EXAMPLES

In the following, round brackets $(\cdot)$ are used for vectors and matrices in the "standard" basis, or simply when we do not want to specify any basis.

1. Let $A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 2 & 2 \end{pmatrix}$ and $V = \{\mathbf{x} = (x_1, x_2, x_3, x_4)^t | Ax = 0\}$. Let us row-reduce A:

$$A \sim \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Solution of the homogeneous system: x_4 free, x_3 free, $x_2 = 0$, $x_1 = -x_3 - x_4$, or:

$$\begin{pmatrix} -x_3 - x_4 \\ 0 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Hence $B = \left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis for V.

2. To find a basis for the row space of $A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 2 & 2 \end{pmatrix}$, we row-reduce A to $\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, whose first two rows form a basis for the row space.

3. Let us find a basis for the column space of the matrix $A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 7 \\ 3 & 6 & 3 \end{pmatrix}$ composed of column vectors of A.

We will work on the row space of $A^t = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 7 & 3 \end{pmatrix}$.

Let us row-reduce the matrix A, **without switching rows**:

$$\begin{aligned} A^t &= \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 7 & 3 \end{pmatrix} && \text{pick row 1: } \mathbf{b}_1^t = (\,1 \quad 2 \quad 3\,) \\ &\sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 1 & 7 & 3 \end{pmatrix} && \text{drop row 2} \\ &\sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 5 & 0 \end{pmatrix} && \text{pick row 3: } \mathbf{b}_2^t = (\,1 \quad 7 \quad 3\,). \end{aligned}$$

The required basis is composed of columns 1 and 3 of A (rows 1 and 3 of A^t). The rank of A is 2.

4. • Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be an orthogonal set of vectors and assume that $\sum_{i=1}^n \alpha_i \mathbf{v}_i = 0$. Then, for any index k:

$$0 = \langle \mathbf{v}_k, \sum_{i=1}^n \alpha_i \mathbf{v}_i \rangle = \sum_{i=1}^n \alpha_i \langle \mathbf{v}_k, \mathbf{v}_i \rangle = \alpha_k \|\mathbf{v}_k\|^2 \Rightarrow \alpha_k = 0.$$

• Let V be the vector space spanned by the functions $\sin x$ and $\cos x$. Define

$$\langle f, g \rangle = \int_0^\pi f(x) g(x)\, dx.$$

Let $f(x) = \sin x + \cos x$ and $g(x) = \sin x - \cos x$. Then:

$$\begin{aligned}
\langle f, g \rangle &= \int_0^\pi (\sin x + \cos x)(\sin x - \cos x)\, dx = \int_0^\pi (\sin^2 x - \cos^2 x)\, dx \\
&= -\int_0^\pi \cos 2x\, dx = -(\sin 2x/2)|_0^\pi = 0 \qquad (f(x) \perp g(x)) \\
\|f\|^2 &= \int_0^\pi (\sin x + \cos x)^2\, dx = \int_0^\pi (1 + \sin 2x)\, dx = \pi \\
[d(f,g)]^2 &= \|f(x) - g(x)\|^2 = \|2\cos x\|^2 = 4\|\cos x\|^2 \\
&= 4\int_0^\pi \cos x \cdot \cos x\, dx = 4\int_0^\pi (1 + \cos 2x)/2\, dx = 2\pi.
\end{aligned}$$

5. Let $V = R^2$ with inner product $\langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_1 + u_2 v_2$. the Cauchy-Schwarz inequality tells us that

$$|u_1 v_1 + u_2 v_2| \le \sqrt{u_1^2 + u_2^2}\sqrt{v_1^2 + v_2^2}.$$

Let $u = (\,1 \quad 2\,)^t$, $v = (\,3 \quad -4\,)^t$ and check that

$$5 = |\langle \mathbf{u}, \mathbf{v} \rangle| \le \|\mathbf{u}\|\, \|\mathbf{v}\| = \sqrt{5} \cdot 5.$$

