

A Variational Inequality Approach to Dynamic Pricing under Competition

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### **Some Applications**

#### **Motivation**

#### Traffic Congestion severe in recent years.

# Congestion costs \$640/traveler in 39 areas U.S. in 1994 implying congestion \$48 billion for 1994!







### **Traffic Assignment**



# How can traffic congestion be managed and alleviated ?

# **Revenue Management**

# Post deregulation era: after 1978

Revenue management at : American Airlines
\$1.4 billion in added revenue in 3 years

• Marriott International: 4.7% increase in room revenue per year from revenue management

# **Revenue Management**

# **Airline Revenue Management:**



- "Control and management of reservations inventory in a way that increases company profitability." (Barry Smith)
- "Selling the right seat to the right customer at the right price at the right time." (Tom Cook)

# Industries Adopting Revenue Management

- Airlines
- Hotels
- Railroads
- Car Rentals
- Cruise lines
- Tour and vacation packages
- Television Networks
- Others: public events ticketing, ....



### **Industry Characteristics**

Advanced reservations system Range of prices for services Customer willingness to pay more/less for certain service guarantee Cancellations and no-shows High fixed cost, low variable cost Highly perishable inventory of "product"



# **Price Segmentation**



# Revenue Management Modeling

### Decisions

– How many seats to sell in each fare class for each itinerary in the airline's schedule?

# Maximize Expected Revenue

- account for uncertainty of demands, no-shows, overbooking, etc.
- account for plane seat configurations, capacity, etc.

### **Revenue Management Modeling**

• Prices for fare classes are given from marketing department.

 Only worry about how to manage the inventory (i.e., plane's total capacity)

How about pricing fare classes?

### Outline

**Nonlinear Optimization and Variational Inequalities** 

- Some key concepts and results in NLPs and VIs
- Connection with problems in mathematical programming

#### **An Application in Multi Period Pricing**

- The Best Response Problem --- The Market Equilibrium Problem
- An Iterative Learning Algorithm
- Insights from Numerical Examples
- The Stochastic Demand Setting

### The Price of Anarchy -- The Price of Competition

• Measuring the loss of efficiency due to competition (Asymmetric Costs)

### History of Optimization

# Fermat, 1638; Newton, 1670 $\min f(x)$ x: scalar $\frac{df(x)}{dx}=0$ Minimum Maximum Euler, 1755 min $f(x_1,\ldots,x_n)$ $\nabla f(x) = 0$

### History of Optimization

Lagrange, 1797

min  $f(x_1,\ldots,x_n)$ 

s.t. 
$$g_k(x_1,\ldots,x_n)=0$$
  $k=1,\ldots,m$ 

Euler, Lagrange Problems in infinite dimensions, calculus of variations.

Kuhn and Tucker 1950s

### History of Optimization

1950s Applications.1960s Large Scale Optimization.Karmakar, 1984 Interior point algorithms.



### Where do NLPs Arise?

### Wide Applicability

### Transportation

- Traffic management, Traffic equilibrium ... Revenue management and Pricing
- Finance Portfolio Management
- Equilibrium Problems

### Where do NLPs Arise?

### Wide Applicability

Engineering

 Data Networks and Routing
 Pattern Classification
 Manufacturing
 Resource Allocation
 Production Planning

# The general problem again

 $f(x): \Re^n \mapsto \Re$ is a continuous (usually differentiable) function of nvariables

$$oldsymbol{g_i}(x) \colon \real^n \mapsto \real, i = 1, \dots, m,$$

 $h_j(x) \colon \Re^n \mapsto \Re, j = 1, \ldots, l$ 

# The general problem again



### An Analogy -- Chutes and Ladders



### Two Issues Where do solutions lie?

**Consider the problem:** 

s.t.	$(x-8)^2 + (y-9)^2 \le 49$
	x ≥ 2
	x ≤ 13
	$\mathbf{x} + \mathbf{y} \le 24$



### Where do solutions lie?

### **Consider the problem:**

Minimize 
$$(x - 11)^2 + (y - 10)^2$$

s.t.	$(x-8)^2 + (y-9)^2 \le 49$
	x ≥ 2
	x ≤ 13
	x + y ≤ 24

### **Solution in the Interior**



### Where do solutions lie?

### **Consider the problem:**

Minimize 
$$(x-14)^2 + (y-14)^2$$

s.t.	$(x-8)^2 + (y-9)^2 \le 49$
	x ≥ 2
	x ≤ 13
	$x + y \le 24$

### **Solution on the Boundary**



### Second Issue Global versus Local Minima

### Minimization in one variable over $2 \le x \le 7$



### **Convex Sets**

- A subset  $old S\subset \Re^n$  is a *convex set* if $x,y\in S\Rightarrow \lambda x+(1-\lambda)y\in S \qquad orall \lambda\in [0,1]$
- If S, T are convex sets, then S ∩ T is a convex set
  Implication: The intersection of any collection of convex sets is a convex set







### **Convex Set**



Not a Convex Set



### Not a Convex Set



### **Convex Set**

### Convex Functions

• A function f(x) is a convex function if  $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$  $orall oldsymbol{x},oldsymbol{y} \qquad orall oldsymbol{\lambda} \in [oldsymbol{0},oldsymbol{1}]$ • A function f(x) is a concave function if  $f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda)f(y)$  $orall oldsymbol{x},oldsymbol{y} \qquad orall oldsymbol{\lambda} \in [oldsymbol{0},oldsymbol{1}]$ 

# **Convex Functions**



**Convexity and Minima** 

 $egin{array}{lll} \min & f(m{x}) \ {\sf s.t.} & m{x} \in \mathcal{F} \end{array}$ 

<u>Theorem</u>: Suppose that  $\mathcal{F}$  is a convex set,  $f : \mathcal{F} \to \Re$ is a convex function, and  $x^*$  is a local minimum of P. Then  $x^*$  is a global minimum of f over  $\mathcal{F}$ .

# **COP and Global Minima**



### **Convexity and Minima**

### Proof

Assume that  $x^*$  is not the global minimum. Let y be the global minimum. From the convexity of  $f(\cdot)$ ,

 $egin{aligned} f(y(\lambda)) &= f(\lambda x^* \!+\! (1\!-\!\lambda)y) \leq \lambda f(x^*) \!+\! (1\!-\!\lambda)f(y) \ &< \lambda f(x^*) + (1\!-\!\lambda)f(x^*) = f(x^*) \end{aligned}$ 

for all  $\lambda \in (0, 1)$ .

Therefore,  $f(y(\lambda)) < f(x^*)$  for all  $\lambda \in (0, 1)$ , and so  $x^*$  is not a local minimum, resulting in a contradiction

# $egin{aligned} COP:&\min f(x)\ ext{s.t. }g_1(x)\leq 0\ ert \ &ert \ &e$

COP

### COP

**COP** is called a *convex optimization problem* if  $f(x), g_1(x), \ldots, g_m(x)$  are convex functions

Note that this implies that the feasible region  $\boldsymbol{\mathcal{F}}$  is a convex set

In *COP* we are minimizing a convex function over a convex set

Implication: If **COP** is a convex optimization problem, then any local minimum will be a global minimum.

