Sampling a continuous signal

The Fourier transform of the Dirac comb with spacing $D$ is the Dirac comb with spacing $\frac{1}{D}$.

- Recall that elementwise multiplication in the time domain is convolution in the frequency domain.
- Sampling therefore generates replicates of the original spectrum known as (spectral) alias:

Sampling generates spectral aliases

- The spectrum of the digitized signal is the infinite superposition of translated copies of the original spectrum.
Sampling generates spectral aliases

- In theory, we can perfectly undo this effect and recover the original signal by low-pass filtering, if the spectral aliases don’t overlap.
- Otherwise, sampling will lose information.

The sampling theorem (wikipedia)

If a function $s(t)$ contains no frequencies higher than $D$ hertz, it is completely determined by giving its ordinates at a series of points spaced $1/(2D)$ seconds apart. The frequency $1/(2D)$ is called Nyquist frequency.

Sampling theorem in 2d

$|S(f_1, f_2)|$
Aliasing example

Aliasing in 2d

Aliasing in 2d
Aliasing in 2d
Aliasing in 2d

Smoothing

- An immediate practical implication of the sampling theorem:
  
  If you want to subsample an image, do smoothing first.

Example

Example from http://redwood.berkeley.edu/bruno/psc129/handouts/aliasing.pdf
Example

Welcome to Joe’s webpage!

Ideal interpolation

To re-generate the original (continuous) signal from the sampled signal, do low-pass filtering.

Applying an ideal low-pass filter amounts to convolving the signal with the Fourier transform of the box function.

Thus, ideal interpolation is convolution of the discrete signal with the sinc-function.

DFT leakage

- We can think of the DFT of a finite signal as the DFT of a periodic signal after multiplying it by a rectangular window.
- The DFT spectrum you get can be thought of as the spectrum of the periodic signal convolved with a sinc-function.
- Because of the zero-crossings of the sinc-function the convolution will have no effect on signal components whose frequencies are integer multiples of the window length.
- For any other components, the convolution will generate additional components in the spectrum.
- This effect is known as leakage.
Leakage example

It is best to think of leakage as *leaking into*: frequency components or noise can leak into the frequency bins that you actually want to detect.

- Leakage cannot be avoided.
- But a window other than the box-window may lead to different, potentially less undesirable, leakage properties.
- This may allow us to re-distribute the effect of leakage across the bins such that it causes the least harm for the application at hand.

Leakage with box window
Leakage with Gaussian window

Leakage with small Gaussian window
Windowing and Short Time Fourier Transform

- The **Short-Time Fourier Transform (STFT)** is an application of window functions:
- Fourier-transform the signal *locally*, then view the resulting set of spectra as a function of time or space.
- In 1d, the result is called **spectrogram**.
- The choice of window is important for the reasons just discussed.
- An STFT using a Gaussian window is also called **Gabor transform**.

Gabor features (2d)

- 2d Gabor functions are highly common in visual recognition tasks.
- They can be defined as the product of two exponential functions (one for the window, one for the phasor):

\[
g(K, \sigma, x_0, y_0, \gamma, u, v, P) = K \exp \left( -\frac{1}{\sigma^2} \left( (x - x_0)^2 + \gamma^2 (y - y_0)^2 \right) \right) \cdot \exp \left( i2\pi (ux + vy) + P \right)
\]

- Many variations are possible (for example, Gabor filters with oriented rather than axis parallel windows, etc.).
In many applications, local Gabor features are used as filters, i.e., they are scanned across the image. This naturally raises the question: What is the frequency response of a Gabor filter?

It is a localized blob in the frequency domain:
- The Fourier transform of a phasor times a Gaussian will be a delta-peak convolved with a Gaussian.
- Thus, Gabor filters are oriented bandpass filters.

(from Bishop, 2006)
- The visual analog of the spectrogram is a 3-dimensional feature map.
Complex cells

- AKA “energy mechanism”
- A Fourier feature pair with 90 deg phase difference is known as **quadrature pair**. (Better than saying “sin/cos”, because phasors in the Fourier expansion may have arbitrary different phase.)

Why spatial pooling computes approximate amplitude spectra

- In a convnet, lower layers learn Gabor-like features.
- Thus, locally, a feature map with ReLU units contains the rectified responses of phase-shifted Fourier components.
- Local max-pooling will keep the absolute value of the best matching phase.
- Local average-pooling will keep an “attenuated” version of this value.
- Overall, pooling therefore computes a “sample-approximation” of the amplitude response.

The uncertainty principle

In 2d: orientation uncertainty

from: Natural Image Statistics (Hyvärinen, Hurri, Hoyer; 2009)
Some recent papers on Fourier analysis in convnets