

Extensional Uniformity for Boolean Circuits*

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Abstract. Imposing an extensional uniformity condition on a non-uniform circuit complexity class \mathcal{C} means simply intersecting \mathcal{C} with a uniform class \mathcal{L} . By contrast, the usual intensional uniformity conditions require that a resource-bounded machine be able to exhibit the circuits in the circuit family defining \mathcal{C} . We say that $(\mathcal{C}, \mathcal{L})$ has the *Uniformity Duality Property* if the extensionally uniform class $\mathcal{C} \cap \mathcal{L}$ can be captured intensionally by means of adding so-called \mathcal{L} -numerical predicates to the first-order descriptive complexity apparatus describing the connection language of the circuit family defining \mathcal{C} .

This paper exhibits positive instances and negative instances of the Uniformity Duality Property.

Keywords: Boolean circuits, uniformity, descriptive complexity.

1 Introduction

A family $\{C_n\}_{n \geq 1}$ of Boolean circuits is *uniform* if the way in which C_{n+1} can differ from C_n is restricted. Generally, uniformity is imposed by requiring that some form of a resource-bounded constructor on input n be able to fully or partially describe C_n (see [1, 5, 8, 14, 19] or refer to [22] for an overview). Circuit-based language classes can then be compared with classes that are based on a finite computing mechanism such as a Turing machine.

Recall the gist of descriptive complexity. Consider the set of words $w \in \{a, b\}^*$ having no b at an even position. This language is described by the FO[\langle, EVEN] formula $\neg \exists i (\text{EVEN}(i) \wedge P_b(i))$. In such a first-order formula, the variables range over positions in w , a predicate P_σ for $\sigma \in \{a, b\}$ holds at i iff $w_i = \sigma$, and a *numerical* predicate, such as the obvious 1-ary EVEN predicate here, holds at its arguments iff these arguments fulfill the specific relation.

The following viewpoint has emerged [3, 5, 6] over two decades: *when a circuit-based language class is characterized using first-order descriptive complexity, the circuit uniformity conditions spring up in the logic in the form of restrictions on the set of numerical predicates allowed.*

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As a well studied example [5, 12], $\text{FO}[\langle, +, \times \rangle] = \text{DLOGTIME-uniform } \text{AC}^0 \subsetneq \text{non-uniform } \text{AC}^0 = \text{FO}[\mathbf{arb}]$, where the latter class is the class of languages definable by first-order formulae entitled to *arbitrary* numerical predicates (we use a logic and the set of languages it captures interchangeably when this brings no confusion).

In a related vein but with a different emphasis, Straubing [21] presents a beautiful account of the relationship between automata theory, formal logic and (non-uniform) circuit complexity. Straubing concludes by expressing the proven fact that $\text{AC}^0 \subsetneq \text{ACC}^0$ and the celebrated conjectures that $\text{AC}^0[q] \subsetneq \text{ACC}^0$ and that $\text{ACC}^0 \subsetneq \text{NC}^1$ as instances of the following conjecture concerning the class REG of regular languages:

$$\mathcal{Q}[\mathbf{arb}] \cap \text{REG} = \mathcal{Q}[\mathbf{reg}]. \quad (1)$$

In Straubing's instances, \mathcal{Q} is an appropriate set of quantifiers chosen from $\{\exists\} \cup \{\exists^{(a,r)} : 0 \leq r < q\}$ and \mathbf{reg} is the set of *regular* numerical predicates, that is, the set of those numerical predicates of arbitrary arity definable in a formal sense by finite automata. We stress the point of view that intersecting $\{\exists\}[\mathbf{arb}] = \text{FO}[\mathbf{arb}]$ with REG to form $\text{FO}[\mathbf{arb}] \cap \text{REG}$ in conjecture (1) amounts to imposing uniformity on the non-uniform class $\text{FO}[\mathbf{arb}]$. And once again, imposing uniformity has the effect of restricting the numerical predicates: it is a proven fact that $\text{FO}[\mathbf{arb}] \cap \text{REG} = \text{FO}[\mathbf{reg}]$, and conjecture (1) expresses the hope that this phenomenon extends from $\{\exists\}$ to other \mathcal{Q} , which would determine much of the internal structure of NC^1 . We ask:

1. Does the duality between uniformity in a circuit-based class and numerical predicates in its logical characterization extend beyond NC^1 ?
2. What would play the role of the regular numerical predicates in such a duality?
3. Could such a duality help understanding classes such as the context-free languages in AC^0 ?

To tackle the first question, we note that intersecting with REG is just one out of many possible ways in which one can “impose uniformity”. Indeed, if \mathcal{L} is any uniform language class, one can replace $\mathcal{Q}[\mathbf{arb}] \cap \text{REG}$ by $\mathcal{Q}[\mathbf{arb}] \cap \mathcal{L}$ to get another uniform subclass of $\mathcal{Q}[\mathbf{arb}]$. For example, consider any “formal language class” (in the loose terminology used by Lange when discussing language theory versus complexity theory [14]), such as the class CFL of context-free languages. Undoubtedly, CFL is a uniform class of languages. Therefore, the class $\mathcal{Q}[\mathbf{arb}] \cap \text{CFL}$ is another uniform class well worth comparing with $\mathcal{Q}[\langle, + \rangle]$ or $\mathcal{Q}[\langle, +, \times \rangle]$. Of course, $\text{FO}[\mathbf{arb}] \cap \text{CFL}$ is none other than the poorly understood class $\text{AC}^0 \cap \text{CFL}$, and when Q is a quantifier given by some word problem of a nonsolvable group, $(\text{FO} + \{Q\})[\mathbf{arb}] \cap \text{CFL}$ is the poorly understood class $\text{NC}^1 \cap \text{CFL}$ alluded to 20 years ago [11].

The present paper thus considers classes $\mathcal{Q}[\mathbf{arb}] \cap \mathcal{L}$ for various \mathcal{Q} and \mathcal{L} . To explain its title, we note that the constructor-based approach defines uniform classes by specifying their properties: such definitions are *intensional* definitions.

By contrast, viewing $\mathcal{Q}[\mathbf{arb}] \cap \text{REG}$ as a uniform class amounts to an *extensional* definition, namely one that selects the members of $\mathcal{Q}[\mathbf{arb}]$ that will collectively form the uniform class. In this paper we set up the extensional uniformity framework and we study classes $\mathcal{Q}[\mathbf{arb}] \cap \mathcal{L}$ for $\mathcal{Q} \supseteq \{\exists\}$.