### Optimality Conditions

**Characterization of convex functions** 

Theorem Let f(x) be continuously differentiable. Then f(x) is convex if and only if

 $abla f(x)'(\overline{x}-x) \leq f(\overline{x}) - f(x)$ 

First order Taylor approximation of *f* is a global underestimator of *f* 

 $f(x) + \nabla f(x)'(\overline{x} - x)$
Problem

 $min_{x\in K}f(x)$ 

•  $K \subseteq \Re^n$  nonempty, closed, convex • f continuously diff/ble over K

If f is convex then local min is also global min

#### **Optimality Conditions - Motivation**



# Generalizes:Zero slope at local min x\*from unconstrained toconstrained problems

1. Necessary If  $x^*$  is a local min of f in  $K \Rightarrow$   $\nabla f(x^*)'(x - x^*) \ge 0, \forall x \in K$  (1) (identifies candidates for local min) 2. Sufficient f convex over K then also if condition (1) is

satisfied then  $x^* = arg \min_{x \in K} f(x)$ 

When  $K = \Re^n$  reduces to  $\nabla f(x^*) = 0$ 







#### First Order Conditions become

 $\nabla f(x^*) = 0$ 

#### **Examples**

# **1st Order Optimality Conditions**



#### Proof

# **1st Order Optimality Conditions**

 $\begin{array}{l} \displaystyle \underbrace{\mathsf{Necessary:}}_{f(x^*) \leq f(y)} \\ \displaystyle \mathsf{If}(x^*) \leq f(y) \\ \displaystyle \mathsf{Let} \text{ any } x \in K, \, \phi(t) = f(x^* + t(x - x^*)), \, t \in [0, \bar{t}] \, (\bar{t} \\ \displaystyle \mathsf{suff. small}) \\ \displaystyle f(x^*) \leq f(y) \Rightarrow \phi(0) \leq \phi(t), \, t \in [0, \bar{t}] \\ \displaystyle \mathsf{Min}_{t \in [0, \bar{t}]} \, \phi(t) \Rightarrow 0 = argmin_{t \in [0, \bar{t}]} \phi(t) \Rightarrow \\ \displaystyle \frac{d\phi(t)}{dt}|_{t=0} \geq 0 \\ \displaystyle \Rightarrow \nabla f(x^*)'(x - x^*) \geq 0, \ \forall x \in K \end{array}$ 

#### Proof

# **1st Order Optimality Conditions**

# $\begin{array}{ll} \underline{\text{Sufficient:}} & \forall x \in K \ \nabla f(x^*)'(x-x^*) \geq \mathbf{0} \\ \text{For } f \text{ convex} \\ \text{Recall: } f(x) \geq f(x^*) + \nabla f(x^*)'(x-x^*) \ \forall x \in K \\ \Rightarrow f(x) - f(x^*) \geq \nabla f(x^*)'(x-x^*) \geq \mathbf{0} \\ \Rightarrow f(x) \geq f(x^*) \ \forall x \in K \text{ i.e } x^* \text{ global min.} \end{array}$





The solution lies at the corner point  $x^* = (13, 11)$ 



#### **Examples**

# **1st Order Optimality Conditions**

 $2(x_1^*-16)(x_1-x_1^*)+2(x_2^*-14)(x_2-x_2^*)=$  $2(13-16)(x_1-13) + 2(11-14)(x_2-11) =$  $2(-3)(x_1-13)+2(-3)(x_2-11)=$  $2(-3)(x_1+x_2-24) \ge 0$ True since  $x_1 + x_2 < 24$ 



 $x^*$  is a local minimum but  $\sqrt{f(x^*)'(x-x^*)} < 0$  for the feasible vector x shown

#### **A MORE GENERAL FORMULATION**

Variational Inequalities

#### Variational Inequality $VI(\mathbf{F},\mathcal{K})$ Find $\mathbf{x}^* \in \mathcal{K}$ such that $\mathbf{F}(\mathbf{x}^*) \ (\mathbf{x}-\mathbf{x}^*) \geq 0 \ \forall \mathbf{x} \in \mathcal{K}$



# **Early History of VIs**

- Study of equilibria in electromagnetism (electrical networks, Kirkhoff's laws)
- Classical mechanics (equilibrium of forces, variational principle)
- Hartman and Stampacchia (1966) (partial differential equations, calculus of variation and control)
- R. Cottle introduces linear complementarity (1964) Karamardian nonlinear complementarity (1969)
- Smith in 1979 in the context of transportation and Dafermos in 1980, defined the variational inequality problem.

# When is VIP equivalent to NLP ??

$$\nabla F(x) = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ & \cdots & \\ \frac{\partial F_n}{\partial x_1} & \cdots & \frac{\partial F_n}{\partial x_n} \end{bmatrix}$$
 symmetric + p.sd.

VI(F,K) equivalent to NLP(f,K) convex

# **Linear Optimization Problems**

$$c^t x \ge c^t x^*$$
, for all  $x \in P$ 

$$F(x) = ??$$

$$f(x) = ??$$

C( )

$$\nabla f(x^*)^t (x - x^*) \ge 0, \text{ for all } x \in K$$
$$F(x^*)^t (x - x^*) \ge 0, \text{ for all } x \in K$$

# **Nonlinear Optimization**



 $oldsymbol{K} = \{oldsymbol{x} \in \Re^{oldsymbol{n}} : oldsymbol{x} \geq oldsymbol{0}\}$ 

Then it follows:

$$abla^{ extsf{F}}_{ asymptot{f}}(x^{*})'(x-x^{*}) \geq 0 \hspace{0.2cm} orall x \geq 0$$

**Optimization Subject to Bounds** 

 $abla f(x^*) \geq 0, \ x^* \geq 0, \ 
abla f(x^*)'x^* = 0 \quad ( ext{NLCP})$ 

 $\Leftrightarrow$ 

# **Optimality Conditions** - R<sup>n</sup><sub>+</sub>



# **Optimality Conditions** - R<sup>n</sup><sub>+</sub>



"**⇒**"

#### **Optimization Subject to Bounds**

#### Proof...

If  $abla f(x^*)'(x-x^*) \geq 0, \ \forall x \in K$ Let  $x = x^* + e_i \ge 0$ ,  $e_i = (0, ..., 1, ..., 0)$  $abla f(x^*)'e_i \geq 0, \ i.e. \ rac{ heta f(x^*)}{ heta x_i} \geq 0$ Repeat  $\forall i$  then  $\nabla f(x^*) \ge 0$  (\*) If  $x_i^* > 0$  then let  $x = x^* - \epsilon e_i \ge 0$  for  $\epsilon$  small  $\epsilon 
abla f(x^*)' e_i \geq 0, \;\; i.e. rac{ heta f(x^*)}{ heta x_i} \leq 0$  $f(*)+(**) \Rightarrow \text{ If } x_i^* > 0 \text{ then } rac{ heta f(x^*)}{ heta x_i} = 0$ i.e.  $\nabla f(x^*)'x^* = 0$  (\*\*\*)

