EVERYTHING... MATH 105 ©Patrice MARCOTTE

1. **System of linear equations** (m equations, n variables):

$$\begin{array}{ccccccc} a_{11}x_1 + & a_{12}x_2 + & \cdots + & a_{1n}x_n = & b_1 \\ a_{21}x_1 + & a_{22}x_2 + & \cdots + & a_{2n}x_n = & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1}x_1 + & a_{m2}x_2 + & \cdots + & a_{mn}x_n = & b_m \end{array}$$

The system is **consistent** if it has at least one solution $\{x_1, x_2, \ldots, x_n\}$.

2. **Geometry of 2×2 systems:** In the xy-plane, the linear equation $ax+by=c$ defines a line. Two equations result in two lines that can either (i) intersect at a unique point (ii) be parallel (iii) be identical.

3. **Elementary row operations:**
   - switch two equations
   - multiply an equation by a *nonzero* number
   - add (or subtract) a multiple of an equation to another equation

   The resulting system is **(row) equivalent** to the original one, hopefully simpler.

4. **Matrix representation** (of a system of linear equations):

Augmented matrix of the system:
$$\begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & \cdots & x_n & & \end{array} \\ \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right) \end{array}$$

5. **Elementary row operations** on a matrix:
   - switch two rows
   - multiply a row by a *nonzero* constant
   - add (or subtract) a multiple of a row to another row

   Two matrices are said to be **(row) equivalent** if they can be obtained from each other through a sequence of elementary row operations.

6. **Gaussian elimination:** Using row operations, simplify the augmented matrix to obtain zeroes in the southwest area. The leftmost nonzero element of each nonzero row is called a **leader**. All elements *below* a leader should be zero. Rows with leaders should appear before zero rows and the leaders should be located on a NW-SE staircase. At the end of the process, read aloud the equations! Ignore the zero rows with zero right-hand sides (0=0!); if a zero row has a nonzero right-hand side, the system has no solution ($0 \neq$ nonzero); Otherwise, set **free** all variables whose columns have no leader and solve for the remaining (**dependent**) variables by back-substitution. The reduction process goes TOP-DOWN while the back-substitution process goes BOTTOM-UP.

7. **Gauss-Jordan elimination:** Once Gaussian elimination is complete, perform row operations to introduce zeroes *above* the leaders (BOTTOM-UP process). Then solve the system as before.

8. **Matrix:** An $m\times n$ matrix is a rectangular array of m **rows** and n **columns**, usually denoted by an uppercase letter. The elements of a matrix A are sometimes denoted A_{ij} or a_{ij} (entry in row i and column j of the matrix).

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

9. **Matrix operations:**

• **Equality** (only for matrices of identical dimensions):

$$A = B \iff A_{ij} = B_{ij} \quad i = 1, \ldots, m \quad j = 1, \ldots, n.$$

• **Addition** (only for matrices of identical dimensions):

$$C = A + B \iff C_{ij} = A_{ij} + B_{ij} \quad i = 1, \ldots, m \quad j = 1, \ldots, n.$$

• **Multiplication by a scalar**

$$B = kA \iff B_{ij} = kA_{ij} \quad i = 1, \ldots, m \quad j = 1, \ldots, n.$$

• **Transpose:** Let A be an $m \times n$ matrix. Its transpose A^t is an $n \times m$ matrix whose columns are the rows of A and vice versa: $A^t_{ij} = A_{ji} \quad i = 1, \ldots, m \quad j = 1, \ldots, n$. A matrix A is **symmetric** if $A^t = A$, **skew-symmetric** if $A^t = -A$. Skew-symmetric matrices must have zeroes on their diagonals. Note that $(A^t)^t = A$.

• **Product:** $A_{m\times p}B_{p\times n} = C_{m\times n}$ and $C_{ij} = A_{i1}B_{1j} + A_{i2}B_{2j} + \ldots + A_{ip}B_{pj}$. To compute C_{ij}, multiply row i of A (length p) and column j of B (length p). There are mn such products. The matrix product is **associative**: $A(BC) = (AB)C = ABC$ but not **commutative** i.e., except in very special cases, $AB \neq BA$ (it is possible that AB is defined but not BA!).

We have: $(AB)^t = B^tA^t$ (the order of the product is reversed!).

10. **Matrix form of a system:** $AX = B$.

$$\overset{A}{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}} \overset{X}{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}} = \overset{B}{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}}$$

11. **Homogeneous system:** $AX = 0$. An homogeneous system possesses at least the **trivial** (obvious) solution $X = 0$.