Certainly, the uniform class \mathcal{L} will determine the class of numerical predicates we have to use when trying to capture $\mathcal{Q}[\mathbf{arb}] \cap \mathcal{L}$, as Straubing does for $\mathcal{L} = \text{REG}$, as an intensionally uniform class. A contribution of this paper is to provide a meaningful definition for the set $\mathcal{L}^{\mathbb{N}}$ of *\mathcal{L} -numerical predicates*. Informally, $\mathcal{L}^{\mathbb{N}}$ is the set of relations over the natural numbers that are definable in the sense of Straubing [21, Section III.2] by a language over a singleton alphabet drawn from \mathcal{L} . When \mathcal{L} is REG, the \mathcal{L} -numerical predicates are precisely Straubing’s regular numerical predicates.

Fix a set \mathcal{Q} of monoidal or groupoidal quantifiers in the sense of [5, 16, 22]. (As prototypical examples, the reader unfamiliar with such quantifiers may think of the usual existential and universal quantifiers, of Straubing’s “there exist r modulo q ” quantifiers, or of threshold quantifiers such as “there exist a majority” or “there exist at least t ”). We propose the *Uniformity Duality Property for $(\mathcal{Q}, \mathcal{L})$* as a natural generalization of conjecture (1):

Uniformity Duality Property for $(\mathcal{Q}, \mathcal{L})$

$$\mathcal{Q}[\mathbf{arb}] \cap \mathcal{L} = \mathcal{Q}[\langle, \mathcal{L}^{\mathbb{N}}] \cap \mathcal{L}.$$

Barrington, Immerman and Straubing [5] have shown that $\mathcal{Q}[\mathbf{arb}]$ equals $\text{AC}^0[\mathcal{Q}]$, that is, non-uniform AC^0 with \mathcal{Q} gates. Behle and Lange [6] have shown that $\mathcal{Q}[\langle, \mathcal{L}^{\mathbb{N}}]$ equals $\text{FO}[\langle, \mathcal{L}^{\mathbb{N}}]$ -uniform $\text{AC}^0[\mathcal{Q}]$, that is, uniform $\text{AC}^0[\mathcal{Q}]$ where the direct connection language of the circuit families can be described by means of the logic $\text{FO}[\langle, \mathcal{L}^{\mathbb{N}}]$. Hence the Uniformity Duality Property can be restated in circuit complexity-theoretic terms as follows:

Uniformity Duality Property for $(\mathcal{Q}, \mathcal{L})$, 2nd form

$$\text{AC}^0[\mathcal{Q}] \cap \mathcal{L} = \text{FO}[\langle, \mathcal{L}^{\mathbb{N}}]\text{-uniform } \text{AC}^0[\mathcal{Q}] \cap \mathcal{L}.$$

By definition, $\mathcal{Q}[\mathbf{arb}] \cap \mathcal{L} \supseteq \mathcal{Q}[\langle, \mathcal{L}^{\mathbb{N}}] \cap \mathcal{L}$. The critical question is whether the reverse inclusion holds. Intuitively, the Uniformity Duality Property states that the “extensional uniformity induced by intersecting $\mathcal{Q}[\mathbf{arb}]$ with \mathcal{L} ” is a strong enough restriction imposed on $\mathcal{Q}[\mathbf{arb}]$ to permit expressing the uniform class using the \mathcal{L} -numerical predicates, or in other words: the extensional uniformity given by intersecting the non-uniform class with \mathcal{L} coincides with the intensional uniformity condition given by first-order logic with \mathcal{L} -numerical predicates. Further motivation for this definition of $\mathcal{Q}[\langle, \mathcal{L}^{\mathbb{N}}] \cap \mathcal{L}$ is as follows:

- when constructors serve to define uniform classes, they have access to input lengths but not to the inputs themselves; a convenient logical analog to this is to use the unary alphabet languages from \mathcal{L} as a basis for defining the extra numerical predicates

- if the closure properties of \mathcal{L} differ from the closure properties of $\mathcal{Q}[\mathbf{arb}]$, then $\mathcal{Q}[\mathbf{arb}] \cap \mathcal{L} = \mathcal{Q}[\langle, \mathcal{L}^{\mathbb{N}}]$ may fail trivially (this occurs for example when $\mathcal{L} = \text{CFL}$ and $\mathcal{Q} = \{\exists\}$ since the non-context-free language $\{a^n b^n c^n : n \geq 0\}$ is easily seen to belong to $\mathcal{Q}[\langle, \mathcal{L}^{\mathbb{N}}]$ by closure under intersection of the latter); hence intersecting $\mathcal{Q}[\langle, \mathcal{L}^{\mathbb{N}}]$ with \mathcal{L} before comparing it with $\mathcal{Q}[\mathbf{arb}] \cap \mathcal{L}$ is necessary to obtain a reasonable generalization of Straubing’s conjecture for classes \mathcal{L} that are not Boolean-closed.

We now state our results, classified, loosely, as foundational observations (F) or technical statements (T). We let \mathcal{L} be *any* class of languages.

- (F) By design, the Uniformity Duality Property for $(\mathcal{Q}, \text{REG})$ is precisely Straubing’s conjecture (1), hence its conjectured validity holds the key to the internal structure of NC^1 .
- (F) The Uniformity Duality Property for $(\{\exists\}, \text{NEUTRAL})$ is precisely the Crane Beach Conjecture [4]; here, NEUTRAL is the class of languages L that have a neutral letter, i.e., a letter e that may be arbitrarily inserted into or deleted from words without changing membership in L . The Crane Beach conjecture, stating that any neutral letter language in $\text{AC}^0 = \text{FO}[\mathbf{arb}]$ can be expressed in $\text{FO}[\langle]$, was motivated by attempts to develop a purely automata-theoretic proof that *Parity*, a neutral letter language, is not in AC^0 . The Crane Beach Conjecture was ultimately refuted [4], but several of its variants have been studied. Thus [4]:
 - the Uniformity Duality Property for $(\{\exists\}, \text{NEUTRAL})$ fails
 - the Uniformity Duality Property for $(\{\exists\}, \text{NEUTRAL} \cap \text{REG})$ holds
 - the Uniformity Duality Property for $(\{\exists\}, \text{NEUTRAL} \cap \{\text{two-letter languages}\})$ holds.
- (T) Our definition for the set $\mathcal{L}^{\mathbb{N}}$ of \mathcal{L} -numerical predicates parallels Straubing’s definition of regular numerical predicates. For kernel-closed language classes \mathcal{L} that are closed under homomorphisms, inverse homomorphisms and intersection with a regular language, we furthermore characterize $\mathcal{L}^{\mathbb{N}}$ as the set of predicates expressible as one generalized unary \mathcal{L} -quantifier applied to an $\text{FO}[\langle]$ -formula. (Intuitively, \mathcal{L} -numerical predicates are those predicates definable in first-order logic with one “oracle call” to a language from \mathcal{L} .)
- (T) We characterize the numerical predicates that surround the context-free languages: first-order combinations of $\text{CFL}^{\mathbb{N}}$ suffice to capture all semilinear predicates over \mathbb{N} ; in particular, $\text{FO}[\langle, +] = \text{FO}[\text{DCFL}^{\mathbb{N}}] = \text{FO}[\text{BC}(\text{CFL})^{\mathbb{N}}]$, where DCFL denotes the deterministic context-free languages and $\text{BC}(\text{CFL})$ is the Boolean closure of CFL .
- (T) We deduce that, despite the fact that $\text{FO}[\text{BC}(\text{CFL})^{\mathbb{N}}]$ contains all the semilinear relations, the Uniformity Duality Property fails for $(\{\exists\}, \mathcal{L})$ in each of the following cases:
 - $\mathcal{L} = \text{CFL}$
 - $\mathcal{L} = \text{VPL}$, the “visibly pushdown languages” recently introduced by [2]
 - $\mathcal{L} = \text{Boolean closure of the deterministic context-free languages}$

- \mathcal{L} = Boolean closure of the linear context-free languages
- \mathcal{L} = Boolean closure of the context-free languages.