#### **Optimization Subject to Bounds**

#### Proof...

# Since $\mathbf{x}^*$ is feasible then (\*)+(\*\*\*) QED

That is:

# $x^* \ge 0, \quad \nabla f(x^*) \ge 0, \quad x^*, \nabla f(x^*) = 0$

#### **Optimization Subject to Bounds**

#### ...Proof...

"⇐"

If  $x^* \ge 0$ ,  $\nabla f(x^*)'x^* = 0$ ,  $\nabla f(x^*) \ge 0$ Then  $\forall x \ge 0$ ,  $\nabla f(x^*)'x \ge 0$ (combining with  $\nabla f(x^*)'x^* = 0$ )  $\Rightarrow$  $\nabla f(x^*)'(x - x^*) \ge 0$ ,  $\forall x \ge 0$ . QED

These results hold more generally for VIs

#### **Optimization Subject to Bounds**

#### ...Proof...

Generalizes to Optimization Subject to Bounds

 $oldsymbol{K} = \{oldsymbol{x} \in \Re^{oldsymbol{n}}: \hspace{0.1cm} oldsymbol{a}_i \leq oldsymbol{x}_i \leq oldsymbol{b}_i, \hspace{0.1cm} i = 1,...,n \}$ 

```
egin{aligned} &oldsymbol{x}^* 	ext{ local min then} \ &rac{\partial f(x^*)}{\partial x_i} \geq \mathbf{0}, 	ext{ if } oldsymbol{x}^*_i = oldsymbol{a}_i \ &rac{\partial f(x^*)}{\partial x_i} \leq \mathbf{0}, 	ext{ if } oldsymbol{x}^*_i = oldsymbol{b}_i \ &rac{\partial f(x^*)}{\partial x_i} = \mathbf{0}, 	ext{ if } oldsymbol{b}_i > oldsymbol{x}^*_i > oldsymbol{a}_i \end{aligned}
```

# Projection on a Convex Set

## Let $z \in \Re^n$ and $K \subseteq \Re^n$ closed, convex set **Projection Problem**

$$\min f(x) = \|z-x\|^2$$

subject to  $x \in K$ 

### **Projection Over a Convex Set**



Projection of *z* to *K* is:

The Closest Point to z in Feasible Region K

# Projection on a Convex Set

#### **Projection Reformulation**

Important Implication:First Order Optimality conditions: $\bigvee f(x^*)'(x - x^*) \ge 0 \quad \forall x \in K$ F

 $egin{array}{lll} \operatorname{Find}\, x^* \in K: & x^* = Pr_K(x^* - 
ho igotimes f(x^*)) \ arepsilon \end{array}$ 

### **Optimality Conditions through Projection**

 $x^* \in K$ :  $\nabla f(x^*)'(x-x^*) \ge 0$ ,  $\forall x \in K$ 





$$x^* = \Pr_K(x^* - \rho \nabla f(x^*))$$

# Projection on a Convex Set

 $egin{aligned} & 
abla f(x^*)'(x-x^*) \geq \mathbf{0}, orall x \in K \Leftrightarrow \ & 
ho > \mathbf{0}, \, 
ho 
abla f(x^*)'(x-x^*) \geq \mathbf{0}, \, orall x \in K \Leftrightarrow \ & ho 
abla f(x^*)'(x-x^*) \leq \mathbf{0}, \, orall x \in K \Leftrightarrow \ & (x^* - 
ho 
abla f(x^*) - x^*)'(x-x^*) \geq \mathbf{0}, \, orall x \in K \Leftrightarrow \ & ext{Let} \ & z = x^* - 
ho 
abla f(x^*) \ & (z-x^*)'(x-x^*) \leq \mathbf{0}, \, orall x \in K \Leftrightarrow \ & x^* = Pr_K(z) = Pr_K(x^* - 
ho 
abla f(x^*)) \text{ QED} \end{aligned}$ 

Proof

# **Connection with Fixed Points**

 $x^* \in K : F(x^*)^t (x - x^*) \ge 0, \forall x \in K$ 



$$\widehat{\mathbf{WHY}}?$$

$$x^* = \Pr_{K}(x^* - \rho F(x^*))$$

$$T(x) = Pr_{K}(x - r F(x))$$

$$\mathbf{x}^* = \mathbf{T}(\mathbf{x}^*)$$

#### How would you solve the VI or NLP as a result?

# Projection on a Convex Set

Motivates the following method

#### Proof

**X**<sup>k+2</sup>

Xk+

 $x^{k+1} - \rho \sqrt{f(x^{k+1})}$  $x^k - \rho \sqrt{f(x^k)}$ 

 $x^{k+1} = Pr_K(x^k - 
ho 
abla f(x^k))$ 

This can be viewed as a constrained version of Steepest Descent  $-\nabla f(x^k)$ 

K

# **Overview of Solution Methods**

- 1) Solve Equivalent Convex Min Problem
- 2) Projection Methods:  $x_{k+1} = \Pr_{K}^{G}(x_{k} \rho G^{-1}F(x_{k}))$ G p.d, sym.  $\rho > 0$ , suf. small
- **3)** Relaxation Methods:  $g_i(x_{k+1}, x_k) = F_i(x_1^k, \dots, x_{i-1}^k, x_i^{k+1}, x_{i+1}^k, \dots, x_n^k)$
- 4) Linearization Methods:  $g(x_{k+1}, x_k) = F(x_k) + A(x_k)(x_{k+1} x_k)$
- 5) General Iterative Scheme:

Given  $x_k$ , find  $x_{k+1}$  in K:  $g(x_{k+1}, x_k)(x - x_{k+1}) \ge 0$  for all x in K

- 6) Frank-Wolfe:  $y_k = Arg \min_{y \in K} F(x_k)y, x_{k+1} in [x_k, y_k]$  via line-search
- 7) Simplicial Decomposition: Column generation approach
- 8) Cutting Plane Type Methods: Ellipsoid Method
- 9) Interior Point Methods



# $\mathsf{F}(\mathsf{x}) = \mathsf{0}$

#### T:K --->K, K closed, convex subset of R<sup>n</sup>

#### F(x)=x-T(x), for all x in K

#### VI(F,K), FP(T,K) same solutions, if any!

# **Relation Between Problems**





- K compact, convex set
- **F** continuous on K

then VI(f, K) at least one solution

VI(f,K) has a solution iff exists R>0: VI(F,K<sub>R</sub>) has  $\mathbf{x_R}^*$  $\| \mathbf{x_R}^* \| < \mathbf{R}$ 

#### F is **coercive**:

 $(F(x) - F(x_0))(x - x_0)/||x - x_0|| \longrightarrow infty$ as  $||x|| \longrightarrow infty$ for x in K, some  $x_0$  in K then VI(F, K) **at least one** solution
# **Conditions on VI Problem**

Monotonicity  $(F(x)-F(y))(x-y) \ge 0$  --- JF p.sd. --- convexity of f

Strict Mon. (F(x)-F(y))(x-y) > 0,  $x \neq y --- JF$  p.d. --- s. convexity of f

Stro. Mon. a>0,  $(F(x)-F(y))(x-y) \ge a ||x-y||^2 --- JF$  u.pd. -- u. conv of f

BUT for LP (c-c)(x-y)=0 ALWAYS!

