1+i = \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \sqrt{2}(\cos\pi/4 + i\sin\pi/4)$. We have:

$$\begin{aligned} & z^3 = 2\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) \\ & z^{\frac{1}{2}} = 2^{\frac{1}{4}}\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right) \\ \text{or} \qquad & z^{\frac{1}{2}} = 2^{\frac{1}{4}}\left(\cos\left(\frac{\pi}{8}+\pi\right) + i\sin\left(\frac{\pi}{8}+\pi\right)\right) = 2^{\frac{1}{4}}\left(\cos\frac{-3\pi}{8} + i\sin\frac{-3\pi}{8}\right)\end{aligned}$$

35. Let $z = 3e^{-i\pi}$. We have: $z = 3(\cos\pi + i\sin\pi) = -3$. Also: $\mathrm{Re}(z) = -3$, $\mathrm{Im}(z) = 0$ and $|z| = 3$.