

EXAMPLES

In the following, round brackets (\cdot) are used for vectors and matrices in the “standard” basis, or simply when we do not want to specify any basis.

1. Let $A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 2 & 2 \end{pmatrix}$ and $V = \{\mathbf{x} = (x_1, x_2, x_3, x_4)^t \mid Ax = 0\}$. Let us row-reduce A :

$$A \sim \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Solution of the homogeneous system: x_4 free, x_3 free, $x_2 = 0$, $x_1 = -x_3 - x_4$, or:

$$\begin{pmatrix} -x_3 - x_4 \\ 0 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Hence $B = \left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis for V .

2. To find a basis for the row space of $A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 2 & 2 \end{pmatrix}$, we row-reduce A to $\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, whose first two rows form a basis for the row space.

3. Let us find a basis for the column space of the matrix $A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 7 \\ 3 & 6 & 3 \end{pmatrix}$ composed of column vectors of A .

We will work on the row space of $A^t = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 7 & 3 \end{pmatrix}$.

Let us row-reduce the matrix A , **without switching rows**:

$$\begin{aligned} A^t &= \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 7 & 3 \end{pmatrix} && \text{pick row 1: } \mathbf{b}_1^t = (1 \ 2 \ 3) \\ &\sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 1 & 7 & 3 \end{pmatrix} && \text{drop row 2} \\ &\sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 5 & 0 \end{pmatrix} && \text{pick row 3: } \mathbf{b}_2^t = (1 \ 7 \ 3). \end{aligned}$$

The required basis is composed of columns 1 and 3 of A (rows 1 and 3 of A^t). The rank of A is 2.

4. • Let $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be an orthogonal set of vectors and assume that $\sum_{i=1}^n \alpha_i \mathbf{v}_i = \mathbf{0}$. Then, for any index k :

$$0 = \langle \mathbf{v}_k, \sum_{i=1}^n \alpha_i \mathbf{v}_i \rangle = \sum_{i=1}^n \alpha_i \langle \mathbf{v}_k, \mathbf{v}_i \rangle = \alpha_k \|\mathbf{v}_k\|^2 \Rightarrow \alpha_k = 0.$$

• Let V be the vector space spanned by the functions $\sin x$ and $\cos x$. Define

$$\langle f, g \rangle = \int_0^\pi f(x)g(x) dx.$$

Let $f(x) = \sin x + \cos x$ and $g(x) = \sin x - \cos x$. Then:

$$\begin{aligned} \langle f, g \rangle &= \int_0^\pi (\sin x + \cos x)(\sin x - \cos x) dx = \int_0^\pi (\sin^2 x - \cos^2 x) dx \\ &= -\int_0^\pi \cos 2x dx = -(\sin 2x/2)|_0^\pi = 0 \quad (f(x) \perp g(x)) \\ \|f\|^2 &= \int_0^\pi (\sin x + \cos x)^2 dx = \int_0^\pi (1 + \sin 2x) dx = \pi \\ [d(f, g)]^2 &= \|f(x) - g(x)\|^2 = \|2 \cos x\|^2 = 4\|\cos x\|^2 \\ &= 4 \int_0^\pi \cos x \cdot \cos x dx = 4 \int_0^\pi (1 + \cos 2x)/2 dx = 2\pi. \end{aligned}$$

5. Let $V = R^2$ with inner product $\langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_1 + u_2 v_2$. the Cauchy-Schwarz inequality tells us that

$$|u_1 v_1 + u_2 v_2| \leq \sqrt{u_1^2 + u_2^2} \sqrt{v_1^2 + v_2^2}.$$

Let $u = (1 \ 2)^t$, $v = (3 \ -4)^t$ and check that

$$5 = |\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \|\mathbf{v}\| = \sqrt{5} \cdot 5.$$