12. **Special matrices:**

• **Square matrix:** matrix with the same number of rows and columns. The **diagonal** elements of a square matrix A are the entries on the main (northwest-southeast) **diagonal**. They are of the form A_{ii}. The common number of rows and columns of a square matrix is the **order** of the matrix. The **trace** of a square matrix of order n is the sum of its diagonal elements: $\mathrm{tr}(A) = a_{11} + a_{22} + \ldots + a_{nn}$. We have that $\mathrm{tr}(A + B) = \mathrm{tr}(A) + \mathrm{tr}(B)$, $\mathrm{tr}(A^t) = \mathrm{tr}(A)$ and, more surprisingly: $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.

• **Identity matrix:** square matrix with 1s along the diagonal and 0s elsewhere: $I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$.

Whenever either product is defined: $AI = A$ and $IA = A$.

• **Zero matrix:** matrix with only zero entries. Whenever the product is defined: $A0 = 0A = 0$.

• **Inverse of a matrix:** the inverse of a square matrix A is the unique (if it exists at all) square matrix denoted A^{-1} such that $A^{-1}A = AA^{-1} = I$. We have: $(AB)^{-1} = B^{-1}A^{-1}$ (note the reverse order!), $(A^t)^{-1} = (A^{-1})^t$ and $A^{-n} = (A^{-1})^n = (A^n)^{-1}$.

Any **left inverse** of a matrix A is *the* inverse of A. Any **right inverse** of A is the inverse of A.

13. **Finding A^{-1}:** An **elementary matrix** is a matrix obtained by performing a single elementary row operation on an identity matrix I. Performing a row operation on a matrix A is equivalent to multiplying A by the corresponding elementary matrix E *from the left*! The row-reduction process can be depicted as follows:

$$A \xrightarrow{\text{op1}} E_1A \xrightarrow{\text{op2}} E_2E_1A \xrightarrow{\text{op3}} \ldots \xrightarrow{\text{op}k} E_k \cdots E_1A = R \quad \text{(reduced matrix)}.$$

If $R = I$ then we have: $A^{-1} = E_k \cdots E_2E_1$ and $A = E_1^{-1}E_2^{-1} \cdots E_k^{-1}$. The inverse A^{-1} can be obtained by performing simultaneously the row operations on A and I:

$$(A|I) \xrightarrow{\text{op1}} (E_1|E_1I = E_1) \xrightarrow{\text{op2}} (E_2E_1A|E_2E_1) \xrightarrow{\text{op3}} \ldots \xrightarrow{\text{op}k} (E_k \cdots E_1A|E_k \cdots E_2E_1) = (I|A^{-1}).$$

If A *cannot* be reduced to I then it is *not* invertible. If A is invertible then the system $AX = B$ has the unique solution $X = A^{-1}B$.

14. **Determinant** (of a square matrix): $\det(A)$ (also denoted $|A|$) is the sum of the signed **elementary products** of A. An elementary product is the product of n elements of A, one from each row and each column. Each elementary product corresponds to a permutation on the integers $\{1, 2, \ldots, n\}$. There are $n! = 1 \cdot 2 \cdots n$ such permutations. The **sign** of an elementary product is $+$ if an even number of switches are required to "undo" the corresponding permutation, $-$ otherwise.

    For 2×2 matrices: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$. The determinant of a **triangular** (in particular diagonal) matrix is equal to the product $\Pi_{i=1}^{n} a_{ii}$ of its diagonal elements. Obviously: $|I| = 1$. Beware: If A is a square matrix of order n, $|kA| = k^n|A|$!

15. **Evaluation of determinants** (by row reduction):

$$A \xrightarrow{\text{switch}} A' = EA \Rightarrow |A'| = -|A| \quad A \xrightarrow{\times\text{row by}k} A' = EA \Rightarrow |A'| = k|A|.$$

The other row operation (add a multiple of a row to another row) leaves the determinant unchanged. This can be used in conjunction with row reduction to evaluate the determinant.

16. **Properties of the determinant:** $|AB| = |A|\,|B|$, $|A^t| = |A|$, $|A^{-1}| = 1/|A|$.

17. **Main theorem:** For a square matrix A, the following statements are equivalent:
    - A is invertible
    - A is row equivalent to I
    - $AX = 0$ has only the trivial solution $X = 0$
    - $AX = B$ is consistent for any right-hand side B
    - $\det(A) \neq 0$.

