

AN INTRODUCTION TO  
BILEVEL PROGRAMMING

Patrice MARCOTTE

DIRO, Université de Montréal

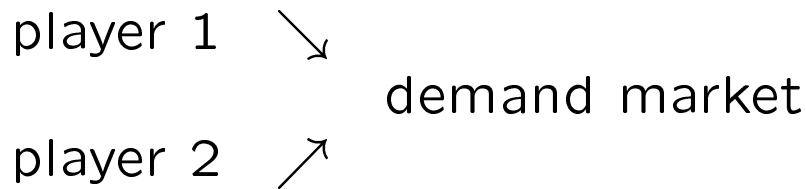
<http://www.iro.umontreal.ca/~marcotte>

Göteborg, March 1999

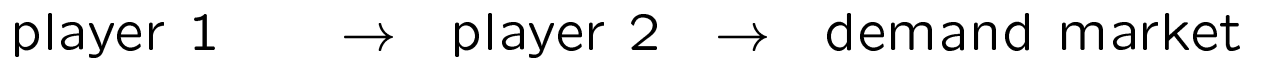
## A two-player game with leader

leader:	$\max_{x \in X} f(x, y)$
follower:	$y \in \arg \min_{y' \in Y} g(x, y')$

### Cournot-Nash

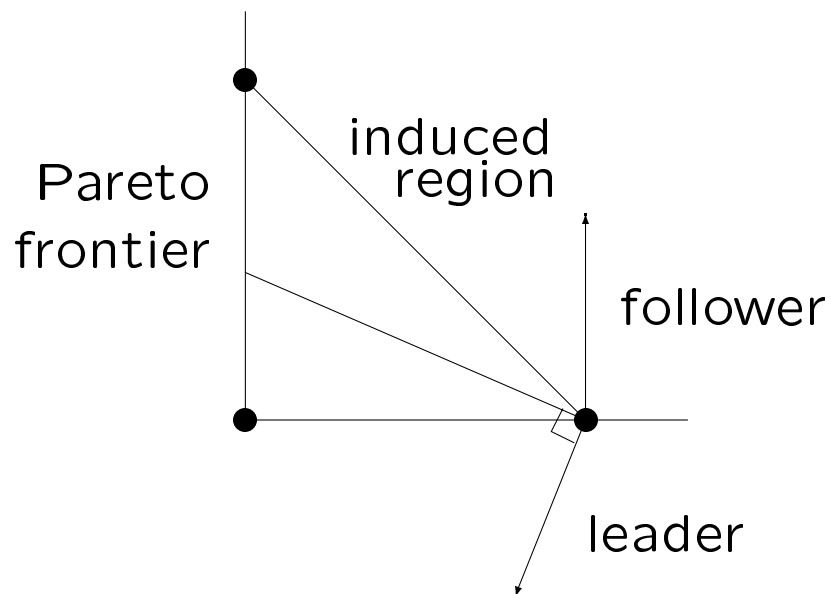
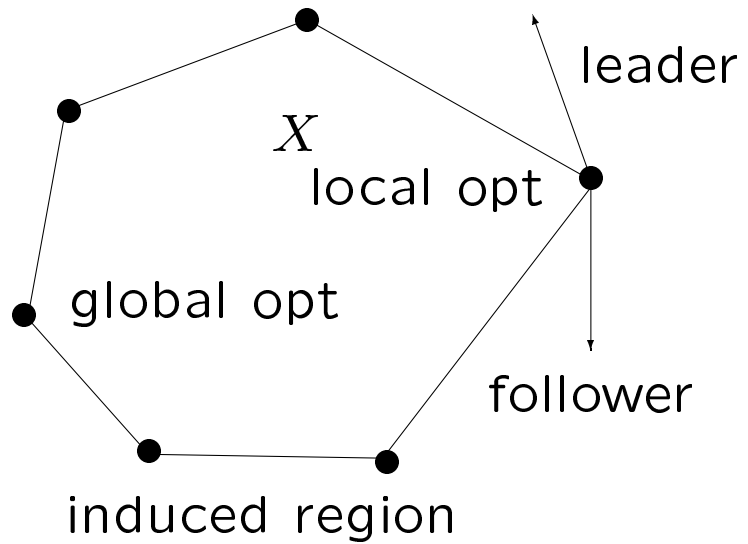


### Stackelberg



# Linear case

$$\begin{array}{ll} \min_x & cx + dy \\ \min_y & ey \\ \text{s.t.} & Ax + By \geq b \end{array}$$



## Exact penalization and bilinear programming

$$\begin{aligned} \min_{x,y,\lambda} \quad & cx + dy + \boxed{K\lambda(Ax + By - b)} \\ \text{s.t.} \quad & Ax + By \geq b \\ & \lambda B = e \\ & \lambda \geq 0 \end{aligned}$$