6.7. • Let $\mathbf{b}_1 = (\cos \theta \ \sin \theta)^t$ and $\mathbf{b}_2 = (-\sin \theta \ \cos \theta)^t$. Then $B = \{\mathbf{b}_1, \mathbf{b}_2\}$ is an orthonormal basis for R^2 . Hence the rotation matrix $\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$ is an orthogonal matrix whose inverse is its transpose $\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$. Any vector in R^2 can be expressed with respect to the basis B . For example:

$$\begin{aligned} \begin{pmatrix} 1 \\ 1 \end{pmatrix} &= \left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} \right\rangle \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} + \left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix} \right\rangle \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix} \\ &= (\cos \theta + \sin \theta) \mathbf{b}_1 + (-\sin \theta + \cos \theta) \mathbf{b}_2 = \begin{bmatrix} \cos \theta + \sin \theta \\ -\sin \theta + \cos \theta \end{bmatrix}_B. \end{aligned}$$

Now let $W = \text{lin}\{\mathbf{b}'_1, \mathbf{b}'_2\}$, where $\mathbf{b}'_1 = (3/5 \ 0 \ 4/5)^t$ and $\mathbf{b}'_2 = (4/5 \ 0 \ -3/5)^t$; $B' = \{\mathbf{b}'_1, \mathbf{b}'_2\}$ is an orthonormal basis for the subspace W of R^3 . Let us compute the (orthogonal) projection \mathbf{p} of $\mathbf{v} = (0 \ 4 \ 4)^t$ onto W :

$$\mathbf{p} = \left\langle \begin{pmatrix} 0 \\ 4 \\ 4 \end{pmatrix}, \begin{pmatrix} 3/5 \\ 0 \\ 4/5 \end{pmatrix} \right\rangle \begin{pmatrix} 3/5 \\ 0 \\ 4/5 \end{pmatrix} + \left\langle \begin{pmatrix} 0 \\ 4 \\ 4 \end{pmatrix}, \begin{pmatrix} 4/5 \\ 0 \\ -3/5 \end{pmatrix} \right\rangle \begin{pmatrix} 4/5 \\ 0 \\ -3/5 \end{pmatrix} = \begin{bmatrix} 16/5 \\ -12/5 \end{bmatrix}_{B'} = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}.$$

The orthogonal component of \mathbf{v} with respect to W is $(0 \ 4 \ 0)^t$.

- Let us find $\mathbf{p} = \text{Proj}_W(v)$, where $\mathbf{v} = (-1, 2, 6, 0)^t$ and $W = \text{lin}\left\{(0 \ 1 \ 0 \ 1)^t, (-1 \ 0 \ 1 \ 2)^t\right\} = \{\mathbf{b}_1, \mathbf{b}_2\}$.

♡ first method: Since $\mathbf{p} \in W$ we can write $\mathbf{p} = \alpha_1 \mathbf{b}_1 + \alpha_2 \mathbf{b}_2$. Also: $(\mathbf{v} - \mathbf{p}) \perp W$ implies that

$$\begin{aligned}\langle \mathbf{v} - \alpha_1 \mathbf{b}_1 - \alpha_2 \mathbf{b}_2, \mathbf{b}_1 \rangle &= 0 \\ \langle \mathbf{v} - \alpha_1 \mathbf{b}_1 - \alpha_2 \mathbf{b}_2, \mathbf{b}_2 \rangle &= 0.\end{aligned}$$

Rearranging terms, we obtain the linear system:

$$\begin{aligned}\alpha_1 \|\mathbf{b}_1\|^2 + \alpha_2 \langle \mathbf{b}_1, \mathbf{b}_2 \rangle &= \langle \mathbf{v}, \mathbf{b}_1 \rangle \\ \alpha_1 \langle \mathbf{b}_1, \mathbf{b}_2 \rangle + \alpha_2 \|\mathbf{b}_2\|^2 &= \langle \mathbf{v}, \mathbf{b}_2 \rangle.\end{aligned}$$

We have: $\|\mathbf{b}_1\|^2 = 2$, $\|\mathbf{b}_2\|^2 = 6$, $\langle \mathbf{b}_1, \mathbf{b}_2 \rangle = 2$, $\langle \mathbf{v}, \mathbf{b}_1 \rangle = 2$ and $\langle \mathbf{v}, \mathbf{b}_2 \rangle = 7$. The linear system is:

$$\begin{aligned}2\alpha_1 + 2\alpha_2 &= 1 \\ 2\alpha_1 + 6\alpha_2 &= 7.\end{aligned}$$

Its solution is $\alpha_1 = -1/4$, $\alpha_2 = 5/4$. Finally: $\mathbf{p} = -1/4 \mathbf{b}_1 + 5/4 \mathbf{b}_2 = (-1/4) \begin{pmatrix} -5 & -1 & 5 & 9 \end{pmatrix}^t$.

