Arithmetic circuits and polynomial replacement systems
(work in progress)

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Caveat to the Web slides

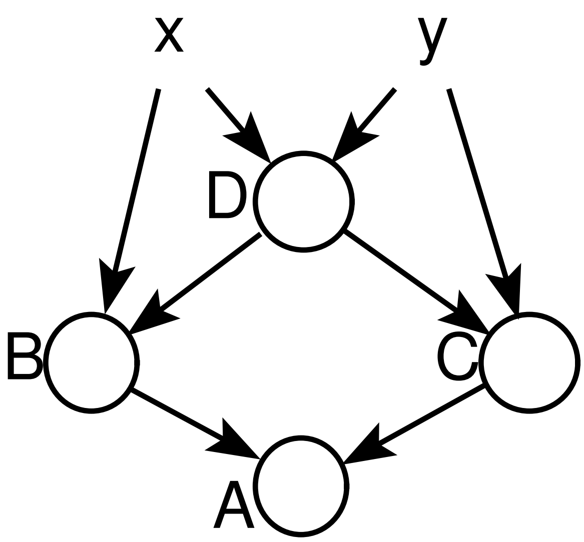
(A few slides, including motivation, are missing which I did not know how to convert from Powerpoint to Postscript, and then to include here.)

(Some overlays, and the hand-drawn colors, for example the AND and OR labels of various diagrams, are sometimes missing as well.)

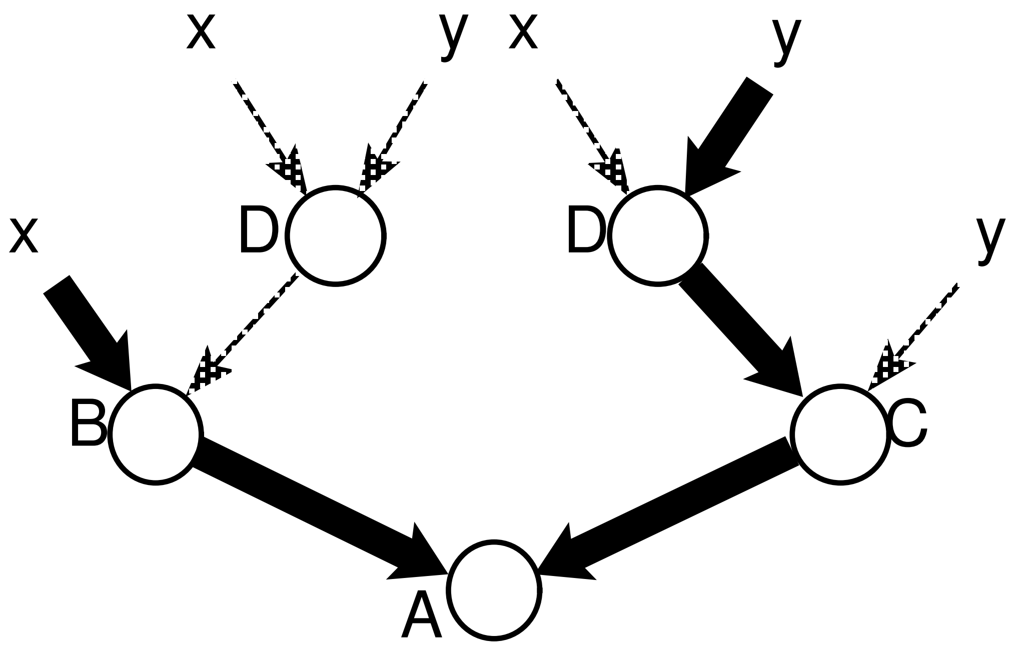
(Motivation is from #P, #L, #LOGFCL, #NC1, #AC0)

(Next circuit has AND at bottom and OR elsewhere.)