Strong-f-Mon. b>0,  $(F(x)-F(y))(x-y) \ge b ||F(x)-F(y)||^2 ---JF(x)-bJF(x)J(F(y))$ Co-coercivity p.sd.



# F <u>strictly monotone</u> $\longrightarrow$ VI(F,K) unique solution if any





**1st Order Optimality Conditions** 

**Optimization over a Simplex** 

 $K = \{x \in \Re^n : x \ge 0, \sum_{i=1}^n \chi_i = d\},\$ where d > 0 in  $\Re$ 

If  $x_i^* > 0$  then  $rac{\partial f(x^*)}{\partial x_i} \leq rac{\partial f(x^*)}{\partial x_j}, \ \forall j$ 

#### **Optimization over a Simplex**

Example:  $f(x) = x_1^2 + x_2^2 + (x_3 + 15)^2$ subject to  $x_1 + x_2 + x_3 = 20, \ x_1, x_2, x_3 \ge 0$ Optimal solution:  $x_1^* = 10 = x_2^*$  and  $x_3^* = 0$ Notice that  $x_1^* > 0$ ,  $x_2^* > 0$  then  $rac{\partial f(x^*)}{\partial x_1}=2x_1^*=20=rac{\partial f(x^*)}{\partial x_2}=2x_2^*$  $\leq rac{\partial f(x^*)}{\partial x_2} = 2(x_3^*+15) = 30.$ 

# **Some Application Areas**

# Transportation

Transportation Planning IVHS Free Flight

### Economics

Spatial Price Equilibria Pure Exchange Equilibria General Economic Equilibria Financial Equilibria

<u>Multi-period Pricing</u>





### **Optimal Routing**



**Optimization over a Simplex** 

**Optimal Routing...** 

Given a data net and a set W of OD pairs w = (i, j) each OD pair w has input traffic dw
Optimal routing problem:

Min 
$$C(x) = \sum_{\rho} c_{\rho}(x) x_{\rho} = c(x)' x$$

$$s.t. \quad \sum_{p\in P_{oldsymbol{w}}} x_p = d_w, \;\; orall w\in W$$

 $oldsymbol{x_p} \geq oldsymbol{0}, \hspace{0.1cm} orall oldsymbol{p} \in oldsymbol{P_w}, \hspace{0.1cm} oldsymbol{w} \in oldsymbol{W}$ 

**System Optimum** 

**Optimization over a Simplex** 

...Optimal Routing...

 $egin{aligned} ext{Optimality conditions:} & & 
abla C(x^*)'(x-x^*) \geq 0, & orall ext{feasible} x & 
onumber \ & 
abla C(x) = (rac{\partial C(x)}{\partial x_1},...,rac{\partial C(x)}{\partial x_p},...rac{\partial C(x)}{\partial x_n}) & 
onumber \end{aligned}$ 

Equivalent Optimality conditions:  $x_p^* > 0 \Rightarrow \frac{\partial C(x^*)}{\partial x_p} \le \frac{\partial C(x^*)}{\partial x_{p'}}, \ \forall p' \in P_w$ 

Used paths have smallest marginal cost

### **1st Order Optimality Conditions**

#### **Optimization over a Simplex**

...Optimal Routing...

# Optimality conditions: $abla C(x^*)'(x-x^*) \ge 0, \quad \forall ext{feasible } x$ $abla C(x) = (rac{\partial C(x)}{\partial x_1}, ..., rac{\partial C(x)}{\partial x_p}, ... rac{\partial C(x)}{\partial x_n})$

Equivalent Optimality conditions:  $x_p^* > 0 \Rightarrow \frac{\partial C(x^*)}{\partial x_p} \leq \frac{\partial C(x^*)}{\partial x_{p'}}, \ \forall p' \in P_w$ 

### **Traffic Assignment**





$$c_{p_1}(x) = x_{p_1} + \frac{1}{4}x_{p_2} + 13, c_{p_2}(x) = 3x_{p_2} + \frac{1}{2}x_{p_1} + 5, c_{p_3}(x) = 2x_{p_3} + 25$$

Behavioral Assumption: No user has ANY incentive to change

Only shortest paths are used !

If  $x_{\rho}^* > 0 \implies c_{\rho}(x^*) = \min_q c_q(x^*)$ 





Arises is various contexts!





$$c_{p_1}(x^*)(x_{p_1} - 6) = 20(x_{p_1} - 6)$$
  

$$c_{p_2}(x^*)(x_{p_2} - 4) = 20(x_{p_2} - 4) \implies c(x^*)^t (x - x^*) \ge 0$$
  

$$c_{p_3}(x^*)(x_{p_3} - 0) \ge 20(x_{p_3} - 0) \qquad \forall x \text{ feasible}$$

### **Conversely?**

# **Decentralized Problem----VIs**

#### p in P, w in W, i in L

- $f = \Sigma_{i in P} F_p$
- $\mathbf{d}_{\mathsf{w}} = \Sigma_{\mathsf{p} \text{ in } \mathsf{P}_{\mathsf{W}}} F_{\mathsf{p}}$
- $C_{p}(f) = \Sigma_{i in P} c_{i}(f)$



### A flow pattern ( $f^*$ , $F^*$ ) is **U-O**

if for all w in W and every p in  $\mathsf{P}_{\mathsf{w}}$ 

$$C_{p}(f^{*}) = v_{w} \text{ if } F^{*}_{p} > 0,$$
  
 $C_{p}(f^{*}) \ge v_{w} \text{ if } F^{*}_{p} = 0$ 

Equivalent to VI(c, K) ?:  $c(f^*)^t(f-f^*) \ge 0, \forall f$ 

 $\mathsf{K}=\{(f,F): f_i=\Sigma_{i \text{ in } P} F_{p_i} \mathsf{d}_w=\Sigma_{p \text{ in } P_W} F_{p_i} F_p \ge 0\}$ 



# Traffic<br/>AssignmentUnderlying ?System Optimum ← User Optimum<br/>SOME

Equivalent to Find  $\mathbf{x}^*$ :  $c(x^*)'(x - x^*) = \sum_{n} c_p(x^*)(x_p - x_p^*)$ Objective C does not exist if the Jacobian matrix Jc is not symmetric