6.7. • Let $\mathbf{b}_1 = (\,\cos\theta \quad \sin\theta\,)^t$ and $\mathbf{b}_2 = (\,-\sin\theta \quad \cos\theta\,)^t$. Then $B = \{\mathbf{b}_1, \mathbf{b}_2\}$ is an orthonormal basis for R^2. Hence the rotation matrix $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ is an orthogonal matrix whose inverse is its transpose $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. Any vector in R^2 can be expressed with respect to the basis B. For example:

$$\begin{aligned}
\begin{pmatrix} 1 \\ 1 \end{pmatrix} &= \left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \right\rangle \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} + \left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \right\rangle \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \\
&= (\cos\theta + \sin\theta) b_1 + (-\sin\theta + \cos\theta) b_2 = \begin{bmatrix} \cos\theta + \sin\theta \\ -\sin\theta + \cos\theta \end{bmatrix}_B.
\end{aligned}$$

Now let $W = \operatorname{lin}\{\mathbf{b}_1', \mathbf{b}_2'\}$, where $\mathbf{b}_1' = (\,3/5 \quad 0 \quad 4/5\,)^t$ and $\mathbf{b}_2' = (\,4/5 \quad 0 \quad -3/5\,)^t$; $B' = \{\mathbf{b}_1', \mathbf{b}_2'\}$ is an orthonormal basis for the subspace W of R^3. Let us compute the (orthogonal) projection $\mathbf{p}$ of $\mathbf{v} = (\,0 \quad 4 \quad 4\,)^t$ onto W:

$$\mathbf{p} = \left\langle \begin{pmatrix} 0 \\ 4 \\ 4 \end{pmatrix}, \begin{pmatrix} 3/5 \\ 0 \\ 4/5 \end{pmatrix} \right\rangle \begin{pmatrix} 3/5 \\ 0 \\ 4/5 \end{pmatrix} + \left\langle \begin{pmatrix} 0 \\ 4 \\ 4 \end{pmatrix}, \begin{pmatrix} 4/5 \\ 0 \\ -3/5 \end{pmatrix} \right\rangle \begin{pmatrix} 4/5 \\ 0 \\ -3/5 \end{pmatrix} = \begin{bmatrix} 16/5 \\ -12/5 \end{bmatrix}_{B'} = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}.$$

The orthogonal component of $\mathbf{v}$ with respect to W is $(\,0 \quad 4 \quad 0\,)^t$.

• Let us find $\mathbf{p} = \mathrm{Proj}_W(v)$, where $\mathbf{v} = (-1, 2, 6, 0)^t$ and $W = \mathrm{lin}\left\{(\,0 \quad 1 \quad 0 \quad 1\,)^t, (\,-1 \quad 0 \quad 1 \quad 2\,)^t\right\} = \{\mathbf{b}_1, \mathbf{b}_2\}$.

♡ <u>first method</u>: Since $\mathbf{p} \in W$ we can write $\mathbf{p} = \alpha_1\mathbf{b}_1 + \alpha_2\mathbf{b}_2$. Also: $(\mathbf{v} - \mathbf{p}) \perp W$ implies that

$$\begin{aligned}\langle \mathbf{v} - \alpha_1\mathbf{b}_1 - \alpha_2\mathbf{b}_2, \mathbf{b}_1\rangle &= 0\\ \langle \mathbf{v} - \alpha_1\mathbf{b}_1 - \alpha_2\mathbf{b}_2, \mathbf{b}_2\rangle &= 0.\end{aligned}$$

Rearranging terms, we obtain the linear system:

$$\begin{aligned}\alpha_1\|\mathbf{b}_1\|^2 + \alpha_2\langle \mathbf{b}_1, \mathbf{b}_2\rangle &= \langle \mathbf{v}, \mathbf{b}_1\rangle\\ \alpha_1\langle \mathbf{b}_1, \mathbf{b}_2\rangle + \alpha_2\|\mathbf{b}_2\|^2 &= \langle \mathbf{v}, \mathbf{b}_2\rangle.\end{aligned}$$

We have: $\|\mathbf{b}_1\|^2 = 2$, $\|\mathbf{b}_2\|^2 = 6$, $\langle \mathbf{b}_1, \mathbf{b}_2\rangle = 2$, $\langle \mathbf{v}, \mathbf{b}_1\rangle = 2$ and $\langle \mathbf{v}, \mathbf{b}_2\rangle = 7$. The linear system is:

$$\begin{aligned}2\alpha_1 + 2\alpha_2 &= 1\\ 2\alpha_1 + 6\alpha_2 &= 7.\end{aligned}$$

Its solution is $\alpha_1 = -1/4$, $\alpha_2 = 5/4$. Finally: $\mathbf{p} = -1/4\mathbf{b}_1 + 5/4\mathbf{b}_2 = (-1/4)\,(\,-5 \quad -1 \quad 5 \quad 9\,)^t$.