Proof trees by example

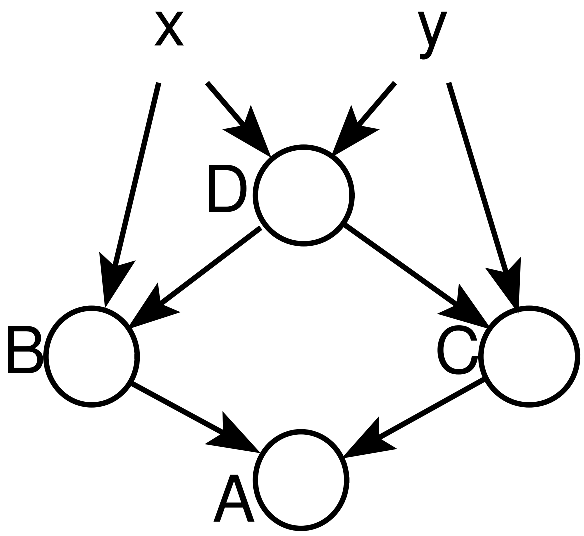


Must first unwind circuit into a tree:

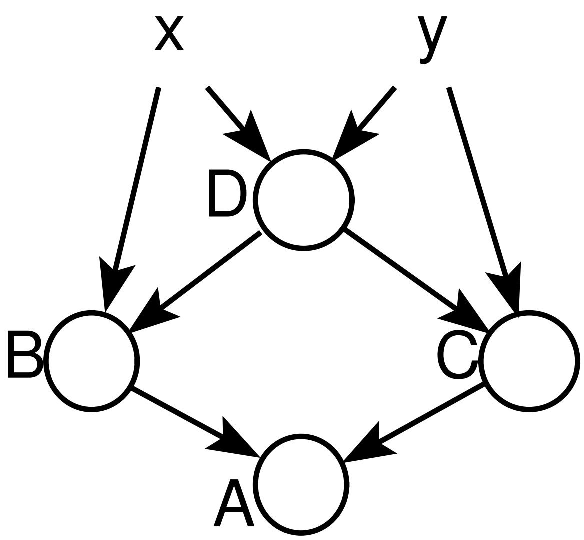


Thick arcs form a proof tree on $x = y = 1$

Known: the number of proof trees in



is given by the $\{+, \times\}$-arithmetic circuit

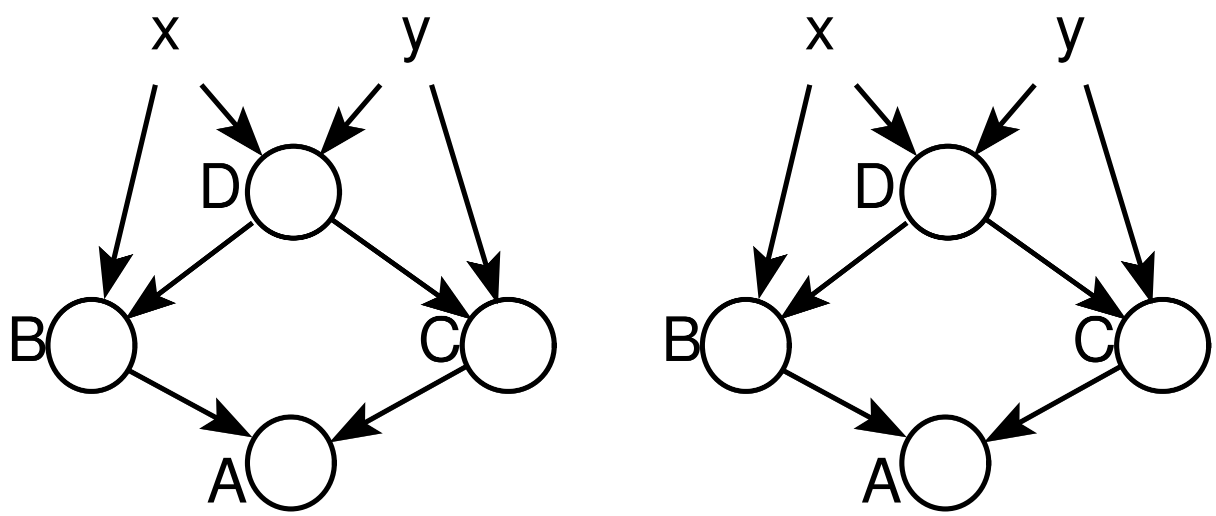


To count proof trees

- Replace $\vee$ with $+$ and $\wedge$ with $\times$
- View circuit as straight-line program
- Compute polynomial “top down”