The crux of the justifications of these negative results is a proof that the complement of the “Immerman language”, used in disproving the Crane Beach Conjecture, is context-free.

- (T) At the opposite end of the spectrum, while it is clear that the Uniformity Duality Property holds for the set of all languages and any \mathcal{Q} , we show that the Uniformity Duality Property already holds for $(\mathcal{Q}, \mathcal{L})$ whenever \mathcal{Q} is a set of groupoidal quantifiers and $\mathcal{L} = \text{NTIME}(n)^{\mathcal{L}}$; thus it holds for, e.g., the rudimentary languages, $\text{DSPACE}(n)$, CSL and PSPACE.

The rest of this paper is organized as follows. Section 2 contains preliminaries. Section 3 defines the \mathcal{L} -numerical predicates and introduces the Uniformity Duality Property formally. The context-free numerical predicates are investigated in Section 4, and the duality property for classes of context-free languages is considered in Section 5. Section 6 shows that the duality property holds when \mathcal{L} is “large enough”. Section 7 concludes with a summary and a discussion. For the sake of brevity, proofs are omitted and will be included in the full version.

2 Preliminaries

2.1 Complexity Theory

We assume familiarity with standard notions in formal languages, automata and complexity theory.

When dealing with circuit complexity classes, all references will be made to the non-uniform versions unless otherwise stated. Thus AC^0 refers of the Boolean functions computed by constant-depth polynomial-size unbounded-fan-in $\{\vee, \wedge, \neg\}$ -circuits. And $\text{DLOGTIME-uniform AC}^0$ refers to the set of those functions in AC^0 computable by a circuit family having a direct connection language decidable in time $O(\log n)$ on a deterministic Turing machine (cf. [5, 22]).

2.2 First-Order Logic

Let \mathbb{N} be the natural numbers $\{1, 2, 3, \dots\}$ and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. A *signature* σ is a finite set of relation symbols with fixed arity and constant symbols. A σ -structure $\mathfrak{A} = \langle \mathcal{U}^{\mathfrak{A}}, \sigma^{\mathfrak{A}} \rangle$ consists of a set $\mathcal{U}^{\mathfrak{A}}$, called *universe*, and a set $\sigma^{\mathfrak{A}}$ that contains an *interpretation* $R^{\mathfrak{A}} \subseteq (\mathcal{U}^{\mathfrak{A}})^k$ for each k -ary relation symbol $R \in \sigma$. We fix the interpretations of the “standard” numerical predicates $<$, $+$, \times , etc. to their natural interpretations. By BIT we will denote the binary relation $\{(x, i) \in \mathbb{N}^2 : \text{bit } i \text{ in the binary representation of } x \text{ is } 1\}$. For logics over strings with alphabet Σ , we will use signatures extending $\sigma_{\Sigma} = \{P_a : a \in \Sigma\}$ and identify $w = w_1 \dots w_n \in \Sigma^*$ with $\mathfrak{A}_w = \langle \{1, \dots, n\}, \sigma^{\mathfrak{A}_w} \rangle$, where $P_a^{\mathfrak{A}_w} = \{i \in \mathbb{N} : w_i = a\}$ for all $a \in \Sigma$. We will not distinguish between a relation symbol and its interpretation, when the meaning is clear from the context.

Let \mathcal{Q} be a set of (first-order) quantifiers. We denote by $\mathcal{Q}[\sigma]$ the set of first-order formulae over σ using quantifiers from \mathcal{Q} only. The set of all $\mathcal{Q}[\sigma]$ -formulae will be referred to as the *logic* $\mathcal{Q}[\sigma]$. In case $\mathcal{Q} = \{\exists\}$ ($\mathcal{Q} = \{\exists\} \cup \mathcal{Q}'$), we will also write $\text{FO}[\sigma]$ ($\text{FO}+\mathcal{Q}'[\sigma]$, respectively). When discussing logics over strings, we will omit the relation symbols from σ_Σ .

Say that a language $L \subseteq \Sigma^*$ is *definable* in a logic $\mathcal{Q}[\sigma]$ if there exists a $\mathcal{Q}[\sigma]$ -formula φ such that $\mathfrak{A}_w \models \varphi \iff w \in L$ for all $w \in \Sigma^*$, and say that a relation $R \subseteq \mathbb{N}^n$ is definable by a $\mathcal{Q}[\sigma]$ -formula if there exists a formula φ with free variables x_1, \dots, x_n that defines R for all sufficiently large initial segment of \mathbb{N} , i. e., if $\langle \{1, \dots, m\}, \sigma \rangle \models \varphi(c_1, \dots, c_n) \iff (c_1, \dots, c_n) \in R$ for all $m \geq c_{\max}$, where $c_{\max} = \max\{c_1, \dots, c_n\}$ [20, Section 3.1]. By abuse of notation, we will write $L \in \mathcal{Q}[\sigma]$ (or $R \in \mathcal{Q}[\sigma]$) to express that a language L (a relation R , resp.) is definable by a $\mathcal{Q}[\sigma]$ -formula and use a logic and the set of languages and relations it defines interchangeably.

3 The Uniformity Duality Property

In order to generalize conjecture (1), we propose Definition 3.2 as a simple generalization of the regular numerical predicates defined using \mathcal{V} -structures by Straubing [21, Section III.2].