## Complexity: subsumes 0 – 1 linear programming

$$\begin{aligned} \min_{x \in \{0,1\}^n} \quad & cx \\ \text{s.t.} \quad & Ax \geq b \end{aligned}$$

$$\begin{aligned} \min_x \quad & cx + M \sum \xi_j \\ \max_{\xi} \quad & \sum \xi_j \\ \text{s.t.} \quad & \xi_j \leq x_j \\ & \xi_j \leq 1 - x_j \\ & 0 \leq x_j \leq 1 \\ & Ax \geq b \end{aligned}$$

## Continuous network design

$$\begin{aligned} \min_{x \in X} \quad & \langle C(x, y), y \rangle \\ \text{s.t.} \quad & \langle C(x, y), y - y' \rangle \leq 0 \quad \forall y' \in Y \end{aligned}$$

### General form

$$\boxed{\begin{aligned} \min_{x \in X} \quad & f(x, y) \\ \text{s.t.} \quad & y \in S(x) \end{aligned}} \rightarrow \text{solution set of MP or VI}$$

### Strong form

$$\boxed{\begin{aligned} \min_{x \in X} \quad & \max_{y \in S(x)} \quad f(x, y) \end{aligned}}$$

### Tax-setting

$$\boxed{\begin{aligned} \max_v \quad & v y_1 \\ \min_{x, y} \quad & (c_1 + v) y_1 + c_2 y_2 \\ \text{s.t.} \quad & A_1 y_1 + A_2 y_2 = b \\ & y_1, y_2 \geq 0 \end{aligned}}$$

## Principal agent

$$\begin{aligned} \max_{x(\cdot)} \quad & f(x(\omega(y)), w(y)) \\ \text{s.t.} \quad & g(x(\omega(y)), y) \geq g_{\min} \end{aligned}$$

$$\max_y \quad g(x(\omega(y)), y)$$

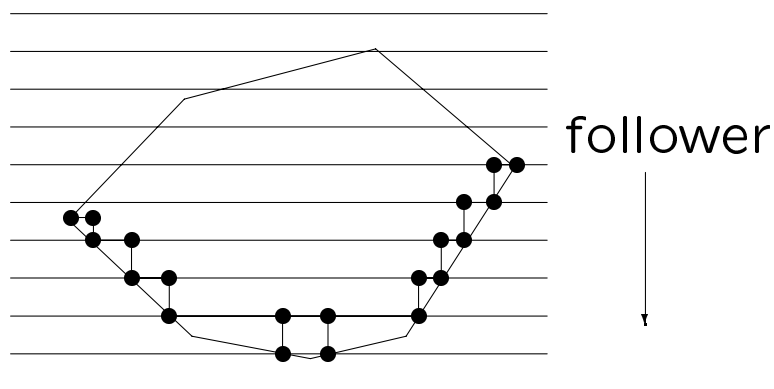
$y$  : effort level

$\omega(y) \in \Omega$  : outcome

$x : \Omega \rightarrow R^n$

## Existence

True if lower level program is well-behaved  
(convex and closed solution set  $S(x)$ )



$$\begin{array}{l} \min_x f(x, y) \\ \\ \min_y g(x, y) \\ \text{s.t. } H(x, y) \leq 0 \end{array}$$

## Value function reformulation

$$v(x) = \min_{y: H(x, y) \leq 0} g(x, y)$$

$$\begin{array}{l} \min_x f(x, y) \\ \text{s.t. } g(x, y) - v(x) = 0 \\ H(x, y) \leq 0 \end{array}$$

Constraint qualification (linear independence of active constraints' gradients) fails generically since:

$$\nabla_x(g(x, y) - v(x)) = \lambda \nabla_x H(x, y).$$

## Kuhn-Tucker reformulation

$$\begin{aligned} \min_{x,y,\lambda} \quad & f(x, y) \\ \text{s.t.} \quad & H(x, y) \leq 0 \\ & \nabla_x g(x, y) + \lambda \nabla_x H(x, y) = 0 \\ & \lambda \geq 0 \\ & \lambda H(x, y) = 0 \end{aligned}$$

## Exact penalty reformulation

$$\begin{aligned} \min_{x,y,\lambda} \quad & f(x, y) + \boxed{K \lambda H(x, y)} \\ \text{s.t.} \quad & H(x, y) \leq 0 \\ & \nabla_x g(x, y) + \lambda \nabla_x H(x, y) = 0 \\ & \lambda \geq 0 \end{aligned}$$

Drawbacks: several stationary points, nonconvex, nondifferentiable.



## Algorithms

- sensibility analysis
- nonconvex NDO
- lower level smoothing (interior points)
- global optimization (B and B)
- heuristics
- exploitation of structure