Identical with optimality conditions of routing problem if we identify the path travel time with the derivative of cost *C* wrt path flow  $\boldsymbol{x}_p$ , i.e.  $\boldsymbol{c}_p(\boldsymbol{x}) = \begin{vmatrix} \frac{\partial C(\boldsymbol{x})}{\partial \boldsymbol{x}_p} \\ \frac{\partial C(\boldsymbol{x})}{\partial \boldsymbol{x}_p} \end{vmatrix}$ 

### Traffic Assignment

### Example: One OD pair connected by 3 paths, with demand d = 10 $J_{C} = \begin{bmatrix} 1 & \frac{1}{4} & 0 \\ \frac{1}{2} & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ $c_{p_1}(x) = x_{p_1} + \frac{1}{4}x_{p_2} + 13,$ $c_{p_2}(x) = 3x_{p_2} + \frac{1}{2}x_{p_1} + 5,$ $c_{p_3}(x) = 2x_{p_3} + 25$ $x_{p_1}^*=6, x_{p_2}^*=4, x_{p_3}^*=0$ $c_{p_1}(x^*) = c_{p_2}(x^*) = 20 \le c_{p_3}(x^*) = 25$ Can we find C(x): $\nabla C(x) = c(x)$ ?

NO, the Jacobian matrix Jc is not symmetric

#### System Optimum vs User Optimum



 $\sum_{p} c_{p}(x^{*})(x_{p}-x^{*}) \ge 0$ , for all x feasible



- Variational Inequalities
- Connections with other problem classes
- Nonlinear Optimization (NLP)
- First Order Optimality Conditions for
- Optimization Subject to Bounds
- Optimization over Simplex
- Projection Problems
- Applications in Transportation



# **Questions?**





#### **Nonlinear Optimization and Variational Inequalities**

# $\checkmark$

#### **Multi Period Pricing**

- The Best Response Problem --- The Market Equilibrium Problem
- An Iterative Learning Algorithm
- Insights from Numerical Examples
- The Stochastic Demand Setting

#### The Price of Anarchy -- The Price of Competition

• Measuring the loss of efficiency due to competition (Asymmetric Costs)

### **Another Application**

#### Multi-Period Pricing for Perishable Products





#### **Total Market**

Single, homogenous product



#### **Total Market**

Market demand for Seller *i* is determined by the function  $h_i(p_1,...,p_l)$ 

**Hotel room** 



Period 1

Period t

Period T

#### Market in a Multi-period Setting Perishable Product

Flight NY-Boston Airlines Advance booking

# **Characteristics**

Oligopolistic market

- (Few Firms Compete for Products)
- Dynamic nature of problem
- Level of inventory

(Prices depend on level of inventory left, demand, prices of competitors)

- Nature of goods (perishable or not)



# **Application Areas**



Airline Industry

Pricing airline tickets under competition

Service Industry

 Car Rentals, Hotels

 Retail Industry

 Filene's - Macy's

 Internet Pricing







### Sample Literature

#### **Revenue Management, Pricing and Inventory Control:**

- Rosen (1965), Harker (1984), Murphy etal (1982)
- Gallego and van Ryzin (1993)
- Bitran and Mondschein (1997)
- Federgruen and Heching (1999)
- Chan et.al (2000)
- Bernstein and Federgruen (2001)
- Inventory: Zipkin (1999)

(BOOK)

- RM: McGill and van Ryzin (1999), Caldentey & Bitran (2002) (REVIEWS)
- Pricing: Elmaghraby & Keskinocak (2002),

Chan et.al., Gilbert & Yano (2003)

Competition: Cachon, Netessine (2003)

(REVIEWS)

(REVIEW)

# The Decentralized Problem



# **Pricing and Allocation**

- How many seats to sell (allocate) in each fare class?
- How to price each fare class?
- Account for uncertainty and competitors and network of legs in the itinerary.

### **The Decentralized Problem**

**User Optimum** 

Demand Non-separable

 $h_{1}(p) = -g_{11}p_{1} + g_{12}p_{2} + h_{1}$  $h_{2}(p) = g_{21}p_{1} - g_{22}p_{2} + h_{2}$ 

Could be Asymmetric g<sub>12</sub> is not g<sub>21</sub>



#### **Questions:**

- How should seller i price the product in the face of competition?
- > What are the equilibrium prices for all competitors in the market?
- > How inefficient does the system become due to competition?

# Multi-period oligopoly models

- Total capacity over horizon is fixed
  - No option to replenish capacity.
  - Inventory holding costs are negligible.
  - No backordering allowed.
  - Typically finite horizon problems.
  - Policies consist of decisions regarding pricing and protection levels.
  - Suited for pricing in transportation, revenue management, communication, energy etc.



#### **Multi Period Pricing**

- The Best Response Problem --- The Market Equilibrium Problem
- An Iterative Learning Algorithm
- Insights from Numerical Examples
- Conclusions

Based on paper with A. Sood entitled: "Competitive Multi-period Pricing for Perishable Products"

# Assumptions

Joint work with

A. Sood

Perfect Information (of their and competitors' demand, total inventories)

Demand depends on current period prices

> For this talk: Demand is deterministic

> Single product and fixed perishability deadline

> Sellers are revenue maximizers over time horizon



#### **Best Response Policy**

#### If : Demand is deterministic


### Cannot be applied...

- The best response problem can not be formulated as a convex optimization problem (except only under linear demand).
- The objective in the best response problem is not quasi-concave (except for a single period problem).
- The feasible space is not a lattice because of the fixed-inventory constraint. Hence the game is not a supermodular game.

### **Best Response Policy**

If : Demand function  $h_i^t$  is s. decreasing w.r.t.  $p_i^t$ 

Note: In the optimal solution,  $d_i^t = h_i^t(p_i^t, \bar{p}_{-i}^t)$ 

# Approach

Best response optimization problem



#### Variational inequality

Variational inequality



Best response optimization problem **Best Response Policy** 

### If *h<sub>i</sub>(.)* is a **concave function** then the Best Response Problem can be **expressed as a** Variational Inequality problem

#### **Demand decreases faster as price increases**

#### Question

# When is the **Best Response Problem** reasonable?

#### **Proposition 1**

#### The Best Response policy exists.



Perfect Information (of their and competitors' demand, total inventories)

Demand depends on current period prices

> For this talk: Demand is deterministic

> Single product

Demand s. decreases with own price

#### Theorem 2

#### The Best Response policy is **unique**. The Variational Inequality has a **unique** solution.

[ $h_i^t(p_i^t, p_{-i}^t)$  is a **strictly decreasing** function of  $p_i^t$ ]

### Market Equilibrium



#### **Nash Equilibrium Policies**

### Questions

- Does such a Nash Equilibrium exist?
- •ls it unique?
- How do we **compute** it?