Definition 3.1. Let $\mathcal{V}_n = \{x_1, \dots, x_n\}$ be a nonempty set of variables and let Σ be a finite alphabet. A \mathcal{V}_n -structure is a sequence

$$\underline{w} = (a_1, V_1) \cdots (a_m, V_m) \in (\Sigma \times \mathfrak{P}(\mathcal{V}_n))^*$$

such that $a_1, \dots, a_m \in \Sigma$ and the nonempty sets among V_1, \dots, V_m form a partition of \mathcal{V}_n (the underscore distinguishes \mathcal{V}_n -structures from ordinary strings). Define $\Gamma_n = \{0\} \times \mathfrak{P}(\mathcal{V}_n)$. We say that a \mathcal{V}_n -structure \underline{w} is *unary* if $\underline{w} \in \Gamma_n^*$, i. e., if $a_1 \cdots a_n$ is defined over the singleton alphabet $\{0\}$; in that case, we define the *kernel* of \underline{w} , $\text{kern}(\underline{w})$, as the maximal prefix of \underline{w} that does not end with $(0, \emptyset)$; to signify that $x_i \in V_{c_i}$ for all $1 \leq i \leq n$, we also write $\text{kern}(\underline{w})$ as $[x_1 = c_1, \dots, x_n = c_n]$ and we let $\underline{w}^{\mathbb{N}}$ stand for (c_1, \dots, c_n) .

We define Struc_n as the language of all such words in Γ_n^* that are unary \mathcal{V}_n -structures and let $\text{Struc} = \bigcup_{n>0} \text{Struc}_n$.

Any set L of unary \mathcal{V}_n -structures naturally prescribes a relation over the natural numbers. Hence, a set of such L prescribes a set of relations, or numerical predicates, over \mathbb{N} .

Definition 3.2. Let $L \subseteq \Gamma_n^*$ be a unary \mathcal{V}_n -language, that is, a set of unary \mathcal{V}_n -structures. Let $L^{\mathbb{N}} = \{\underline{w}^{\mathbb{N}} : \underline{w} \in L\}$ denote the relation over \mathbb{N}^n defined by L . Then the \mathcal{L} -numerical predicates are defined as

$$\mathcal{L}^{\mathbb{N}} = \{L^{\mathbb{N}} : L \in \mathcal{L} \text{ and } L \subseteq \text{Struc}\}.$$

We say that a language L is *kernel-closed* if, for every $\underline{w} \in L$, $\text{kern}(\underline{w}) \in L$. We further say that a language class \mathcal{L} is *kernel-closed* if, for every $L \in \mathcal{L}$ there exists an $L' \in \mathcal{L}$ such that $L^{\mathbb{N}} = L'^{\mathbb{N}}$ and L' is kernel-closed.

We point out the following facts, where we write $\equiv_q r$ for the unary predicate $\{x : x \equiv r \pmod q\}$.

Proposition 3.3. *Let APER and NEUTRAL denote the set of aperiodic languages and the set of languages having a neutral letter respectively. Then*

1. $\text{APER}^{\mathbb{N}} = \text{FO}[\langle],$
2. $\text{REG}^{\mathbb{N}} = (\text{AC}^0 \cap \text{REG})^{\mathbb{N}} = \text{FO}[\langle, \{\equiv_q r : 0 \leq r < q\}] = \mathbf{reg},$ and
3. $\text{NEUTRAL}^{\mathbb{N}} \subseteq \text{FO}[\langle].$

Having discussed the \mathcal{L} -numerical predicates, we can state the property expressing the dual facets of uniformity, namely, intersecting with an a priori uniform class on the one hand, and adding the corresponding numerical predicates to first-order logics on the other.

Property 3.4 (Uniformity Duality for $(\mathcal{Q}, \mathcal{L})$). *Let \mathcal{Q} be a set of quantifiers and let \mathcal{L} be a language class, then*

$$\mathcal{Q}[\mathbf{arb}] \cap \mathcal{L} = \mathcal{Q}[\langle, \mathcal{L}^{\mathbb{N}}] \cap \mathcal{L}.$$

As $\mathcal{Q}[\mathbf{arb}] = \text{AC}^0[\mathcal{Q}]$ [5] and $\mathcal{Q}[\langle, \mathcal{L}^{\mathbb{N}}] = \text{FO}[\langle, \mathcal{L}^{\mathbb{N}}]$ -uniform $\text{AC}^0[\mathcal{Q}]$ [6], the above property equivalently states that

$$\text{AC}^0[\mathcal{Q}] \cap \mathcal{L} = \text{FO}[\langle, \mathcal{L}^{\mathbb{N}}]$$
-uniform $\text{AC}^0[\mathcal{Q}] \cap \mathcal{L}.$

As a consequence of Proposition 3.3 (1–2), the Uniformity Duality Property is equivalent to the instances of the Straubing conjectures obtained by setting \mathcal{Q} and \mathcal{L} as we expect, for example $\mathcal{Q} \subseteq \{\exists\} \cup \{\exists^{(q,r)} : 0 \leq r < q\}$ and $\mathcal{L} = \text{REG}$ yield exactly (1). Similarly, as a consequence of Proposition 3.3 (3), the Uniformity Duality Property is equivalent to the Crane Beach Conjecture if $\text{FO}[\langle] \subseteq \mathcal{L}$. Property 3.4 is thus false when $\mathcal{Q} = \{\exists\}$ and \mathcal{L} is the set NEUTRAL of all neutral letter languages. For some other classes, the Crane Beach Conjecture and thus Property 3.4 hold: consider for example the case $\mathcal{L} = \text{REG} \cap \text{NEUTRAL}$ [4], or the case $\mathcal{Q} = \{\exists\}$ and $\mathcal{L} \subseteq \text{NEUTRAL} \cap \text{FO}[+]$. Accordingly the Uniformity Duality Property both generalizes the conjectures of Straubing et al. and captures the intuition underlying the Crane Beach Conjecture. Encouraged by this unification, we will take a closer look at the Uniformity Duality in the case of first-order logic and context-free languages in the next section.

In the rest of this section, we present an alternative characterization of $\mathcal{L}^{\mathbb{N}}$ using $\text{FO}[\langle]$ -transformations and unary Lindström quantifiers. This is further justification for our definition of \mathcal{L} -numerical predicates. The reader unfamiliar with this topic may skip to the end of Section 3.

Digression: Numerical Predicates and Generalized Quantifiers

Generalized or Lindström quantifiers provide a very general yet coherent approach to extending the descriptive complexity of first-order logics [17]. Since we only deal with unary Lindström quantifiers over strings, we will restrict our definition to this case.