◇ second method: Gram-Schmidtize $\{\mathbf{b}_1, \mathbf{b}_2\}$ and use a well-known (!) formula. We easily obtain $\mathbf{b}'_1 = \frac{1}{\sqrt{2}}$

$$\begin{aligned}\mathbf{b}'_1 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} & \mathbf{b}'_2 &= \text{normalize} \left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \\ 2 \end{pmatrix} - \left\langle \begin{pmatrix} -1 \\ 0 \\ 1 \\ 2 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\rangle \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\} \\ & & &= \text{normalize} \left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\} \\ & & &= \text{normalize} \left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix} \right\} = \frac{1}{2} \begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix} \\ \mathbf{p} &= \langle \mathbf{v}, \mathbf{b}'_1 \rangle \mathbf{b}'_1 + \langle \mathbf{v}, \mathbf{b}'_2 \rangle \mathbf{b}'_2 = \frac{2}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} + \frac{5}{2} \cdot \frac{1}{2} \begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -5/4 \\ -1/4 \\ 5/4 \\ 9/4 \end{pmatrix}.\end{aligned}$$

The distance from \mathbf{v} to W is the norm of the orthogonal component $\mathbf{v} - \mathbf{p}$:

$$\text{distance} = \|\mathbf{v} - \mathbf{p}\| = \left\| \begin{pmatrix} \frac{1}{4} & \frac{9}{4} & \frac{19}{4} & -\frac{9}{4} \end{pmatrix}^t \right\| = \frac{1}{4} \sqrt{1^2 + 9^2 + 19^2 + (-9)^2} = \frac{\sqrt{524}}{4} = \sqrt{131}.$$

- Let $B = \{\mathbf{b}_1, \mathbf{b}_2\} = \{1, x\}$ be a basis for P_1 and $\langle f, g \rangle = \int_0^1 f(t)g(t) dt$. Let us orthogonalize B into $B' = \{\mathbf{b}'_1, \mathbf{b}'_2\}$:

$$\begin{aligned}\|\mathbf{b}_1\|^2 &= \int_0^1 1 \cdot 1 dt = 1 \text{ and } \mathbf{b}'_1 = \mathbf{b}_1 \\ \mathbf{b}'_2 &= \text{normalize} \left\{ x - \left(\int_0^1 t \cdot 1 dt \right) \cdot 1 \right\} \\ &= \text{normalize} \{x - 1/2\} \\ \text{since } \|x - 1/2\|^2 &= \int_0^1 (t - 1/2)^2 dt = 1/12 \\ \text{we have } \mathbf{b}'_2 &= \sqrt{12}(x - 1/2).\end{aligned}$$

8. Consider the two basis $B = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ and $B' = \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$ for R^2 . We have:

$$[\mathbf{v}]_{B'} = \begin{bmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}_{B'} & \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{B'} \end{bmatrix} [\mathbf{v}]_B = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}_{B'} [\mathbf{v}]_B.$$

For example, if $\mathbf{v} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}_B = \begin{bmatrix} 3 \\ 4 \end{bmatrix}_B$, then

$$[\mathbf{v}]_{B'} = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}_{B \rightarrow B'} [\mathbf{v}]_B = \begin{bmatrix} -7 \\ 10 \end{bmatrix}_{B'} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}_I.$$

9. Let P_2 denote the vector space of second-order polynomials and let $B = \{1, x, x^2\}$ denote its standard basis.

Let $B' = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$ be a basis of R^2 . Consider the linear transformation from P_2 into R^2 defined as:

$T(a_0 + a_1x + a_2x^2) = \begin{pmatrix} a_0 + 2a_2 \\ a_0 + a_1 + 3a_2 \end{pmatrix}$. The columns of the transformation matrix $A_{B \rightarrow B'}$ are

$$T(1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}_{B'}, \quad T(x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}_{B'}, \quad T(x^2) = \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}_{B'}.$$

$$A_{B \rightarrow B'} = [T(B)]_{B'} = \begin{bmatrix} 1 & 1 & 3 \\ 0 & -1 & -1 \end{bmatrix}_{B \rightarrow B'}.$$

Let $\mathbf{v} = a_0 + a_1x + a_2x^2 \in P_2$. In the basis B , $\mathbf{v} = [a_0 \ a_1 \ a_2]_B$; T can be expressed matrixially as:

$$[T(\mathbf{v})]_{B'} = A[\mathbf{v}]_B = \begin{bmatrix} 1 & 1 & 3 \\ 0 & -1 & -1 \end{bmatrix}_{B \rightarrow B'} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}_B = \begin{bmatrix} a_0 + a_1 + 3a_2 \\ -a_1 - a_2 \end{bmatrix}_{B'} = \begin{bmatrix} a_0 + 2a_2 \\ a_0 + a_1 + 3a_2 \end{bmatrix}_I$$

where I denotes the standard basis of R^2 .

10. The kernel of the preceding transformation can be obtained by row-reducing the transformation matrix:

$$\begin{array}{ccc|c} a_0 & a_1 & a_2 & \\ \hline \begin{pmatrix} 1 & 1 & 3 & | & 0 \\ 0 & -1 & -1 & | & 0 \end{pmatrix} & \sim & \begin{pmatrix} 1 & 0 & 2 & | & 0 \\ 0 & 1 & 1 & | & 0 \end{pmatrix} & \end{array}$$

The solution to this homogeneous system is: $a_2 = 0$, $a_1 = -a_2$ and $a_0 = -2a_2$. Consequently, a solution can

be written as: $\begin{bmatrix} -2a_2 \\ -a_2 \\ a_2 \end{bmatrix}_B$ or $a_2 \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}_B$, and the polynomial $\mathbf{v} = -2 - x + x^2$ is a basis for the kernel of T .

You should try to change the target basis B' (take the standard basis for instance) and check that the kernel is still the same!

11. • The range of the preceding transformation T can be obtained by computing the column space of the transformation matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 0 & -1 & -1 \end{bmatrix}_{B \rightarrow B'}$ or, equivalently, the row space of $A^t = \begin{pmatrix} 1 & 0 \\ 1 & -1 \\ 3 & -1 \end{pmatrix}$. A basis

for the row space of A^t is $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}_{B'}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}_{B'} \right\} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$. There are others.

♡ Let $V \xrightarrow{T_1} V_1 \xrightarrow{T_2} V_2$ and $T = T_2 \circ T_1$. We can write:

$$\text{rank}(T) = \dim(T_2 \circ T_1)(V) \leq \dim(T_1(V)) = \text{rank}(T_1)$$

Also: $\mathbf{v}_2 \in \text{range}(T) \Rightarrow \mathbf{v}_2 = T_2(\mathbf{v}_1)$ where $\mathbf{v}_1 = T_1(\mathbf{v}) \Rightarrow \mathbf{v}_2 \in \text{range}(T_2)$. Therefore: $\text{range}(T) \subseteq \text{range}(T_2)$, i.e. $\text{rank}(T) \leq \text{rank}(T_2)$. Putting all this together yields

$$\text{rank}(T) \leq \min \{ \text{rank}(T_1), \text{rank}(T_2) \}.$$

For matrix transformations, this yields the interesting result: $\text{rank}(AB) \leq \min \{ \text{rank}(A), \text{rank}(B) \}$. If $A = \begin{pmatrix} 1 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \end{pmatrix}^t$, then $AB = (0)$ and $0 = \text{rank}(AB) \leq \min \{ \text{rank}(A), \text{rank}(B) \} = \min \{ 1, 1 \} = 1$. Now let A be an $m \times n$ matrix with $m > n$. We have $\text{rank}(A) \leq n$. Furthermore, AA^t is a square matrix of order m whose rank, by the preceding result, has to be less than $n < m$ and we can conclude that A is *not* invertible, without evaluating the product AA^t !