- Evaluate program at desired input

Proof trees in our example



Straight-line program: $\begin{bmatrix} D & \to & x+y \\ C & \to & D+y \\ B & \to & x+D \\ A & \to & BC \end{bmatrix}$

Counting:

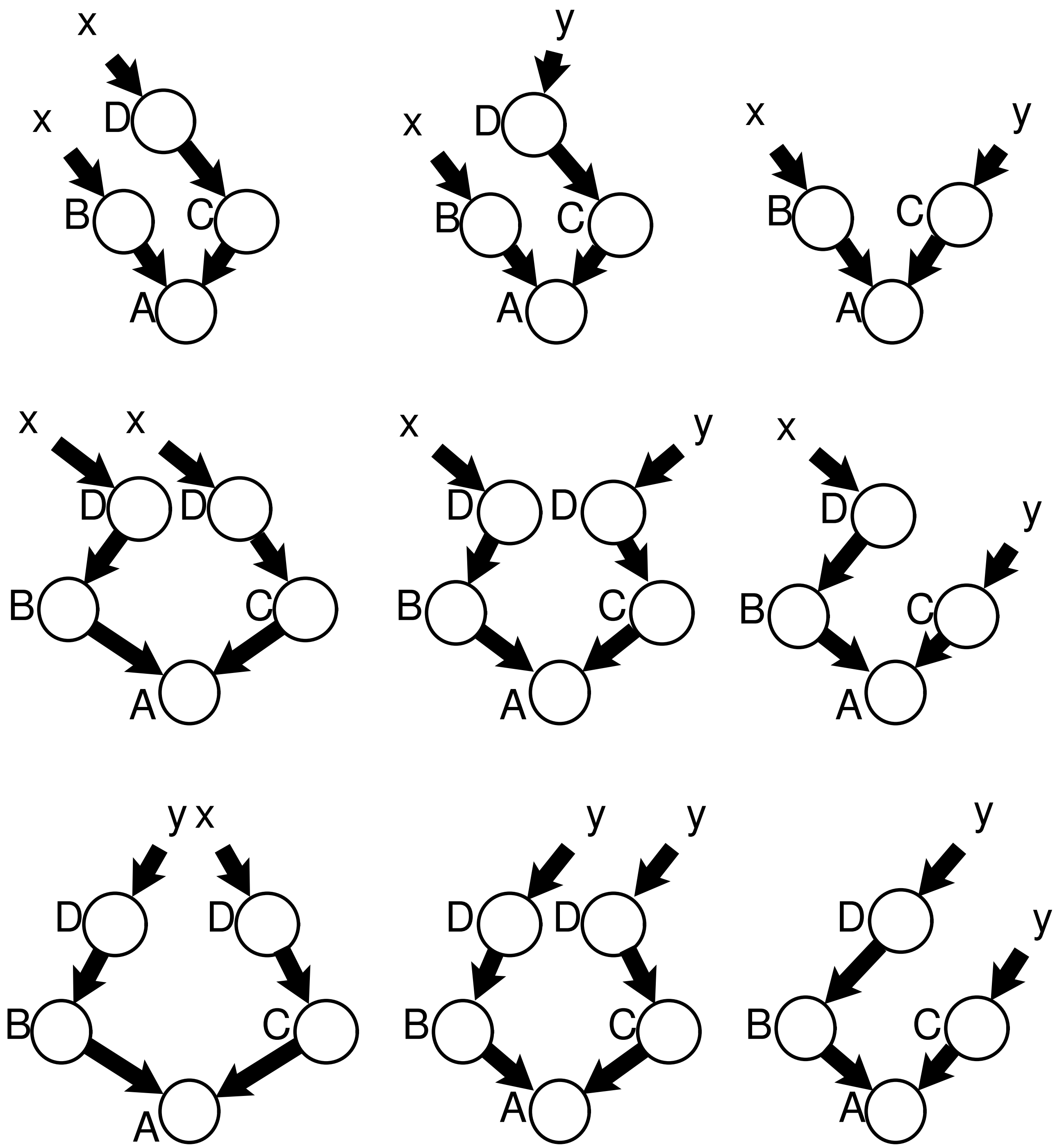
$$\begin{array}{rcl} A & \to & BC \to (x+D)C \\ & \to & (x+D)(D+y) \\ & & \\ & \to & (2x+y)(x+2y) \\ & & \\ & \to & 9 \text{ on input } x=y=1 \end{array}$$