18. **Cofactor expansion of A:** The **minor** M_{ij} corresponding to the entry A_{ij} is the determinant of the submatrix obtained after deleting row i and column j of A; the **cofactor** of A_{ij} is equal to $C_{ij} = (-1)^{i+j}M_{ij}$. We have:

$$\det(A) = \sum_{i=1}^{n} A_{ki}C_{ki}$$

for any row index k. This is called the cofactor expansion of $|A|$ along row k. There are similar expansions along columns, since $|A^t| = |A|$.

19. **Adjoint:** the adjoint $\text{adj}(A)$ of a matrix A is the transpose of its matrix of cofactors $\text{co}(A)$. We have:

$$A^{-1} = (1/\det(A))\text{adj}(A).$$

20. **Cramer's rule** for solving the system $AX = B$, whenever A is invertible: $x_j = \det(A_j)/\det(A)$ where A_j is the matrix obtained by replacing the entries in column j of A by the numbers $b_1, b_2, \ldots, b_n$.

21. Vectors: A vector $\mathbf{v}$ in 2D or 3D is represented by an **arrow** with **tail** P and **head** Q, alternatively denoted $\mathbf{v} = \overrightarrow{PQ}$. Two vectors are **equal** if they have the same **direction** and **length** (the heads and tails need not occupy the same locations). Let $P = (p_1, p_2)$, $Q = (q_1, q_2)$ and $A = (q_1 - p_1, q_2 - p_2)$ in 2D. If one sets the tail at the origin O of the axis, we have: $\mathbf{v} = \overrightarrow{PQ} = \overrightarrow{OA} = (q_1 - p_1, q_2 - p_2)$. Indeed, if the tail of the vector $\mathbf{v}$ is O, $\mathbf{v}$ can be identified by specifying the coordinates of its head node A; the vector $\overrightarrow{PQ}$ has been **translated** to the origin, preserving its length and direction in the process (you should draw a picture at this point). A similar result holds for vectors in 3D.

The **coordinate** vectors are $\mathbf{i} = (1,0,0)$, $\mathbf{j} = (0,1,0)$ and $\mathbf{k} = (0,0,1)$. The **zero** vector is denoted $\mathbf{0} = (0,0,0)$ (in 3D) and the **opposite** of $\mathbf{v} = (v_1, v_2, v_3)$ is $-\mathbf{v} = (-v_1, -v_2, -v_3)$.

22. Operations on vectors (in 3D, similar in 2D):

- **Addition:** geometrically: $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ (third side of the triangle ABC)
  algebraically: $\mathbf{u} + \mathbf{v} = (u_1, u_2, u_3) + (v_1, v_2, v_3) = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$
- **Multiplication by a scalar** k**:** geometrically: length multiplied by $|k|$, direction unchanged if $k > 0$, reversed if $k < 0$.
  algebraically: $k\mathbf{v} = k(v_1, v_2, v_3) = (kv_1, kv_2, kv_3)$.
- **Norm (length):** $\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$ (Pythagoras' theorem)
  $\|\mathbf{AB}\| = \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2}$ (distance between A and B).

A vector whose norm is one is called a **unit** vector. The coordinate vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are unit vectors.

- **Dot product:** geometrically: $\mathbf{u} \cdot \mathbf{v} = \|u\|\,\|v\| \cos(\mathbf{u}, \mathbf{v})$
  algebraically: $\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$.

Two vectors make an **accute** angle if their dot product is positive, an **obtuse** angle if it is negative. They are **orthogonal** (perpendicular) if their dot product is zero, and we write: $u \perp v$.

23. Cross product (*only* in 3D):

– geometrically: $\mathbf{w} = \mathbf{u} \times \mathbf{v}$ is a vector of length $\|\mathbf{u}\|\,\|\mathbf{v}\|\,|\sin(\mathbf{u}, \mathbf{v})|$, orthogonal to both $\mathbf{u}$ and $\mathbf{v}$, and oriented according to the **screwdriver's** rule: turn a screwdriver from $\mathbf{u}$ towards $\mathbf{v}$ and observe the direction of the screw.

– algebraically: $\mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$.

Remark. $\|\frac{1}{2}\mathbf{u} \times \mathbf{v}\|$ = area of the triangle whose sides are $\mathbf{u}$ and $\mathbf{v}$.

24. Change of coordinates: Let $P = (p_1, p_2, p_3)$ in the xOy system of axis, i.e. $\overrightarrow{OP} = (p_1, p_2, p_3)$. If $O' = (o'_1, o'_2, o'_3)$ is the origin of the system $x'O'y'$ then we have: $\overrightarrow{O'P} = \overrightarrow{OP} - \overrightarrow{OO'} = (p_1 - o'_1, p_2 - o'_2, p_3 - o'_3)$ and $P = (p_1 - o'_1, p_2 - o'_2, p_3 - o'_3)$ in the new system.