12. In the transformation of 10.11.12., it is readily checked that

$$\dim(\ker(T)) + \dim(\text{range}(T)) = 1 + 2 = 3 = \dim(P_2).$$

13. Let $B = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ and $B' = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$ be two basis for R^2 . Consider the transformation from R^2 into itself defined by:

$$\left[T \left(\begin{bmatrix} x \\ y \end{bmatrix}_B \right) \right]_B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_B \begin{bmatrix} x \\ y \end{bmatrix}_B.$$

The transition matrix from B' to B is $P^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}_B$, and the transition matrix from B to B' is its inverse

$P = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}_{B'}$. The transformation matrix with respect to the basis B' is obtained by performing the product:

$$A' = PAP^{-1} = \begin{bmatrix} 5 & 1 \\ 2 & 0 \end{bmatrix}_{B'}.$$

Consider the vector $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{B'} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}_B$. We have: $\left[T \left(\begin{bmatrix} 0 \\ 2 \end{bmatrix}_B \right) \right]_B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_B \begin{bmatrix} 0 \\ 2 \end{bmatrix}_B = \begin{bmatrix} 4 \\ 8 \end{bmatrix}_B$ and

$\left[T \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}_{B'} \right) \right]_{B'} = \begin{bmatrix} 5 & 1 \\ 2 & 0 \end{bmatrix}_{B'} \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{B'} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}_{B'} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}_B$, as expected.

14. Let $A = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}$ and $P = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$. We have:

$$B = PAP^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}.$$

The matrix B is similar to A and has a simpler (diagonal) form.

15. Let $A = \begin{pmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$. Then: $\det(\lambda I - A) = \begin{vmatrix} \lambda - 3 & 2 & 0 \\ 2 & \lambda - 3 & 0 \\ 0 & 0 & \lambda - 5 \end{vmatrix} = (\lambda - 1)(\lambda - 5)^2$. Let us find the eigenspace corresponding to $\lambda = 5$, by solving the system $A - \lambda I = 0$:

$$\begin{array}{ccc|c} x_1 & x_2 & x_3 & \\ \hline 2 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \sim \begin{array}{ccc|c} \frac{1}{2} & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \Rightarrow x_3 \text{ free, } x_2 \text{ free } x_1 = -x_2$$

and the general solution can be written $x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. The set $B_5 = \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis for the eigenspace associated with $\lambda = 5$.

We repeat the process for $\lambda = 1$:

$$\lambda I - A = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix} \quad \text{and} \quad \begin{array}{ccc|c} x_1 & x_2 & x_3 & \\ \hline 2 & -2 & 0 & 0 \\ -2 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \end{array} \sim \begin{array}{ccc|c} \frac{1}{2} & -1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}.$$

The solution to the above system is: $x_3 = 0$, x_2 free and $x_1 = x_2$, and $B_1 = \left\{ (1 \ 1 \ 0)^t \right\}$ is a basis for the eigenspace corresponding to the eigenvalue $\lambda = 1$.

16. • Let $T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$. Three independent eigenvectors are (see 15) $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. We have: $P = \begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $P^{-1} = \frac{1}{2} \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 2 \\ 1 & 1 & 0 \end{pmatrix}$ and $P^{-1}AP = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Note that if we change the order of the eigenvalues, we have to change the order of the columns of P accordingly.

• Consider the matrix $A = \begin{pmatrix} -3 & 2 \\ -2 & 1 \end{pmatrix}$ whose characteristic polynomial is $(\lambda + 1)^2$. Its only eigenvalue is $\lambda = -1$ and the associated eigenspace is $\text{lin} \{(1 \ 1)\}$, whose dimension is 1. The matrix does *not* have two independent eigenvectors and cannot be diagonalized.