Our starting point

What counting classes arise if we do not unwind circuits into trees before counting their proofs?

Natural notion of proof <u>circuits</u>

- how to count them?
- is their count expressible as small $\{+, \times\}$-circuits?



The 9 proof trees on input $x = y = 1$

(Overlay to explain which two proof trees amongst the 9 proof trees on the previous slide do not give rise to a valid proof circuit.)

To count proof **circuits**

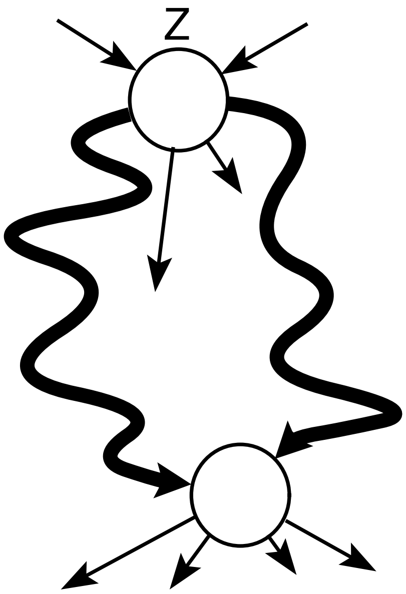
- Replace $\vee$ with $+$ and $\wedge$ with $\times$
- View circuit as straight-line program
- Compute polynomial "top down"

  but knock down any nontrivial power Z^i of a variable to Z before replacing Z

- Evaluate program at desired input

(Overlay slide 6 in order to explain how the “knock down exponents” algorithm yields 7 proof circuits in our example, as D^2 becomes D in the straight-line program evaluation.)

Why does this work?



Because Z^2 reflects two distinct paths from Z into an AND gate $\Rightarrow$ the contribution should be Z alone.

NB: replace Z only after applying $Z^2 \rightarrow Z$ to completion, and only when no descendant of Z remains.

Counting proof circuits...

...is in #P:

guess at an OR gate

recurse at an AND gate

never explore twice $\Rightarrow$ poly time.

...cannot be #P-hard under many-one polytime reduction, unless P=NP, since for any NP machine M:

$$
\begin{aligned}
x \in L(M) \ \text{ iff } \ & \#\text{ACCEPT}(M, x) > 0 \\
\text{ iff } \ & \#\text{PFCIRCUITS}(C, y) > 0 \\
\text{ iff } \ & \text{the circuit } C \text{ accepts } x
\end{aligned}
$$

for "$f_M : x \mapsto \langle C, y \rangle$" the reduction.

Theorem: Counting proof circuits is #P-complete under <u>Turing</u> reduction.

Cor 1. Our "exponents knock down" algorithm for counting proof circuits cannot work in poly time unless P=NP.

Cor 2. No poly size $\{+, \times\}$-algebraic circuit expresses in general the number of proof circuits of a boolean circuit C unless P=NP.

Proof of **2**: C has at most $2^{|C|}$ proof circuits, and evaluating a $\{+, \times\}$-circuit modulo $2^{|C|}$ could be done in P.