Definition 3.5. Let $\Delta = \{a_1, \dots, a_t\}$ be an alphabet, $\varphi_1, \dots, \varphi_{t-1}$ be $\text{FO}[\langle \cdot \rangle]$ -formulae, each with $k + 1$ free variables x_1, x_2, \dots, x_k, y , and let \vec{x} abbreviate x_1, x_2, \dots, x_k . Further, let $\text{STRUCT}(\sigma)$ denote the set of finite structures $\mathfrak{A} = \langle \mathcal{U}^{\mathfrak{A}}, \sigma^{\mathfrak{A}} \rangle$ over σ . Then $\varphi_1, \dots, \varphi_{t-1}$ define an $\text{FO}[\langle \cdot \rangle]$ -transformation

$$[\varphi_1(\vec{x}), \dots, \varphi_{t-1}(\vec{x})]: \text{STRUCT}(\{\langle \cdot, x_1, \dots, x_k \rangle\}) \rightarrow \Delta^*$$

as follows: Let $\mathfrak{A} \in \text{STRUCT}(\{\langle \cdot, x_1, \dots, x_k \rangle\})$, $x_i^{\mathfrak{A}} = c_i \in \mathcal{U}^{\mathfrak{A}}$, $1 \leq i \leq k$, and $s = |\mathcal{U}^{\mathfrak{A}}|$, then $[\varphi_1(\vec{x}), \dots, \varphi_{t-1}(\vec{x})](\mathfrak{A}) = v_1 \cdots v_s \in \Delta^*$, where

$$v_i = \begin{cases} a_1, & \text{if } \mathfrak{A} \models \varphi_1(c_1, \dots, c_k, i), \\ a_j, & \text{if } \mathfrak{A} \models \varphi_j(c_1, \dots, c_k, i) \wedge \bigwedge_{l=1}^{j-1} \neg \varphi_l(c_1, \dots, c_k, i), 1 < j < t, \\ a_t, & \text{if } \mathfrak{A} \models \bigwedge_{l=1}^{t-1} \neg \varphi_l(c_1, \dots, c_k, i). \end{cases}$$

A language $L \subseteq \Delta^*$ and an $\text{FO}[\langle \cdot \rangle]$ -transformation $[\varphi_1(\vec{x}), \dots, \varphi_{t-1}(\vec{x})]$ now naturally define a (unary) Lindström quantifier $\mathcal{Q}_L^{\text{un}}$ via

$$\mathfrak{A} \models \mathcal{Q}_L^{\text{un}} y [\varphi_1(\vec{x}, y), \dots, \varphi_{t-1}(\vec{x}, y)] \iff [\varphi_1(\vec{x}), \dots, \varphi_{t-1}(\vec{x})](\mathfrak{A}) \in L.$$

Finally, the set of relations definable by formulae $\mathcal{Q}_L^{\text{un}} y [\varphi_1(\vec{x}, y), \dots, \varphi_{t-1}(\vec{x}, y)]$, where $L \in \mathcal{L}$ and $\varphi_1, \dots, \varphi_{t-1} \in \text{FO}[\langle \cdot \rangle]$, will be denoted by $\mathcal{Q}_L^{\text{un}} \text{FO}[\langle \cdot \rangle]$.

The notation $[\varphi_1(\vec{x}), \dots, \varphi_{t-1}(\vec{x})]$ is chosen to distinguish the variables in \vec{x} from y ; the variables in \vec{x} are interpreted by \mathfrak{A} whereas y is utilized in the transformation.

Theorem 3.6. Let \mathcal{L} be a kernel-closed language class which is closed under homomorphisms, inverse homomorphisms and intersection with regular languages, then $\mathcal{L}^{\mathbb{N}} = \mathcal{Q}_{\mathcal{L}}^{\text{un}} \text{FO}[\langle \cdot \rangle]$; that is, the \mathcal{L} -numerical predicates correspond to the predicates definable using a unary Lindström quantifier over \mathcal{L} and an $\text{FO}[\langle \cdot \rangle]$ -transformation.

We stress that the above result provides a logical characterization of the \mathcal{L} -numerical predicates for all kernel-closed classes \mathcal{L} forming a cone, viz. a class of languages \mathcal{L} closed under homomorphisms, inverse homomorphisms and intersection with regular languages [10]. As the closure under these operations is equivalent to the closure under rational transductions (i. e., transductions performed by finite automata [7]), we obtain:

Corollary 3.7. Let \mathcal{L} be kernel-closed and closed under rational transductions, then $\mathcal{L}^{\mathbb{N}} = \mathcal{Q}_{\mathcal{L}}^{\text{un}} \text{FO}[\langle \cdot \rangle]$.

4 Characterizing the Context-Free Numerical Predicates

In order to examine whether the Uniformity Duality Property for first-order logics holds in the case of context-free languages, we first need to consider the counterpart of the regular numerical predicates, that is, $\text{CFL}^{\mathbb{N}}$. Our results in this section

will relate $\text{CFL}^{\mathbb{N}}$ to addition w. r. t. to first-order combinations, and are based upon a result by Ginsburg [9]. Ginsburg showed that the number of repetitions per fragment in bounded context-free languages corresponds to a subset of the semilinear sets. For a start, note that addition is definable in $\text{DCFL}^{\mathbb{N}}$.

Lemma 4.1. *Addition is definable in $\text{DCFL}^{\mathbb{N}}$.*

Next, we restate the result of Ginsburg in order to prepare ground for the examination of the context-free numerical predicates. In the following, let w^* abbreviate $\{w\}^*$ and say that a language $L \subseteq \Sigma^*$ is *bounded* if there exists an $n \in \mathbb{N}$ and $w_1, \dots, w_n \in \Sigma^+$ such that $L \subseteq w_1^* \cdots w_n^*$.

Definition 4.2. *A set $R \subseteq \mathbb{N}_0^n$ is stratified if*

1. *each element in R has at most two non-zero coordinates,*
2. *there are no integers i, j, k, l and $x = (x_1, \dots, x_n), x' = (x'_1, \dots, x'_n)$ in R such that $1 \leq i < j < k < l \leq n$ and $x_i x'_j x_k x'_l \neq 0$.*

Moreover, a set $S \subseteq \mathbb{N}^n$ is said to be stratified semilinear if it is expressible as a finite union of linear sets, each with a stratified set of periods; that is, $S = \bigcup_{i=1}^m \{\vec{\alpha}_{i0} + \sum_{j=1}^{n_i} k \cdot \vec{\alpha}_{ij} : k \in \mathbb{N}_0\}$, where $\vec{\alpha}_{i0} \in \mathbb{N}^n$, $\vec{\alpha}_{ij} \in \mathbb{N}_0^n$, $1 \leq j \leq n_i$, $1 \leq i \leq m$, and each $P_i = \{\vec{\alpha}_{ij} : 1 \leq j \leq n_i\}$ is stratified.

Theorem 4.3 ([9, Theorem 5.4.2]). *Let Σ be an alphabet and $L \subseteq w_1^* \cdots w_n^*$ be bounded by $w_1, \dots, w_n \in \Sigma^+$. Then L is context-free if and only if the set*

$$E(L) = \{(e_1, \dots, e_n) \in \mathbb{N}_0^n : w_1^{e_1} \dots w_n^{e_n} \in L\}$$

is a stratified semilinear set.