25. Vector arithmetic: All the usual properties of addition and scalar multiplication are preserved. Furthermore we have:

$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$	$\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$
$\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$	$\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$
$k(\mathbf{u} \cdot \mathbf{v}) = (k\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (k\mathbf{v})$	$(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w})$
$\|k\mathbf{v}\| = \|k\|\,\|\mathbf{v}\|$	$k(\mathbf{u} \times \mathbf{v}) = (k\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (k\mathbf{v})$
$\|\mathbf{v}\| \geq 0$	$\mathbf{v} \times \mathbf{v} = \mathbf{0}$
$\|\mathbf{v}\| = 0 \iff \mathbf{v} = \mathbf{0}$	$\|\mathbf{u} \times \mathbf{v}\|^2 = \|\mathbf{u}\|^2\,\|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2$ (Lagrange's identity)
$\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$	$\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{0}$

26. Orthogonal projection of $\mathbf{u}$ along $\mathbf{v}$:

Let us decompose $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$ where $\mathbf{w}_1$ is aligned with $\mathbf{v}$ and $\mathbf{w}_2 \perp \mathbf{v}$. We have: $\mathbf{w}_1 = k\mathbf{v}$ and $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = \mathbf{u} - k\mathbf{v}$. Also:

$$\mathbf{w}_2 \perp \mathbf{v} \Rightarrow (\mathbf{u} - k\mathbf{v}) \cdot \mathbf{v} = \mathbf{0} \Rightarrow k = (\mathbf{u} \cdot \mathbf{v})/\|\mathbf{v}\|^2.$$

Therefore: $\mathbf{w}_1 = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v}$ and $\mathbf{w}_2 = \mathbf{u} - \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v}$.

27. Plane in 3D: Let $P_0 = (x_0, y_0, z_0)$ be any given point on the plane and $\mathbf{n}$ a nonzero vector orthogonal (**normal**) to the plane. If $P = (x, y, z)$ is any point on the plane then $\mathbf{n} \perp \overrightarrow{P_0P}$, i.e.

$$(a, b, c) \cdot (x - x_0, y - y_0, z - z_0) = 0 \quad \text{or} \quad a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

This is the **point-normal** equation of the plane. The **general** equation of a plane is

$$ax + by + cz + d = 0$$

and the vector (a, b, c) is normal to the plane.

28. Line in 3D: Let $P_0 = (x_0, y_0, z_o)$ be a given point on the line and $\mathbf{v}$ a nonzero vector along the line. If $P = (x, y, z)$ is any point on the line then we have $\overrightarrow{P_0P} = t\mathbf{v}$ ($\overrightarrow{P_0P}$ is aligned with $\mathbf{v}$), i.e.

$$\begin{aligned} x - x_0 &= ta \\ y - y_0 &= tb \\ z - z_0 &= tc \end{aligned} \qquad \text{or} \qquad \begin{aligned} x &= x_0 + ta \\ y &= y_0 + tb \\ z &= z_0 + tc \end{aligned} \qquad \text{(parametric equations)}$$

If a, b and c are all nonzero, we can write the **symmetric** equation of the line as:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

29. Euclidian space: This is the set R^n of ordered n-tuples $\mathbf{u} = (u_1, \ldots, u_n)$, also called Euclidian vectors. We generalize the operations from 2D (R^2) and 3D (R^3):

$$\begin{aligned} &\mathbf{u} \cdot \mathbf{v} = u_1v_1 + \ldots + u_nv_n \\ &\|\mathbf{u}\|^2 = u_1^2 + \ldots + u_n^2 \\ &d(u, v) = \|u - v\| = \sqrt{(u_1 - v_1)^2 + \ldots + (u_n - v_n)^2}. \end{aligned}$$

All operations that were valid in 2D and 3D (except the vector product) are still valid in R^n.

A vector $\mathbf{x}$ in R^n can be represented as an $n \times 1$ matrix $\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$. A system of m equations and n variables can then be written $A\mathbf{x} = \mathbf{b}$ where A is an $m \times n$ matrix and $\mathbf{b}$ a vector of R^m.