♢ <u>second method</u>: Gram-Schmidtize $\{\mathbf{b}_1, \mathbf{b}_2\}$ and use a well-known (!) formula. We easily obtain $\mathbf{b}'_1 = \frac{1}{\sqrt{2}}$

$$\mathbf{b}'_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\0\\1\end{pmatrix} \qquad \mathbf{b}'_2 = \mathrm{normalize}\left\{\begin{pmatrix}-1\\0\\1\\2\end{pmatrix} - \left\langle\begin{pmatrix}-1\\0\\1\\2\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\0\\1\end{pmatrix}\right\rangle \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\0\\1\end{pmatrix}\right\}$$

$$= \mathrm{normalize}\left\{\begin{pmatrix}-1\\0\\1\\2\end{pmatrix} - \begin{pmatrix}0\\1\\0\\1\end{pmatrix}\right\}$$

$$= \mathrm{normalize}\left\{\begin{pmatrix}-1\\-1\\1\\1\end{pmatrix}\right\} = \frac{1}{2}\begin{pmatrix}-1\\-1\\1\\1\end{pmatrix}$$

$$\mathbf{p} = \langle \mathbf{v}, \mathbf{b}'_1\rangle\mathbf{b}'_1 + \langle \mathbf{v}, \mathbf{b}'_2\rangle\mathbf{b}'_2 = \frac{2}{\sqrt{2}}\begin{pmatrix}0\\1\\0\\1\end{pmatrix} + \frac{5}{2}\cdot\frac{1}{2}\begin{pmatrix}-1\\-1\\1\\1\end{pmatrix} = \begin{pmatrix}-5/4\\-1/4\\5/4\\9/4\end{pmatrix}.$$

The distance from $\mathbf{v}$ to W is the norm of the orthogonal component $\mathbf{v} - \mathbf{p}$:

$$\text{distance} = \|\mathbf{v} - \mathbf{p}\| = \|\,(\,\tfrac{1}{4} \quad \tfrac{9}{4} \quad \tfrac{19}{4} \quad -\tfrac{9}{4}\,)^t\,\| = \frac{1}{4}\sqrt{1^2 + 9^2 + 19^2 + (-9)^2} = \frac{\sqrt{524}}{4} = \sqrt{131}.$$

• Let $B = \{\mathbf{b}_1, \mathbf{b}_2\} = \{1, x\}$ be a basis for P_1 and $\langle f, g\rangle = \int_0^1 f(t)g(t)\,dt$. Let us orthogonalize B into $B' = \{\mathbf{b}'_1, \mathbf{b}'_2\}$:

$$\begin{aligned}\|\mathbf{b}_1\|^2 &= \int_0^1 1\cdot 1\,dt = 1 \text{ and } \mathbf{b}'_1 = \mathbf{b}_1\\ \mathbf{b}'_2 &= \mathrm{normalize}\left\{x - \left(\int_0^1 t\cdot 1\,dt\right)\cdot 1\right\}\\ &= \mathrm{normalize}\,\{x - 1/2\}\\ \text{since } \|x - 1/2\|^2 &= \int_0^1 (t - 1/2)^2\,dt = 1/12\\ \text{we have } \mathbf{b}'_2 &= \sqrt{12}\,(x - 1/2).\end{aligned}$$

8. Consider the two basis $B = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ and $B' = \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$ for R^2. We have:

$$[\mathbf{v}]_{B'} = \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}_{B'} \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{B'} \right] [\mathbf{v}]_B = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}_{B'} [\mathbf{v}]_B.$$

For example, if $\mathbf{v} = \begin{pmatrix} 3 \\ 7 \end{pmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}_B$, then

$$[\mathbf{v}]_{B'} = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}_{B \to B'} [\mathbf{v}]_B = \begin{bmatrix} -7 \\ 10 \end{bmatrix}_{B'} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}_I.$$