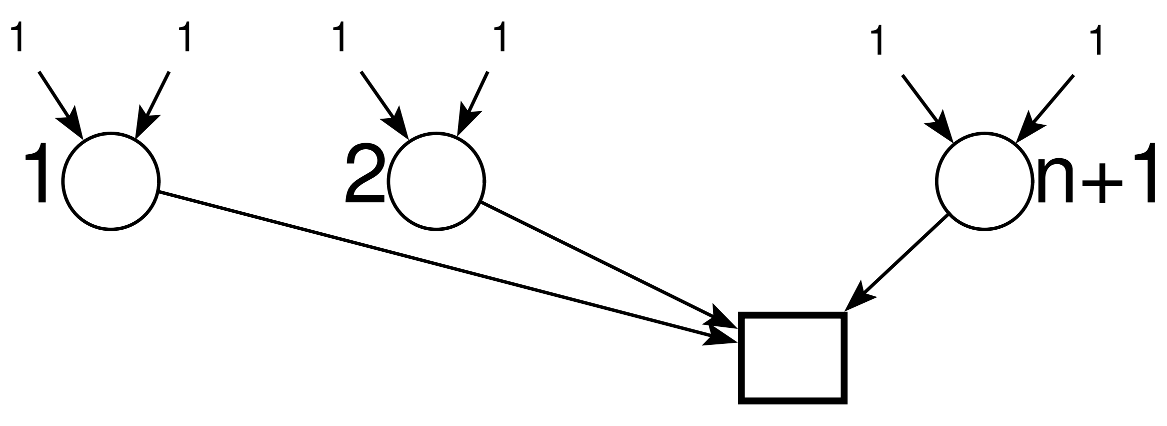
#P-hardness proof:

From Valiant's #3SAT:

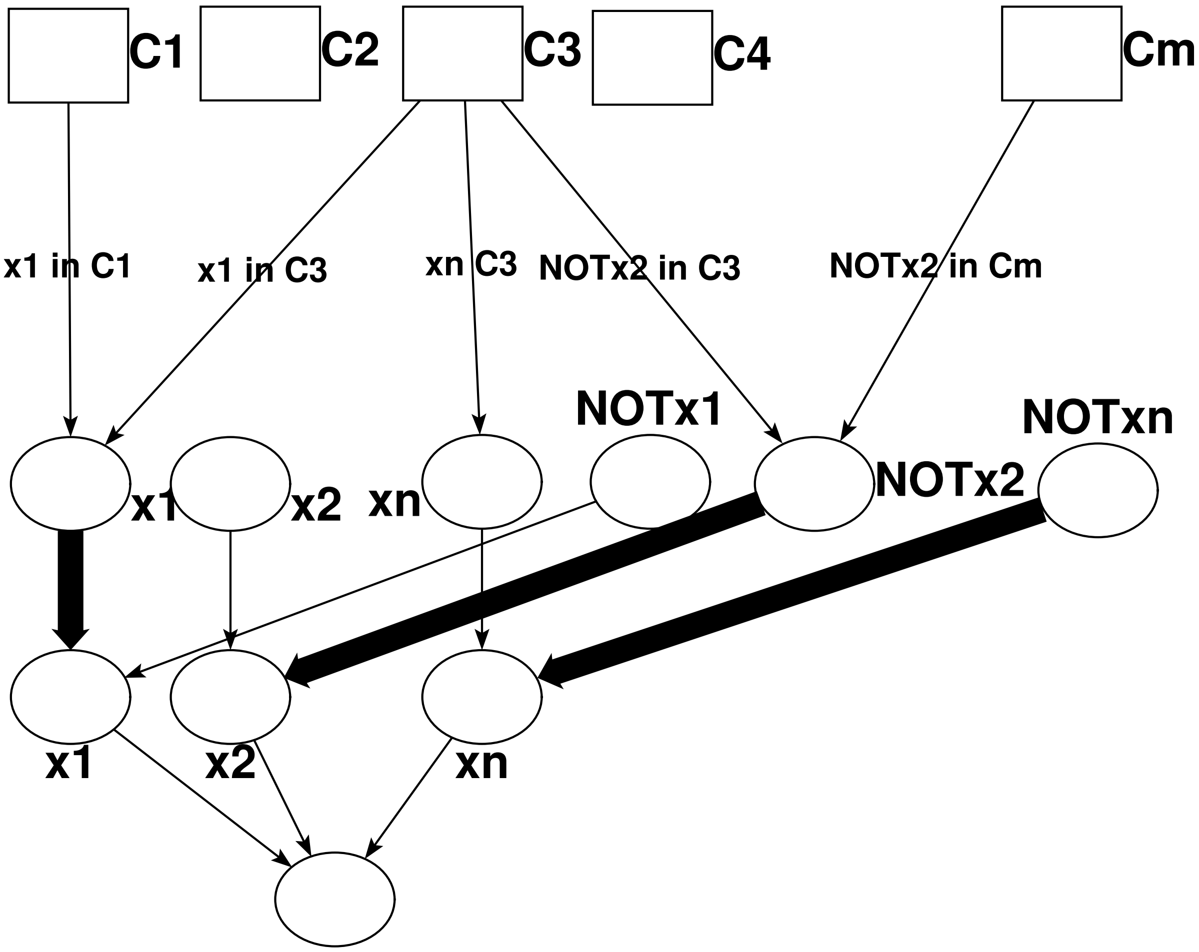
clauses $C1, C2, \ldots, Cm$,

variables $x1, x2, \ldots xn$.

Will use gadgets



which have 2^{n+1} proof circuits.



Thick is a boolean assignment.

Thick generates $2^{k\cdot(n+1)}$ proof circuits iff **thick** satisfies exactly k clauses.

Thick satisfies all 3SAT clauses $\Rightarrow$

contributes $2^{m\cdot(n+1)}$ proof circuits.

Thick $\neg$satisfy all 3SAT clauses $\Rightarrow$

contribution $\leq 2^{(m-1)\cdot(n+1)}$.

If b boolean assignments satisfy the 3SAT instance:

$$
\begin{aligned}
b \cdot 2^{m\cdot(n+1)} &\leq \#\text{proof circuits} \\
&\leq b \cdot 2^{m\cdot(n+1)} + \\
&\quad\ (2^n - b) \cdot 2^{(m-1)\cdot(n+1)} \\
&< (b+1) \cdot 2^{m\cdot(n+1)}.
\end{aligned}
$$

Our counting example: alternate viewpoint

Replacement rules: $\begin{bmatrix} D & \to & x+y \\ D^2 & \to & D \\ C & \to & D+y \\ C^2 & \to & C \\ B & \to & x+D \\ B^2 & \to & B \\ A & \to & BC \\ A^2 & \to & A \end{bmatrix}$

A yields a <u>set</u> of polynomials in $\mathbb{N}[x, y]$:
$\{(2x+y)(x+2y), x^2+3xy+x+y+y^2\}$.

Largest evaluation = # proof trees
smallest evaluation = # proof circuits.

PRS

A **polynomial replacement system** is $S = \big(\{x_1, \ldots, x_n\}, \{x_{n+1}, \ldots, x_m\}, q, R\big)$ where

- $\{x_1, \ldots, x_n\}$ are *terminal variables*,

- $\{x_{n+1}, \ldots, x_m\}$ are *nonterminal vars*,

- $q \in \mathbb{N}[x_1, \ldots, x_m]$ is *start polynomial*,

- R is a finite set of *replacement rules*, i. e. pairs of polynomials in $\mathbb{N}[x_1, \ldots, x_m]$.

How are rules applied?