30. Abstract vector spaces. A set V is a vector space if its elements (vectors) can be added and multiplied by scalars in such a way that the following axioms are satisfied:

1. $\mathbf{u} + \mathbf{v} \in V$
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
4. $\exists \mathbf{0} | \mathbf{0} + \mathbf{u} = \mathbf{u} + \mathbf{0} = \mathbf{u} \quad \forall \mathbf{u} \in V$
5. $\exists {-\mathbf{u}} | \mathbf{u} + -\mathbf{u} = -\mathbf{u} + \mathbf{u} = \mathbf{0}$
6. $k\mathbf{u} \in V$
7. $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$
8. $(k + l)\mathbf{u} = k\mathbf{u} + l\mathbf{v}$
9. $k(l\mathbf{u}) = (kl)\mathbf{u}$
10. $1\mathbf{u} = \mathbf{u}$

Theorem:

1. $0\mathbf{u} = \mathbf{0}$
2. $k\mathbf{0} = \mathbf{0}$
3. $(-1)\mathbf{u} = -\mathbf{u}$
4. if $k\mathbf{u} = \mathbf{0}$ then $k = 0$ or $\mathbf{u} = \mathbf{0}$.

31. Theorem: A subset W of a vector space V is a **subspace** of V if it is **closed** under addition and scalar multiplication, i.e.:

$$\mathbf{u}, \mathbf{v} \in W \Rightarrow \mathbf{u} + \mathbf{v} \in W$$
$$\mathbf{u} \in W \Rightarrow k\mathbf{u} \in W.$$

The set P_n of polynomials of order n or less is a subspace of the space of functions.

32. Linear independence: Let $S = \{\mathbf{v}_1, \dots \mathbf{v}_r\}$ be a set of vectors. If there exist scalars $\alpha_1, \dots, \alpha_r$ (not all zero) such that $\sum_{i=1}^{r} \alpha_i \mathbf{v}_i = \mathbf{0}$, then we say that the set S is linearly **dependent**. Otherwise it is linearly **independent**. If $r > n$, a set of r vectors in R^n is always dependent. Any set containing the zero vector is dependent.

33. Complex number: ordered pair of real numbers: $z = (a, b)$. The number a, the component along the horizontal axis, is the real part $\mathrm{Re}(z)$. The number b, the component along the vertical axis, is the imaginary part $\mathrm{Im}(z)$. Additions and scalar multiplication are defined as for vectors in R^2.

If we denote $1 = (1, 0)$ and $i = (0, 1)$, we can write: $z = (a, b) = a + bi$. Let us define: $i^2 = -1$ (just for the fun of it). Then:

$$z_1 z_2 = (a_1 + b_1 i)(a_2 + b_2 i) = (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1)i.$$

The **modulus** of $z = a + bi$ is the length of the vector (a, b), denoted

$$|z| = \sqrt{a^2 + b^2}.$$

The **conjugate** of $z = a + bi$ is the symmetric with respect to the real axis: $\overline{z} = a - bi$. We have:

$$z\overline{z} = |z|^2 \qquad \overline{z_1 \pm z_2} = \overline{z}_1 \pm \overline{z}_2 \qquad \overline{z_1/z_2} = \overline{z}_1/\overline{z}_2 \qquad \overline{z_1 z_2} = \overline{z}_1 \overline{z}_2 \qquad \overline{\overline{z}} = z.$$

34. Polar form of a complex number:

$$z = x + iy = \frac{1}{\sqrt{x^2 + y^2}} \left(\frac{x}{\sqrt{x^2 + y^2}} + i \frac{y}{\sqrt{x^2 + y^2}} \right) = r(\cos\theta + i \sin\theta)$$

where the angle θ is the **argument** $\arg(z)$ of z. By convention, one may take $\pi < \theta \leq \pi$. We have:

$$\begin{aligned} z_1 z_2 &= r_1(\cos\theta_1 + i\sin\theta_1) r_2(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)) \\ &= r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)), \end{aligned}$$

i.e. that $|z_1 z_2| = r_1 r_2$ and $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$!

De Moivre formula: $(\cos\theta + i\sin\theta)^n = \cos n\theta + i \sin n\theta$.

nth root of a complex number:

$$z^{1/n} = r^{1/n} \left(\cos\frac{\theta + 2k\pi}{n} + i \sin\frac{\theta + 2k\pi}{n} \right) \quad k = 0, 1, \dots, n-1.$$

35. Complex exponential: $e^{i\theta} = \cos\theta + i\sin\theta$, by definition. We can write:

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}.$$

The usual properties of the exponential hold. In particular:

$$z_1 z_2 = r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1 + \theta_2)}.$$

Also: $z\overline{z} = re^{-i\theta}$. Finally check that: $z\overline{z} = re^{i\theta} re^{-i\theta} = r^2 e^0 = r^2$.

THE END...