9. Let P_2 denote the vector space of second-order polynomials and let $B = \{1, x, x^2\}$ denote its standard basis. Let $B' = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$ be a basis of R^2. Consider the linear transformation from P_2 into R^2 defined as: $T(a_0 + a_1 x + a_2 x^2) = \begin{pmatrix} a_0 + 2a_2 \\ a_0 + a_1 + 3a_2 \end{pmatrix}$. The columns of the transformation matrix $A_{B \to B'}$ are

$$T(1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}_{B'} \qquad T(x) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}_{B'} \qquad T(x^2) = \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}_{B'}.$$

$$A_{B \to B'} = [T(B)]_{B'} = \begin{bmatrix} 1 & 1 & 3 \\ 0 & -1 & -1 \end{bmatrix}_{B \to B'}.$$

Let $\mathbf{v} = a_0 + a_1 x + a_2 x^2 \in P_2$. In the basis B, $\mathbf{v} = [\, a_0 \quad a_1 \quad a_2 \,]_B$; T can be expressed matricially as:

$$[T(\mathbf{v})]_{B'} = A[\mathbf{v}]_B = \begin{bmatrix} 1 & 1 & 3 \\ 0 & -1 & -1 \end{bmatrix}_{B \to B'} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}_B = \begin{bmatrix} a_0 + a_1 + 3a_2 \\ -a_1 - a_2 \end{bmatrix}_{B'} = \begin{bmatrix} a_0 + 2a_2 \\ a_0 + a_1 + 3a_2 \end{bmatrix}_I$$

where I denotes the standard basis of R^2.

10. The kernel of the preceding transformation can be obtained by row-reducing the transformation matrix:

$$\begin{array}{c} \begin{matrix} a_0 & a_1 & a_2 \end{matrix} \\ \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -1 & -1 & 0 \end{array} \right) \end{array} \sim \begin{array}{c} \begin{matrix} a_0 & a_1 & a_2 \end{matrix} \\ \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \end{array} \right) \end{array}.$$

The solution to this homogeneous system is: $a_2 = 0$, $a_1 = -a_2$ and $a_0 = -2a_2$. Consequently, a solution can be written as: $\begin{bmatrix} -2a_2 \\ -a_2 \\ a_2 \end{bmatrix}_B$ or $a_2 \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}_B$, and the polynomial $\mathbf{v} = -2 - x + x^2$ is a basis for the kernel of T. You should try to change the target basis B' (take the standard basis for instance) and check that the kernel is still the same!

11. • The range of the preceding transformation T can be obtained by computing the column space of the transformation matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 0 & -1 & -1 \end{bmatrix}_{B \to B'}$ or, equivalently, the row space of $A^t = \begin{pmatrix} 1 & 0 \\ 1 & -1 \\ 3 & -1 \end{pmatrix}$. A basis for the row space of A^t is $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}_{B'}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}_{B'} \right\} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$. There are others.

♡ Let $V \xrightarrow{T_1} V_1 \xrightarrow{T_2} V_2$ and $T = T_2 \circ T_1$. We can write:

$$\text{rank}\,(T) = \dim(T_2 \circ T_1)(V) \leq \dim(T_1(V)) = \text{rank}\,(T_1)$$

Also: $\mathbf{v}_2 \in \text{range}\,(T) \Rightarrow \mathbf{v}_2 = T_2(\mathbf{v}_1)$ where $\mathbf{v}_1 = T_1(\mathbf{v}) \Rightarrow \mathbf{v}_2 \in \text{range}\,(T_2)$. Therefore: $\text{range}\,(T) \subseteq \text{range}\,(T_2)$, i.e. $\text{rank}\,(T) \leq \text{rank}\,(T_2)$. Putting all this together yields

$$\text{rank}\,(T) \leq \min\{\text{rank}\,(T_1), \text{rank}\,(T_2)\}.$$

For matrix transformations, this yields the interesting result: $\text{rank}\,(AB) \leq \min\{\text{rank}\,(A), \text{rank}\,(B)\}$. If $A = \begin{pmatrix} 1 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \end{pmatrix}^t$, then $AB = (0)$ and $0 = \text{rank}\,(AB) \leq \min\{\text{rank}\,(A), \text{rank}\,(B)\} = \min\{1, 1\} = 1$. Now let A be an $m \times n$ matrix with $m > n$. We have $\text{rank}\,(A) \leq n$. Furthermore, AA^t is a square matrix of order m whose rank, by the preceding result, has to be less than $n < m$ and we can conclude that A is *not* invertible, without evaluating the product AA^t!