Most general form:

$$p_1 \underset{S}{\Rightarrow} p_2 \quad \text{iff} \quad \begin{array}{l} \exists (p_3, p_4) \in R, \\ \exists p_5(\overline{x}, y) \in \mathbb{N}[x_1, \ldots, x_m, y] \\ \text{such that} \\ p_1(\overline{x}) = p_5(\overline{x}, p_3(\overline{x})) \text{ and} \\ p_2(\overline{x}) = p_5(\overline{x}, p_4(\overline{x})). \end{array}$$

Restricted form:

$$p_1 \underset{S}{\rightarrow} p_2 \quad \text{iff} \quad \begin{array}{l} \exists (p_3, p_4) \in R, \\ \exists p_5(\overline{x}), p_6(\overline{x}) \in \mathbb{N}[x_1, \ldots, x_m] \\ \text{such that} \\ p_1(\overline{x}) = p_5(\overline{x}) \cdot p_3(\overline{x}) + p_6(\overline{x}) \\ p_2(\overline{x}) = p_5(\overline{x}) \cdot p_4(\overline{x}) + p_6(\overline{x}). \end{array}$$

Lemma: The transitive reflexive closures of both forms are equivalent.

Of interest

PROBLEMS:

$$poly(S) = \Big\{p(x_1, \ \ldots \ , x_n) \ \Big| \ \exists p' \text{ s.t. } q \overset{*}{\underset{S}{\Rightarrow}} p' \text{ and } p(x_1, \ldots, x_n) = p'(x_1, \ldots, x_n, a_1, \ldots, a_m) \text{ for all } a_1, \ldots, a_m \in \mathbb{N}\Big\}.$$

$$range(S) = \{p(a) \mid p \in poly(S), a \in \mathbb{N}^n\}$$

$$eval(S) = \{(a, p(a)) \mid p \in poly(S), a \in \mathbb{N}^n\}.$$

PRS FAMILIES: simple, acyclic, deterministic, idempotent.

Simple and deterministic and acyclic:

$\{+, \times\}$-circuits over $\mathbb{N}$

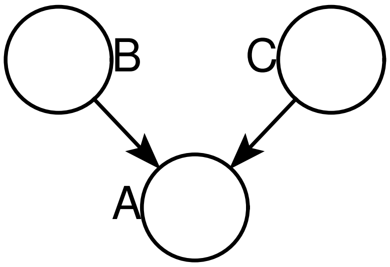
Simple and idempotent and acyclic:

"min $eval(S)$" counts proof circuits.

Simple and acyclic:

$\{\cup, +, \times\}$-circuits over $\mathbb{N}$

i.e. $eval(\cdot) \equiv_m^{\text{log}}$ $\mathbb{N}$-CIRCUIT$\{\cup, +, \times\}$

since 

simulated by $\begin{array}{ccc} A & \to & B \\ A & \to & C. \end{array}$

Note [StMe73]: $\mathbb{N}$-EXPRESSION$\{\cup, +, \times\}$ is NP-complete.

Complexity

S simple and deterministic

- $\forall S, poly(S)$ is P-complete.

In P: canonize S off line:

remove $A \overset{*}{\underset{S}{\Rightarrow}} \alpha A \beta$

remove $A \overset{*}{\underset{S}{\Rightarrow}} A + 0, A \overset{*}{\underset{S}{\Rightarrow}} 0 + A$

remove $A \overset{*}{\underset{S}{\Rightarrow}} A \times 1, A \overset{*}{\underset{S}{\Rightarrow}} 1 \times A$

Then a bounded number of equivalent SLPs

remain, since S is fixed.

Hardness: add to S a prs for CVP.

- $poly(\cdot) \in$ co-RP.

Straight-line program equivalence [IbMo83].

S simple and deterministic (cont)

- $\exists S, range(S)$ is NP-complete.

$range(\cdot) \in$ NP: remove cycles; input has n bits.

NP-hard: from the language QUADRATIC of $(a, b, c) \in \mathbb{N}^3$ s.t. $ax^2 + by = c$ for $x, y \in \mathbb{N}$. Fix

$$\langle \cdot, \cdot, \cdot \rangle : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}$$

an injective map. Then $(a, b, c) \in$ QUADRATIC

iff $\exists x, y, \langle a, b, ax^2 + by \rangle = \langle a, b, c \rangle$

iff $\exists u, v, x, y, \langle u, v, ux^2 + vy \rangle = \langle a, b, c \rangle$.