Theorem 4.3 relates the bounded context-free languages to a strict subset of the semilinear sets. The semilinear sets are exactly those sets definable by $\text{FO}[+]$ -formulae. There are however sets in $\text{FO}[+]$ that are undefinable in $\text{CFL}^{\mathbb{N}}$: e. g., if $R = \{(x, 2x, 3x) : x \in \mathbb{N}\}$ was definable in $\text{CFL}^{\mathbb{N}}$ then $\{a^n b^n c^n : n \in \mathbb{N}\} \in \text{CFL}$. Hence, $\text{FO}[+]$ can not be captured by $\text{CFL}^{\mathbb{N}}$ alone. Yet, addition is definable in $\text{CFL}^{\mathbb{N}}$, therefore we will in the following investigate the relationship between first-order logic with addition, $\text{FO}[+]$, and the Boolean closure of CFL , $\text{BC}(\text{CFL})$.

Theorem 4.4. $\text{BC}(\text{CFL}^{\mathbb{N}}) \subseteq \text{BC}(\text{CFL})^{\mathbb{N}} \subseteq \text{FO}[+]$.

That is, the relations definable in the Boolean closure of the context-free unary V_n -languages are captured by $\text{FO}[+]$. Hence, $\text{FO}[\text{BC}(\text{CFL})^{\mathbb{N}}] \subseteq \text{FO}[+]$. Now Lemma 4.1 yields the following corollary.

Corollary 4.5. $\text{FO}[\text{DCFL}^{\mathbb{N}}] = \text{FO}[\text{CFL}^{\mathbb{N}}] = \text{FO}[\text{BC}(\text{CFL})^{\mathbb{N}}] = \text{FO}[+]$.

We note that in particular, for any $k \in \mathbb{N}$, the inclusion $(\bigcap_k \text{CFL})^{\mathbb{N}} \subsetneq \text{FO}[+]$ holds, where $\bigcap_k \text{CFL}$ denotes the languages definable as the intersection of $\leq k$ context-free languages: this is deduced from embedding numerical predicates

derived from the infinite hierarchy of context-free languages by Liu and Weiner into $\text{CFL}^{\mathbb{N}}$ [18]. Hence,

$$\text{CFL}^{\mathbb{N}} \subsetneq \cdots \subsetneq (\bigcap_k \text{CFL})^{\mathbb{N}} \subsetneq (\bigcap_{k+1} \text{CFL})^{\mathbb{N}} \subsetneq \cdots \subsetneq (\bigcap \text{CFL})^{\mathbb{N}} \subseteq \text{FO}[+].$$

Unfortunately, we could neither prove nor refute $\text{FO}[+] \subseteq \text{BC}(\text{CFL})^{\mathbb{N}}$. The difficulty in comparing $\text{FO}[+]$ and $\text{BC}(\text{CFL})^{\mathbb{N}}$ comes to some extent from the restriction on the syntactic representation of tuples in CFL; viz., context-free languages may only compare distances between variables, whereas the tuples defined by unary \mathcal{V}_n -languages count positions from the beginning of a word. This difference matters only for language classes that are subject to similar restrictions as the context-free languages (e. g., the regular languages are not capable of counting, the context-sensitive languages have the ability to convert between these two representations). To account for this special behavior, we will render precisely $\text{CFL}^{\mathbb{N}}$ in Theorem 4.6.

But there is more to be taken into account. Consider, e. g., the relation $R = \{(x, x, x) : x \in \mathbb{N}\}$. R is clearly definable in $\text{CFL}^{\mathbb{N}}$, yet the set $E(L)$ of the defining language L , $L^{\mathbb{N}} = R$, is not stratified semilinear. Specifically, duplicate variables and permutations of the variables do not increase the complexity of a unary \mathcal{V}_n -language L but affect $L^{\mathbb{N}}$.

Let t be an order type of $\vec{x} = (x_1, \dots, x_n)$ and say that a relation $R \subseteq \mathbb{N}^n$ has order type t if, for all $\vec{x} \in R$, \vec{x} has order type t . For \vec{x} of order type t , let $\vec{x}' = (x'_1, \dots, x'_m)$, $m \leq n$, denote the variables in \vec{x} with mutually distinct values and let π_t denote a permutation such that $x'_{\pi_t(i)} < x'_{\pi_t(i+1)}$, $1 \leq i < m$. We define functions $\text{sort} : \mathfrak{P}(\mathbb{N}^n) \rightarrow \mathfrak{P}(\mathbb{N}^m)$ and $\text{diff} : \mathfrak{P}(\mathbb{N}^n) \rightarrow \mathfrak{P}(\mathbb{N}_0^n)$ as

$$\begin{aligned} \text{sort}(R) &= \{\pi_t(\vec{x}') : \vec{x} \in R \text{ has order type } t\}, \\ \text{diff}(R) &= \left\{ (x_i)_{1 \leq i \leq n} : \left(\sum_{j=1}^i x_j \right)_{1 \leq i \leq n} \in R \right\}. \end{aligned}$$

The function sort rearranges the components of R in an ascending order and eliminates duplicates, whereas diff transforms a tuple (x_1, \dots, x_n) with $x_1 < x_2 < \cdots < x_n$ into $(x_1, x_2 - x_1, x_3 - x_2 - x_1, \dots, x_n - \sum_{i=1}^{n-1} x_i)$, a representation more “suitable” to CFL (cf. $E(L)$ in Theorem 4.3).

Theorem 4.6. *Let $R \subseteq \mathbb{N}^n$. $R \in \text{CFL}^{\mathbb{N}}$ if and only if there exists a partition $R = R_1 \cup \cdots \cup R_k$ such that each $\text{diff}(\text{sort}(R_i))$, $1 \leq i \leq k$, is a stratified semilinear set.*

5 The Uniformity Duality and Context-Free Languages

Due to the previous section, we may express the Uniformity Duality Property for context-free languages using Corollary 4.5 in the following more intuitive way: let $\mathcal{Q} = \{\exists\}$ and \mathcal{L} be such that $\text{FO}[\mathcal{L}^{\mathbb{N}}] = \text{FO}[\langle, +]$ (e. g., $\text{DCFL} \subseteq \mathcal{L} \subseteq \text{BC}(\text{CFL})$), then the Uniformity Duality Property for $(\{\exists\}, \mathcal{L})$ is equivalent to

$$\text{FO}[\text{arb}] \cap \mathcal{L} = \text{FO}[\langle, +] \cap \mathcal{L}. \quad (2)$$

We will hence examine whether (2) holds, and see that this is not the case.

For a binary word $u = u_{n-1}u_{n-2}\cdots u_0 \in \{0,1\}^*$, we write \widehat{u} for the integer $u_{n-1}2^{n-1} + \cdots + 2u_1 + u_0$. Recall the Immerman language $L_I \subseteq \{0,1,a\}^*$, that is, the language consisting of all words of the form $x_1ax_2a\cdots ax_{2^n}$, where $x_i \in \{0,1\}^n$, $\widehat{x}_i + 1 = \widehat{x}_{i+1}$, $1 \leq i < 2^n$, and $x_1 = 0^n$, $x_{2^n} = 1^n$. For example, $00a01a10a11 \in L_I$ and $000a001a010a011a100a101a110a111 \in L_I$. We prove that despite its definition involving arithmetic, L_I is simply the complement of a context-free language.