# Approach

Variational inequality



Joint variational inequality

Variational inequality



Joint variational inequality

Best response Optimization problem



Joint variational inequality

#### **Theorem 4**

#### There exists a **unique** market equilibrium.

[h<sub>i</sub>(p) is concave in p]
[-h(p) is strictly monotone in p]

#### **Monotonicity Conditions**

• Strict Monotonicity

$$[F(x^{1}) - F(x^{2})]^{T}[x^{1} - x^{2}] > 0 \quad \forall x^{1} \neq x^{2}$$

S. Monotonicity F = gradient *obj* — S. Convexity of *obj* 

# Lemma

Let  $(p^*, d^*)$  be a solution to joint variational inequality.  $(p^*, d^*)$  satisfies the following relation:

$$d_i^{t*} = h_i^t(p_i^{t*}, p_{-i}^{t*}), \quad \forall \ i \in \mathbf{I}, \ t \in \mathbf{T}.$$

### • Demand more sensitive to own price than competitors' prices

#### and

# • Own demand decreases less than competitors' demand increases through own price increase



### **Naive Optimizers**



### Question

If all sellers are **naive optimizers** and the process is repeated several times, do the policies **converge** to an **equilibrium?** 

**Iterative learning** 

### **Theorem 5**

Iterative learning will **converge** to Nash equilibrium prices.

 $\begin{bmatrix} -h(p_i, p_{-i}) \text{ is strongly monotone } (\alpha) \text{ in } p_i \end{bmatrix}$  $\begin{bmatrix} h(p_i, p_{-i}) \text{ is Lipschitz continuous } (L) \text{ in } p_{-i} \end{bmatrix}$  $\begin{bmatrix} \alpha > L \end{bmatrix}$ 

#### **Monotonicity Conditions**

• Strict Monotonicity

$$[F(x^{1}) - F(x^{2})]^{T}[x^{1} - x^{2}] > 0 \quad \forall x^{1} \neq x^{2}$$

Strong Monotonicity

$$[F(x^{1}) - F(x^{2})]^{T}[x^{1} - x^{2}] \ge \alpha ||x^{1} - x^{2}||^{2} \quad \forall x^{1}, x^{2}$$

• Lipschitz Continuity

$$||F(x^{1}) - F(x^{2})|| \le L||x^{1} - x^{2}|| \quad \forall x^{1}, x^{2}$$

### **Convergence conditions**

For any given 
$$\bar{\mathbf{p}}_i$$
,  $\mathbf{h}_i(\bar{\mathbf{p}}_i, \mathbf{p}_{-i})$  is Lipschitz continuous with respect to  $\mathbf{p}_{-i}$  with parameter  $\mathcal{L}$ .

$$\|\mathbf{h}_i(ar{\mathbf{p}}_i, \hat{\mathbf{p}}_{-i}) - \mathbf{h}_i(ar{\mathbf{p}}_i, ar{\mathbf{p}}_{-i})\| \leq \mathcal{L} \|\hat{\mathbf{p}}_{-i} - ar{\mathbf{p}}_{-i}\|$$

2. For any given 
$$\bar{\mathbf{p}}_{-i}$$
,  $-\mathbf{h}_i(\mathbf{p}_i, \bar{\mathbf{p}}_{-i})$  is strongly mono-  
tone with respect to  $\mathbf{p}_i$  with parameter  $\alpha$   
 $(-\mathbf{h}_i(\hat{\mathbf{p}}_i, \bar{\mathbf{p}}_{-i}) + \mathbf{h}_i(\check{\mathbf{p}}_i, \bar{\mathbf{p}}_{-i})) \cdot (\hat{\mathbf{p}}_i - \check{\mathbf{p}}_i) \geq \alpha \|\hat{\mathbf{p}}_i - \check{\mathbf{p}}_i\|^2$ 

### 3. $\alpha > L$

Example  

$$h_1^{\dagger}(p) = h_1^{\dagger} - g_{11}^{\dagger} p_1^{\dagger} + g_{12}^{\dagger} p_2^{\dagger}$$
  
 $h_2^{\dagger}(p) = h_2^{\dagger} - g_{22}^{\dagger} p_2^{\dagger} + g_{21}^{\dagger} p_1^{\dagger}$ 



#### Solving the Best Response Sub-Problems

#### At Step k: solve Best Response Sub-Problem:

$$\begin{array}{ccc} \max_{\mathbf{d}_{i},\mathbf{p}_{i}} & \sum_{t=1}^{T} d_{i}^{t} \cdot p_{i}^{t} & \mathbf{d}_{i}^{t}, \mathbf{p}_{i}^{t} \\ \text{such that} & d_{i}^{t} \leq h_{i}^{t}(p_{i}^{t},(p_{-i}^{t})_{k-1}) & \forall t \in \mathbf{T} \\ & \sum_{t=1}^{T} d_{i}^{t} \leq C_{i} & \mathbf{i=1,...,N} \\ & d_{i}^{t}, & p_{i}^{t} \geq \mathbf{0} & \forall t \in \mathbf{T}. \end{array}$$

### **Demand** $h_i^t(., (p_{-i}^t)_{k-1}) = h_{i,k-1}^t(.)$ s. decreas. $\rightarrow$ invertible

$$p_i^t = \bar{p}_{i,k-1}^t(d_i^t) = h_{i,k-1}^t^{-1}(d_i^t)$$

$$\begin{array}{ll} \max_{\mathbf{d}_{i}} & \sum_{t=1}^{T} d_{i}^{t} \cdot \bar{p}_{i}^{t}(d_{i}^{t}) \\ \text{such that} & \sum_{t=1}^{T} d_{i}^{t} \leq C_{i} \\ & d_{i}^{t} \geq \mathbf{0} \qquad \forall t \in \mathbf{T}. \end{array}$$

#### Solving the Best Response Sub-Problems

#### At Step k: solve Best Response Sub-Problem:

$$\begin{array}{ll} \max_{\mathbf{d}_{i}} & \sum_{t=1}^{T} d_{i}^{t} \cdot \bar{p}_{i}^{t}(d_{i}^{t}) \\ \text{such that} & \sum_{t=1}^{T} d_{i}^{t} \leq C_{i} \\ & d_{i}^{t} \geq \mathbf{0} \qquad \forall t \in \mathbf{T}. \end{array}$$

$$\max_{\mathbf{d}_{i}} \sum_{t=1}^{T} \pi_{i}^{t} = \sum_{t=1}^{T} d_{i}^{t} \cdot \bar{p}_{i}^{t}(d_{i}^{t})$$
$$d_{i}^{t} \ge \mathbf{0} \qquad \forall t \in \mathbf{T}.$$

$$\max_{\mathbf{d}_i} \ \pi_i^t = d_i^t \cdot \bar{p}_i^t(d_i^t) \\ d_i^t \ge \mathbf{0}$$

For all t=1,...T

#### Check if total demand does not exceed capacity



# Numerical examples

2 seller  $I = \{1, 2\}$ 10 period  $T = \{1, 2, \dots, 10\}$ 

The demand is linear in prices and symmetric with respect to both sellers and varies with time:

$$h_i^t = D_{\mathsf{base}}^t - \beta^t p_i^t + \alpha^t p_{-i}^t \quad \forall i \in \mathbf{I}$$