Take $\langle \alpha, \beta, \gamma \rangle = \gamma + (\beta + \gamma)^2 + (\alpha + \beta + \gamma)^3$.

- $eval(\cdot)$ is P-complete.

Note: artificial upper bound since a circuit C can have $2^{2^{|C|}}$ proof trees.

S simple and acyclic

- $\forall S, poly(S)$ is P-complete.

  In P: $poly(S)$ finite, rest as before.

- $poly[\cdot]$ NP-hard, in $\exists \cdot$ co-RP $=$ MA.

  NP-hardness: from SUM-OF-SUBSET. Rules

$$\begin{array}{rcl} x_k & \to & x_{k-1} \\ x_k & \to & x_{k-1} + b \end{array}$$

  nondeterministically add b to subset.

- $eval[\cdot]$ NP-complete.

  Again SUM-OF-SUBSET.

S simple and acyclic (cont)

- $eval(\cdot) \equiv_m^{\mathsf{log}} \mathbb{N}$-CIRCUIT$\{\cup, +, \times\}$.

Complexity: **OPEN**

$eval(\cdot) \in$ PSPACE.

Succinct $eval(\cdot)$ PSPACE-hard.

More open questions

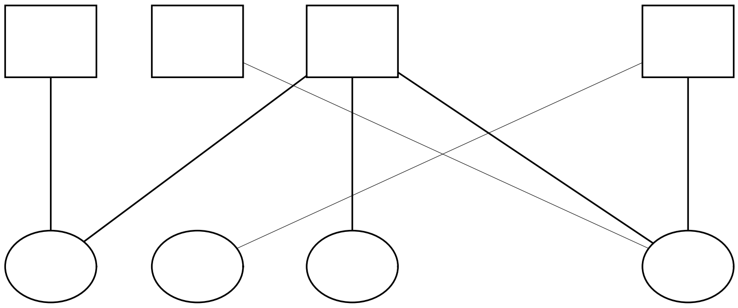
- Useful (beyond unification)?

- Expressibility as various prs? Links with Valiant's definability? Analogy with Valiant?

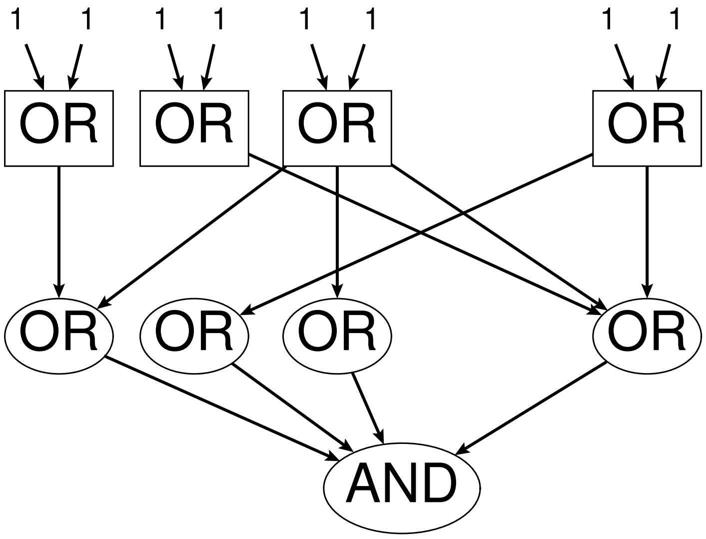
- Multivariate rules, acyclic prs, etc.

- The lurid case of depth-3 circuits versus the "$Z^2 \to 0$" rule...

The case of depth-3 circuits: details



From given $2n$-node bipartite graph, construct



The prs using rules $Z^2 \to 0$ instead of $Z^2 \to Z$ computes $2^n\cdot$ (# perfect matchings).

Yet: is computing the number of proof circuits for depth-3 circuits #P-hard?