Lemma 5.1. *The complement $\overline{L_I}$ of the Immerman language is context-free.*

For a language $L \subseteq \Sigma^*$, let $\text{Neutral}(L)$ denote L supplemented with a neutral letter $e \notin \Sigma$, i.e., $\text{Neutral}(L)$ consists of all words in L with possibly arbitrary repeated insertions of the neutral letter. The above Lemma implies that $\text{Neutral}(L_I) \in \text{BC}(\text{CFL})$. From [4] we know that $\text{Neutral}(L_I) \in \text{FO}[\mathbf{arb}] \setminus \text{FO}[\langle, +]$. This finally leads to the following:

Theorem 5.2. $\text{FO}[\mathbf{arb}] \cap \text{BC}(\text{CFL}) \supsetneq \text{FO}[\langle, +] \cap \text{BC}(\text{CFL})$.

Theorem 5.2 implies that the Uniformity Duality Property fails for $\mathcal{Q} = \{\exists\}$ and $\mathcal{L} = \text{BC}(\text{CFL})$, since $\text{FO}[\langle, \text{BC}(\text{CFL})^{\mathbb{N}}] = \text{FO}[\langle, +]$. Yet, it even provides a witness for the failure of the duality property in the case of $\mathcal{L} = \text{CFL}$, as the context-free language $\text{Neutral}(\overline{L_I})$ lies in $\text{FO}[\mathbf{arb}] \setminus \text{FO}[\langle, +]$. We will state this result as a corollary further below. For now, consider the modified Immerman language R_I defined as L_I except that the successive binary words are reversed in alternance, i.e.,

$$R_I = \{\dots, 000a(001)^R a010a(011)^R a100a(101)^R a110a(111)^R, \dots\}.$$

R_I is the intersection of two deterministic context-free languages. Even more, the argument in Lemma 5.1 can actually be extended to prove that the complement of R_I is a linear CFL. Hence,

Theorem 5.3. 1. $\text{FO}[\mathbf{arb}] \cap \text{BC}(\text{DCFL}) \supsetneq \text{FO}[\langle, +] \cap \text{BC}(\text{DCFL})$.
2. $\text{FO}[\mathbf{arb}] \cap \text{BC}(\text{LinCFL}) \supsetneq \text{FO}[\langle, +] \cap \text{BC}(\text{LinCFL})$.

The role of neutral letters in the above theorems suggests taking a closer look at $\text{Neutral}(\text{CFL})$. As the Uniformity Duality Property for $(\{\exists\}, \text{Neutral}(\text{CFL}))$ would have it, all neutral-letter context-free languages in AC^0 would be regular and aperiodic. This is, however, not the case as witnessed by $\text{Neutral}(\overline{L_I})$. Hence,

Corollary 5.4. *In the case of $\mathcal{Q} = \{\exists\}$, the Uniformity Duality Property fails in all of the following cases.*

1. $\mathcal{L} = \text{CFL}$,
2. $\mathcal{L} = \text{BC}(\text{CFL})$,
3. $\mathcal{L} = \text{BC}(\text{DCFL})$,
4. $\mathcal{L} = \text{BC}(\text{LinCFL})$,
5. $\mathcal{L} = \text{Neutral}(\text{CFL})$.

Remark 5.5. The class VPL of visibly pushdown languages [2] has gained prominence recently because it shares with REG many useful properties. But despite having access to a stack, the VPL-numerical predicates coincide with $\text{REG}^{\mathbb{N}}$, for each word may only contain constantly many characters different from $(0, \emptyset)$. It follows that the Uniformity Duality Property fails for VPL and first-order quantifiers: consider, e. g., $L = \{a^n b^n : n > 0\} \in \text{FO}[\mathbf{arb}] \cap (\text{VPL} \setminus \text{REG})$ then $L \in \text{FO}[\mathbf{arb}] \cap \text{VPL}$ but $L \notin \text{FO}[<, \text{VPL}^{\mathbb{N}}] \cap \text{VPL}$.

6 The Duality in Higher Classes

We have seen that the context-free languages do not exhibit our conjectured Uniformity Duality. In this section we will show that the Uniformity Duality Property holds if the extensional uniformity condition imposed by intersecting with \mathcal{L} is quite loose, in other words, if the language class \mathcal{L} is powerful.

Recall the notion of non-uniformity introduced by Karp and Lipton [13].

Definition 6.1. For a complexity class \mathcal{L} , denote by \mathcal{L}/poly the class \mathcal{L} with polynomial advice. That is, \mathcal{L}/poly is the class of all languages L such that, for each L , there is a function $f: \mathbb{N} \rightarrow \{0, 1\}^*$ with

1. $|f(x)| \leq p(|x|)$, for all x , and
2. $L^f = \{\langle x, f(|x|) \rangle : x \in L\} \in \mathcal{L}$,

where p is a polynomial depending on \mathcal{L} . Without loss of generality, we will assume $|f(x)| = |x|^k$ for some $k \in \mathbb{N}$.

Note that, using the above notation, $\text{DLOGTIME-uniform AC}^0/\text{poly} = \text{AC}^0$. As we further need to make the advice strings accessible in a logic, we define the following predicates.

Following [5], we say that a Lindström quantifier Q_L is *groupoidal*, if $L \in \text{CFL}$.

Definition 6.2. Let \mathcal{Q} be any set of groupoidal quantifiers. Further, let $L \in \text{DLOGTIME-uniform AC}^0[\mathcal{Q}]/\text{poly}$ and let f be the function for which $L^f \in \text{DLOGTIME-uniform AC}^0[\mathcal{Q}]$. Let $r = 2kl + 1$, where k and l are chosen such that the circuit family recognizing L in $\text{DLOGTIME-uniform AC}^0$ has size n^l and $|f(x)| = |x|^k$. We define $\text{ADVICE}_{L, \mathcal{Q}}^f \in \text{FO} + \mathcal{Q}[\mathbf{arb}]$ to be the ternary relation

$$\text{ADVICE}_{L, \mathcal{Q}}^f = \{(i, n, n^r) : \text{bit } i \text{ of } f(n) \text{ equals } 1\},$$

and denote the set of all relations $\text{ADVICE}_{L, \mathcal{Q}}^f$, for $L \in \mathcal{L}$, by $\text{ADVICE}_{\mathcal{L}, \mathcal{Q}}$.

The intention of $\text{ADVICE}_{L, \mathcal{Q}}^f$ is to encode the advice string as a numerical relation. A point in this definition that will become clear later is the third argument of the $\text{ADVICE}_{L, \mathcal{Q}}^f$ -predicate; it will pad words in the corresponding unary \mathcal{V}_n -language to the length of the advice string. This padding will be required for Theorem 6.4.