### Numerical examples

Model parameters held constant	Parameters varied
$D_{base} = \{110, 100, 100, 100, 90, 90, 100, 100, 80, 60\}$ $\beta = \{1.2, 1.2, 1.1, 1.0, 0.9, 0.8, 0.7, 0.6, 0.5, 0.4\}$ $\alpha = \{1.0, 1.1, 1.0, 0.8, 0.8, 0.7, 0.5, 0.4, 0.4, 0.4\}$	$\begin{array}{l} \{C1, C2\} = \\ \{3000, 2000\}, \\ \{3000, 500\}, \\ \{1000, 500\} \end{array}$

#### Trend in pricing policies with varying Capacities.

## Numerical examples (varying capacity)



### Numerical examples (varying capacity)



# Numerical examples (varying capacity)



# Numerical examples

	Λ	<i>Nodel parameters held constant</i>	Model parameters varied
$D_{f base} lpha lph$		$\{110, 100, 100, 100, 90, 90, 100, 100, 80, 60\}$ $\{1.2, 1.2, 1.1, 1.0, 0.9, 0.8, 0.7, 0.6, 0.5, 0.4\}$ $\{1.0, 1.1, 1.0, 0.8, 0.8, 0.7, 0.5, 0.4, 0.4, 0.4\}$	Starting estimate of prices $\forall i \in \mathbf{I} \text{ and } t \in \mathbf{T}$
$\{C_1, C_2\}$	=	{1000,500}	$p_i^* = 0, 150, 300, 450$

# Numerical examples (varying starting est.)



Movement of pricing policies in iterations of the algorithm with varying initial estimates for starting prices.

# Numerical examples

	M	odel parameters held constant	Model parameters varied
${}^{D}_{base}_{eta}$	=	$ \{ 110, 105, 100, 95, 90, 85, 80, 75, 75, 85 \} \\ \{ 1.2, 1.15, 1.1, 1.05, 1, .95, .9, .85, .8, 0.75 \} $	Starting estimate of prices $orall i \in \mathbf{I}$ and $t \in \mathbf{T}$
$\{C1, C2\}$	=	$\frac{13}{32}\beta$ {1000, 500}	$p_i^t = 0, 50, 100, 150$

### Numerical examples (varying starting est.)



Practical convergence behavior of the algorithm with varying initial estimates for starting prices.

# Numerical examples

Model parameters held constant	Model parameters varied
$D_{base} = \{110, 105, 100, 95, 90, 85, 80, 75, 75, 85\}$ $\beta = \{1.2, 1.15, 1.1, 1.05, 1, .95, .9, .85, .8, 0.75\}$	$\alpha = k\beta$
$\{C1, C2\} = \{1000, 500\}$	where, $k = \frac{1}{32}, \frac{1}{32}, \frac{1}{52}, \frac{1}{32}$

### Numerical examples (vary price sensitivities)



Practical convergence behavior of the algorithm with varying relative price sensitivities.

#### **Insights from Numerical Examples**

- 1. Prices decrease as the total inventory increases.
- 2. Prices are higher in periods of low price sensitivity
- 3. But the **revenue** of the seller over the entire horizon **increases** (but diminishing returns) as total **inventory increases**.
- 4. But an **increase** in the total **inventory** of a **competitor** results in **lower** revenues for the seller.
- 5. Algorithm converges to equilibrium prices very fast
- 6. Dependence of Algorithm on starting policies.
- 7. Relative ratio of demand sensitivities to price affects rate of convergence. Prices converge to equilibrium at a geometric rate.
- 1. Introduced a **general dynamic optimization model** of pricing for **perishable** products that incorporates **competition**.
- 2. We characterized **Nash equilibrium** pricing using a **variational inequality formulation**. This allowed us to establish **existence** and **uniqueness** of the Nash equilibrium pricing policies.
- 3. Incorporated **nonlinear**, **asymmetric** demands in a **multi-period** setting

- 4. We introduced an **algorithm** for computing Nash equilibrium pricing policies. We proved its **convergence** and discussed the **practical implementation** of this algorithm.
- 5. We tested our findings through some **numerical examples** and discussed some **insights**.

# **Stochastic Demand Model**

- Demand observed when the price is set at p for a realization x of a random variable X  $\frac{d(p,x) = \theta(p) + \mu(p)x}{d(p,x)}$
- $P(X = x) = \phi(x), E[X] = 1$

•  $\theta$  and  $\mu$  are functions of price

# **Demand Model Effects**

Additive form of demand d(p,x) = θ(p) + x
Multiplicative form of demand d(p,x) = μ(p)x

# $d(p,x) = \theta(p) + \mu(p)x$

# **Stochastic Demand Model**

 Expected demand
 E<sub>X</sub>(d(p, X)) = θ(p) + μ(p)
 Price response of demand
 μ(p) ↓ in p
 θ(p) ↓ in p
 θ(p) ↓ in p

# **Modeling Competition**

#### Many-seller case

 $d_i(p_i, p_{-i}, x) = \theta(p_i, p_{-i}) + x\mu(p_i, p_{-i})$ 

#### **Two-seller case**

 $d_1(p_1, p_2, x) = \theta(p_1, p_2) + x\mu(p_1, p_2)$  $d_2(p_2, p_1, x) = \theta(p_2, p_1) + x\mu(p_2, p_1)$ 

# **Modeling Competition**

Expected demand

E<sub>X</sub>(d<sub>i</sub>(p<sub>i</sub>, p<sub>-i</sub>, X)) = θ(p<sub>i</sub>, p<sub>-i</sub>) + μ(p<sub>i</sub>, p<sub>-i</sub>)

Price response of demand

μ(p<sub>i</sub>, p<sub>-i</sub>)↓ in p<sub>i</sub> and ↑ in p<sub>-i</sub>
θ(p<sub>i</sub>, p<sub>-i</sub>)↓ in p<sub>i</sub> and ↑ in p<sub>-i</sub>



#### Total capacity C<sub>i</sub> and T periods horizon

Arbitrary stochasticity

$$P(X = x) = \phi(x), x \ge 0$$

Linear demand-price relation

$$\mu_i(p_i, p_{-i}) = \hat{d}_i - \beta_i p_i + \alpha_i p_{-i}$$
  
$$\theta_i(p_i, p_{-i}) = \check{d}_i - \gamma_i p_i + \delta_i p_{-i}$$
  
$$\alpha_i, \beta_i, \gamma_i, \delta_i \ge 0$$

# How to handle total capacity C<sub>i</sub>

How do we avoid selling more than we have total capacity C<sub>i</sub>?