12. In the transformation of **10.11.12.**, it is readily checked that

$$\dim(\ker(T)) + \dim(\text{range}(T)) = 1 + 2 = 3 = \dim(P_2).$$

13. Let $B = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ and $B' = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$ be two basis for R^2. Consider the transformation from R^2 into itself defined by:

$$\left[T\left(\begin{bmatrix} x \\ y \end{bmatrix}_B \right) \right]_B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_B \begin{bmatrix} x \\ y \end{bmatrix}_B.$$

The transition matrix from B' to B is $P^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}_B$, and the transition matrix from B to B' is its inverse $P = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}_{B'}$. The transformation matrix with respect to the basis B' is obtained by performing the product:

$$A' = PAP^{-1} = \begin{bmatrix} 5 & 1 \\ 2 & 0 \end{bmatrix}_{B'}.$$

Consider the vector $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{B'} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}_B$. We have: $\left[T\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix}_B \right) \right]_B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_B \begin{bmatrix} 0 \\ 2 \end{bmatrix}_B = \begin{bmatrix} 4 \\ 8 \end{bmatrix}_B$ and $\left[T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}_{B'} \right) \right]_{B'} = \begin{bmatrix} 5 & 1 \\ 2 & 0 \end{bmatrix}_{B'} \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{B'} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}_{B'} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}_B$, as expected.

14. Let $A = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}$ and $P = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$. We have:

$$B = PAP^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}.$$

The matrix B is similar to A and has a simpler (diagonal) form.

15. Let $A = \begin{pmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$. Then: $\det(\lambda I - A) = \begin{vmatrix} \lambda-3 & 2 & 0 \\ 2 & \lambda-3 & 0 \\ 0 & 0 & \lambda-5 \end{vmatrix} = (\lambda-1)(\lambda-5)^2$. Let us find the eigenspace corresponding to $\lambda = 5$, by solving the system $A - \lambda I = 0$:

$$\begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 \end{matrix} \\ \left(\begin{array}{ccc|c} 2 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \end{array} \sim \left(\begin{array}{ccc|c} \underline{1} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow x_3 \text{ free }, x_2 \text{ free } x_1 = -x_2$$

and the general solution can be written $x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. The set $B_5 = \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis for the eigenspace associated with $\lambda = 5$.

We repeat the process for $\lambda = 1$:

$$\lambda I - A = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix} \quad \text{and} \quad \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 \end{matrix} \\ \left(\begin{array}{ccc|c} 2 & -2 & 0 & 0 \\ -2 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right) \end{array} \sim \left(\begin{array}{ccc|c} \underline{1} & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \underline{1} & 0 \end{array}\right).$$

The solution to the above system is: $x_3 = 0$, x_2 free and $x_1 = x_2$, and $B_1 = \left\{ \begin{pmatrix} 1 & 1 & 0 \end{pmatrix}^t \right\}$ is a basis for the eigenspace corresponding to the eigenvalue $\lambda = 1$.

16. • Let $T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$. Three independent eigenvectors are (see **15**) $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. We have: $P = \begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $P^{-1} = \frac{1}{2} \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 2 \\ 1 & 1 & 0 \end{pmatrix}$ and $P^{-1}AP = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Note that if we change the order of the eigenvalues, we have to change the order of the columns of P accordingly.

• Consider the matrix $A = \begin{pmatrix} -3 & 2 \\ -2 & 1 \end{pmatrix}$ whose characteristic polynomial is $(\lambda+1)^2$. Its only eigenvalue is $\lambda = -1$ and the associated eigenspace is $\operatorname{lin}\{(1 \quad 1)\}$, whose dimension is 1. The matrix does *not* have two independent eigenvectors and cannot be diagonalized.