17. Consider the matrix $A = \begin{pmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$ of example 15. Its eigenspaces have respective basis

$$B_5 = \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}, B_1 = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\} \quad \text{and their orthonormal counterparts are } B'_5 = \left\{ \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

and $B'_1 = \left\{ \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \right\}$. Let $P = \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix}$ and check that

$$\begin{aligned} P^t A P &= \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} -\frac{5}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{5}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 5 & 0 \end{pmatrix} = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

18.

$$\begin{aligned} y_1' &= y_1 + y_2 & y_1(0) &= 1 \\ y_2' &= 4y_1 - 2y_2 & y_2(0) &= 6 \end{aligned}$$

$$Y' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} Y = AY \quad Y(0) = C = \begin{pmatrix} 1 \\ 6 \end{pmatrix}$$

We have: $\det(A - \lambda I) = \begin{vmatrix} 1 - \lambda & 1 \\ 4 & -2 - \lambda \end{vmatrix} = \lambda^2 + \lambda - 6 = (\lambda + 3)(\lambda - 2)$. The eigenvector corresponding to $\lambda = -3$ is $(-1 \ 4)^t$ and the eigenvector corresponding to $\lambda = 2$ is $(1 \ 1)^t$. The matrix $P = \begin{pmatrix} -1 & 1 \\ 4 & 1 \end{pmatrix}$ diagonalizes A . Indeed:

$$P^{-1}AP = \frac{1}{5} \begin{pmatrix} -1 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix}.$$

We solve next the system $u_1' = -3u_1$, $u_2' = 2u_2$ whose general solution is $u_1 = d_1 e^{-3x}$, $u_2 = d_2 e^{2x}$, and:

$$Y(x) = PU(x) = \begin{pmatrix} -1 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -d_1 e^{-3x} + d_2 e^{2x} \\ 4d_1 e^{-3x} + d_2 e^{2x} \end{pmatrix}.$$

Finally, $Y(0) = \begin{pmatrix} -d_1 + d_2 \\ 4d_1 + d_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \end{pmatrix}$ implies that $d_1 = 1$ and $d_2 = 2$. The particular solution is

$$\begin{aligned} y_1 &= -e^{-3x} + 2e^{2x} \\ y_2 &= 4e^{-3x} + 2e^{2x}. \end{aligned}$$

19. •

$$\begin{aligned} \langle \cos kx, \cos lx \rangle &= \int_0^{2\pi} \cos kx \cos lx \, dx = \frac{1}{2} \int_0^{2\pi} (\cos(k+l)x + \cos(k-l)x) \, dx \\ &= \left[\frac{1}{2(k+l)} \sin(k+l)x + \frac{1}{2(k-l)} \sin(k-l)x \right]_0^{2\pi} \\ &= 0 \text{ if } k \neq l \end{aligned}$$

If $k = l$:

$$\|\cos kx\|^2 = \int_0^{2\pi} \cos^2 kx \, dx = \int_0^{2\pi} \frac{1 + \cos(2kx)}{2} \, dx = \pi.$$

Similar calculations show (exercise) that :

$$\begin{aligned} \langle \sin kx, \sin lx \rangle &= 0 \text{ if } k \neq l \\ \|\sin kx\|^2 &= \pi \quad \forall k \\ \langle \cos kx, \sin lx \rangle &= 0 \quad \forall k, l. \end{aligned}$$

• Let us approximate the function $f(x) = x$ by a trigonometric polynomial of order k .

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_0^{2\pi} f(x) \, dx = \frac{1}{\pi} \int_0^{2\pi} x \, dx = 2\pi \\ a_k &= \frac{1}{\pi} \int_0^{2\pi} x \sin kx \, dx = \frac{1}{\pi} \int_0^{2\pi} x \, d\frac{\sin kx}{k} = \frac{1}{\pi} x \frac{\sin kx}{k} \Big|_0^{2\pi} - \frac{1}{\pi} \int_0^{2\pi} \frac{\sin kx}{k} \, dx = 0 \\ b_k &= \frac{1}{\pi} \int_0^{2\pi} x \cos kx \, dx = -\frac{1}{\pi} \int_0^{2\pi} x \, d\frac{\cos kx}{k} = -\frac{1}{\pi} x \frac{\cos kx}{k} \Big|_0^{2\pi} + \frac{1}{\pi} \int_0^{2\pi} \frac{\cos kx}{k} \, dx = -\frac{2}{k}. \end{aligned}$$

For $n = 2$ one has: $x \approx \pi - (2/1) \sin x - (2/2) \sin 2x = \pi - 2 \sin x - \sin 2x$.