Theorem 6.3. Let \mathcal{L} be a language class and \mathcal{Q} be a set of groupoidal quantifiers. Then the Uniformity Duality Property for $(\{\exists\} \cup \mathcal{Q}, \mathcal{L})$ holds if $\text{BIT} \in \mathcal{L}^{\mathbb{N}}$ and $\text{ADVICE}_{\mathcal{L}, \mathcal{Q}} \in \mathcal{L}^{\mathbb{N}}$.

We can now give a lower bound beyond which the Uniformity Duality Property holds. Let $\text{NTIME}(n)^\mathcal{L}$ denote the class of languages decidable in linear time by nondeterministic Turing machines with oracles from \mathcal{L} .

Theorem 6.4. *Let \mathcal{Q} be any set of groupoidal quantifiers and suppose $\mathcal{L} = \text{NTIME}(n)^\mathcal{L}$. Then the Uniformity Duality Property for $(\{\exists\} \cup \mathcal{Q}, \mathcal{L})$ holds.*

Corollary 6.5. *Let \mathcal{Q} be any set of groupoidal quantifiers. The Uniformity Duality Property holds for $(\{\exists\} \cup \mathcal{Q}, \mathcal{L})$ if \mathcal{L} equals the deterministic context-sensitive languages $\text{DSPACE}(n)$, the context-sensitive languages CSL, the rudimentary languages (i. e., the linear time hierarchy [24]), PH, PSPACE, or the recursively enumerable languages.*

7 Conclusion

For a set \mathcal{Q} of quantifiers and a class \mathcal{L} of languages, we have suggested that $\mathcal{Q}[\mathbf{arb}] \cap \mathcal{L}$ defines an (extensionally) uniform complexity class. After defining the notion of \mathcal{L} -numerical predicates, we have proposed comparing $\mathcal{Q}[\mathbf{arb}] \cap \mathcal{L}$ with its subclass $\mathcal{Q}[\langle, \mathcal{L}^\mathbb{N}] \cap \mathcal{L}$, a class equivalently defined as the (intensionally) uniform circuit class $\text{FO}[\langle, \mathcal{L}^\mathbb{N}]$ -uniform $\text{AC}^0[\mathcal{Q}] \cap \mathcal{L}$.

We have noted that the duality property, defined to hold when both classes above are equal, encompasses Straubing’s conjecture (1) as well as some positive and some negative instances of the Crane Beach Conjecture.

We have then investigated the duality property in specific cases with $\mathcal{Q} = \{\exists\}$. We have seen that the property fails for several classes \mathcal{L} involving the context-free languages. Exhibiting these failures has required new insights, such as characterizations of the context-free numerical predicates and a proof that the complement of the Immerman language is context-free, but these failures have prevented successfully tackling complexity classes such as $\text{AC}^0 \cap \text{CFL}$. Restricting the class of allowed relations on the left hand side of the uniformity duality property from \mathbf{arb} to a subclass might lead to further insight and provide positive examples of this modified duality property (and address, e.g., the class of context-free languages in different uniform versions of AC^0). Methods from embedded finite model theory should find applications here.

More generally, the duality property widens our perspective on the relationship between uniform circuits and descriptive complexity beyond the level of NC^1 . We have noted for example that the property holds for any set of groupoidal quantifiers $\mathcal{Q} \supseteq \{\exists\}$ and complexity classes \mathcal{L} that are closed under nondeterministic linear-time Turing reductions.

A point often made is that a satisfactory uniformity definition should apply comparable resource bounds to a circuit family and to its constructor. For instance, although P-uniform NC^1 has merit [1], the classes AC^0 -uniform NC^1 and NC^1 -uniform NC^1 [5] seem more fundamental, provided that one can make sense of the apparent circularity. As a by-product of our work, we might suggest $\text{FO}[\langle, \mathcal{L}^\mathbb{N}] \cap \mathcal{L}$ as the minimal “uniform subclass of \mathcal{L} ” and thus as a meaningful (albeit restrictive) definition of \mathcal{L} -uniform \mathcal{L} . Our choice of $\text{FO}[\langle]$ as the

“bottom class of interest” is implicit in this definition and results in the containment of \mathcal{L} -uniform \mathcal{L} in (non-uniform) AC^0 for any \mathcal{L} . Progressively less uniform subclasses of \mathcal{L} would be the classes $\mathcal{Q}[\langle, \mathcal{L}^{\mathbb{N}}] \cap \mathcal{L}$ for $\mathcal{Q} \supseteq \{\exists\}$.

Restating hard questions such as conjecture (1) in terms of a unifying property does not make these questions go away. But the duality property raises further questions. As an example, can the duality property for various $(\mathcal{Q}, \mathcal{L})$ be shown to hold or to fail when \mathcal{Q} includes the majority quantifier? This could help develop incisive results concerning the class TC^0 . To be more precise, let us consider $\mathcal{Q} = \{\exists, MAJ\}$. The majority quantifier is a particular groupoidal (or, *context-free*) quantifier [16], hence it seems natural to consider the Uniformity Duality Property for $(\{\exists, MAJ\}, CFL)$:

$$FO+MAJ[\mathbf{arb}] \cap CFL = FO+MAJ[\langle, +] \cap CFL. \quad (3)$$

It is not hard to see that the Immerman language in fact is in $FO+MAJ[\langle, +]$, hence our Theorem 5.2 that refutes (2), the Uniformity Duality Property for $(FO, BC(CFL))$, does not speak to whether (3) holds. (Another prominent example that refutes (2) is the “Wotschke language” $W = \{(a^n b)^n : n \geq 0\}$, again a co-context-free language [23]. Similar to the case of the Immerman language we observe that $W \in FO+MAJ[\langle, +]$, hence W does not refute (3) either.)

Observe that $FO+MAJ[\mathbf{arb}] = TC^0$ [5] and that, on the other hand, $FO+MAJ[\langle, +] = MAJ[\langle] = FO[+]$ -uniform linear fan-in TC^0 [6, 15]. Let us call this latter class sTC^0 (for *small* TC^0 or *strict* TC^0). It is known that $sTC^0 \subsetneq TC^0$ [16]. Hence we conclude that if (3) holds, then in fact $TC^0 \cap CFL = sTC^0 \cap CFL$. Thus, if we can show that some language in the Boolean closure of the context-free languages is not in sTC^0 , we have a new TC^0 lower bound. Thus, to separate TC^0 from a superclass it suffices to separate sTC^0 from a superclass, a possibly less demanding goal. This may be another reason to look for appropriate uniform classes \mathcal{L} such that

$$FO+MAJ[\mathbf{arb}] \cap \mathcal{L} = FO+MAJ[\langle, +] \cap \mathcal{L}.$$

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