17. Consider the matrix $A = \begin{pmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$ of example **15**. Its eigenspaces have respective basis $B_5 = \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$, $B_1 = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$ and their orthonormal counterparts are $B_5' = \left\{ \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ and $B_1' = \left\{ \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \right\}$. Let $P = \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix}$ and check that

$$\begin{aligned} P^tAP &= \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} -\frac{5}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{5}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 5 & 0 \end{pmatrix} = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

18.

$$\begin{array}{ll} y_1' = y_1 + y_2 & y_1(0) = 1 \\ y_2' = 4y_1 - 2y_2 & y_2(0) = 6 \end{array}$$

$$Y' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} Y = AY \qquad Y(0) = C = \begin{pmatrix} 1 \\ 6 \end{pmatrix}$$

We have: $\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{vmatrix} = \lambda^2 + \lambda - 6 = (\lambda + 3)(\lambda - 2)$. The eigenvector corresponding to $\lambda = -3$ is $(-1 \quad 4)^t$ and the eigenvector corresponding to $\lambda = 2$ is $(1 \quad 1)^t$. The matrix $P = \begin{pmatrix} -1 & 1 \\ 4 & 1 \end{pmatrix}$ diagonalizes A. Indeed:

$$P^{-1}AP = \frac{1}{5}\begin{pmatrix} -1 & 1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix}.$$

We solve next the system $u_1' = -3u_1$, $u_2' = 2u_2$ whose general solution is $u_1 = d_1 e^{-3x}$, $u_2 = d_2 e^{2x}$, and:

$$Y(x) = PU(x) = \begin{pmatrix} -1 & 1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -d_1 e^{-3x} + d_2 e^{2x} \\ 4d_1 e^{-3x} + d_2 e^2 x \end{pmatrix}.$$

Finally, $Y(0) = \begin{pmatrix} -d_1 + d_2 \\ 4d_1 + d_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \end{pmatrix}$ implies that $d_1 = 1$ and $d_2 = 2$. The particular solution is

$$\begin{aligned} y_1 &= -e^{-3x} + 2e^{2x} \\ y_2 &= 4e^{-3x} + 2e^{2x}. \end{aligned}$$

19. •

$$\begin{aligned} \langle \cos kx, \cos lx \rangle &= \int_0^{2\pi} \cos kx \cos lx \, dx = \frac{1}{2}\int_0^{2\pi} (\cos(k+l)x + \cos(k-l)x) \, dx \\ &= \left[\frac{1}{2(k+l)} \sin(k+l)x + \frac{1}{2(k-l)} \sin(k-l)x \right]_0^{2\pi} \\ &= 0 \text{ if } k \neq l \end{aligned}$$

If $k = l$:

$$\| \cos kx \|^2 = \int_0^{2\pi} \cos^2 kx \, dx = \int_0^{2\pi} \frac{1 + \cos(2kx)}{2} \, dx = \pi.$$

Similar calculations show (exercise) that :

$$\begin{aligned} \langle \sin kx, \sin lx \rangle &= 0 \text{ if } k \neq l \\ \| \sin kx \|^2 &= \pi \qquad \forall k \\ \langle \cos kx, \sin lx \rangle &= 0 \qquad \forall k, l. \end{aligned}$$

• Let us approximate the function $f(x) = x$ by a trigonometric polynomial of order k.

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_0^{2\pi} f(x) \, dx = \frac{1}{\pi}\int_0^{2\pi} x \, dx = 2\pi \\ a_k &= \frac{1}{\pi}\int_0^{2\pi} x \, \sin kx \, dx = \frac{1}{\pi}\int_0^{2\pi} x \, d\frac{\sin kx}{k} = \frac{1}{\pi} x \frac{\sin kx}{k}\Big|_0^{2\pi} - \frac{1}{\pi}\int_0^{2\pi} \frac{\sin kx}{k} \, dx = 0 \\ b_k &= \frac{1}{\pi}\int_0^{2\pi} x \, \sin kx \, dx = -\frac{1}{\pi}\int_0^{2\pi} x \, d\frac{\cos kx}{k} = -\frac{1}{\pi} x \frac{\cos kx}{k}\Big|_0^{2\pi} + \frac{1}{\pi}\int_0^{2\pi} \frac{\cos kx}{k} \, dx = -\frac{2}{k}. \end{aligned}$$

For $n = 2$ one has: $x \approx \pi - (2/1)\sin x - (2/2)\sin 2x = \pi - 2\sin x - \sin 2x$.