 Penalty for over-sale at end of horizon (Overbooking penalty)





- More on the stochastic demand case
- Joint pricing and inventory control --- Competition

- Many products (network case)
- Delay of price to affect demand
- Previous prices also affecting demand
- Demand Learning



# **Questions?**





**Nonlinear Optimization and Variational Inequalities** 



**Multi Period Pricing** 



The Price of Anarchy -- The Price of Competition

- Motivation (Transportation, Multi Period Pricing)
- VIs for Decentralized Systems vs NLP for Centralized Systems
- A **Bound** when Costs are Linear (the role of **Asymmetry**)
- A Bound when Costs are Nonlinear (the role of Non linearity)

## **Motivation** --- Transportation

#### **Costs --- Travel times**

## $c_i(f_i)$ on arc i depends on flow $f_i$ on arc i (separable)



#### **Drivers take the "best route" for themselves!**

If car drivers are selfish

**Optimize for themselves!** 



**How many** of the two cars will use arc 1 Versus arc 2?

If car drivers are dictated so that total cost is minimum (social optimization)



**How many** of the two cars will be asked to use arc 1 versus arc 2?



User Opt. Total Cost = 4/3 Social Opt. Total Cost

#### **Social Optimum**

#### User Opt. Total Cost = 4/3 Social Opt. Total Cost





# **Motivation** --- **Transportation**

## Costs --- Travel times

c<sub>i</sub>(f<sub>i</sub>) on arc i depends on flow f<sub>i</sub> on arc i (separable)
 c<sub>i</sub>(f) on arc i depends on flow vector f
 (I.e. neighboring arcs influence traffic)



(non-separable)

# (non-separable)

 $c_{1}(f) = g_{11}f_{1} + g_{12}f_{2} + g_{13}f_{3} + \dots + b_{1}$  $c_{2}(f) = g_{21}f_{1} + g_{22}f_{2} + g_{23}f_{3} + \dots + b_{2}$ 

.....

#### (i.e. neighboring arcs influence traffic)



> upstream effects> downstream effects> intersections

# **Decentralized Problem----VIs**

#### p in P, w in W, i in L

- $f_i = \Sigma_{i \text{ in P}} F_i$
- $\mathbf{d}_{\mathsf{w}} = \Sigma_{\mathsf{p} \text{ in } \mathsf{P}_{\mathsf{W}}} \boldsymbol{F}_{\boldsymbol{p}}$
- $C_{p}(f) = \Sigma_{i in P} c_{i}(f)$



#### A flow pattern ( $f^*$ , $F^*$ ) is **U-O**

if for all w in W and every p in  $\mathsf{P}_\mathsf{w}$ 

$$C_{p}(f^{*}) = v_{w} \text{ if } F^{*}_{p} > 0,$$
  
 $C_{p}(f^{*}) \ge v_{w} \text{ if } F^{*}_{p} = 0$ 

Equivalent to VI(c, K) :  $c(f^*)^t(f - f^*)^t)$ 

$$c(f^*)^t(f-f^*) \ge 0, \forall f$$

 $\mathsf{K}=\{(f,F): f_i=\Sigma_{i \text{ in } P} F_{p_i} d_w = \Sigma_{p \text{ in } P_w} F_{p_i} F_p \ge 0\}$ 

## The Centralized Problem ---- NLP

# **System Optimum**

 $\operatorname{Min}_{\mathbf{f}} \Sigma_{\operatorname{i} \operatorname{in} \mathbf{L}} \mathbf{C}_{\operatorname{i}}(f) f_{i}$ 

such that

$$\mathsf{K}=\{(f,F): f_{i}=\Sigma_{i \text{ in } P} F_{\rho,} \mathsf{d}_{w}=\Sigma_{p \text{ in } P_{w}} F_{\rho,} F_{\rho} \geq 0\}$$

# How bad is User-Optimization ?



#### **Roughgarden and Tardos (2001)**

constant terms  $b \ge 0$ ,  $g_1, g_2 > 0$ 

$$OF_{uo}/OF_{so} \leq 4/3$$

 $c_1(f) = g_1 f_1 + b_1$  $c_2(f) = g_2 f_2 + b_2$  $G = \begin{bmatrix} g_1 & 0 \\ & & \\ 0 & g_2 \end{bmatrix}$ 

G pd, diagonal matrix

1

2

 $c_{1}(f) = g_{11}f_{1} + g_{12}f_{2} + b_{1}$  $c_{2}(f) = g_{12}f_{1} + g_{22}f_{2} + b_{2}$ 



$$g_{12} = g_{21}$$

c(z)=Gf+b  $f=(f_1,f_2)$ 

G pd, symmetric matrix (contribution from arc i to j same as arc j to i)







# c(f)=Gf+b $f=(f_1,f_2)$

G pd, asymmetric matrix (contribution not the same)

#### **Sample Literature**

#### **Transportation and Game Theory:**

- Dafermos (1969, 1980, 1984)
- Florian and Hearn (1993), Nagurney(2000)

(REVIEWS)

- Papadimitriou and Koutsoupias (1999)
- Roughgarden and Tardos, Roughgarden (2001)
- Johari and Tsitsiklis (2002)

**Hotel room** 



Period 1

Period t

Period T

#### Market in a Multi-period Setting Perishable Product

Flight NY-Boston Airlines Advance booking

# The Decentralized Problem



# **The Decentralized Problem**

**User Optimum** 

Demand Non-separable

 $h_{1}(p) = -g_{11}p_{1} + g_{12}p_{2} + h_{1}$  $h_{2}(p) = g_{21}p_{1} - g_{22}p_{2} + h_{2}$ 

Could be Asymmetric g<sub>12</sub> is not g<sub>21</sub>

# The Centralized Problem ?

# **System Optimum**

$$\text{Max}_{\text{d,p}} \Sigma_{\text{i in L}} \text{profit seller}_{\text{i}}$$

# **Often not legal!**

# How bad is system due to competition and lack of coordination?

 $F_{1}(z) = g_{11}z_{1} + g_{12}z_{2} + b_{1}$  $F_{2}(z) = g_{21}z_{1} + g_{22}z_{2} + b_{2}$ 





G pd, asymmetric matrix

# **A Bound between SO and UO** $F(z)=Gz+b \quad z=(z_{1},z_{2},...,z_{n})$

 $F(z^*)'(z-z^*) \ge 0$ , for all z feasible

Define: S = G + G' symmetrized matrix (pd) 2 Define:  $c = ||S^{-1}G||_{S}$ 



# **Transportation**

# c<sup>2</sup> describes how congestion in neighboring roads affects your travel time

# This phenomenon also applies to pricing with competition

# Contributions

> Understand the price of "competition"

> how bad is system due to lack of coordination?

Non-separable payoffs (costs)

> the role of asymmetry (non-separable)

\* symm. ( $c^2=1$ )  $\rightarrow$  4/3,

- \* little asymm. ( $c^2 <= 2$ )  $\rightarrow 4/(4-c^2)$ ,
- \* quite asymm. ( $c^2 > 2$ )  $\rightarrow c^2$
- Nonlinear payoffs (costs)

The role of non-linearity (measure non-linearity)
Version of the paper to appear at the IPCO conference Proceedings
Also full version under review in MOR

# **Questions?**



