



DIRO
IFT 3205

TRAITEMENT DU SIGNAL

TRANSFORMÉE DE FOURIER DISCRÈTE

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TF DISCRÈTE

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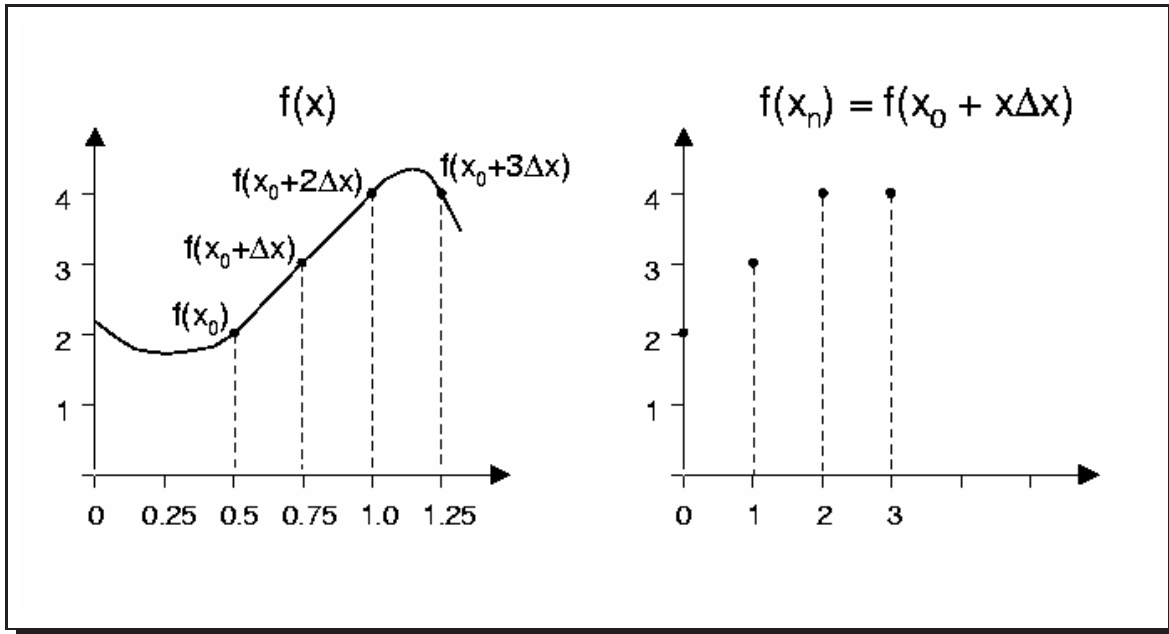
TF DISCRÈTE

TF DISCRÈTE 1D (1)

CAS CONTINU (RAPPEL)

$$\mathcal{F}[f(x)] = F(u) = \int_{-\infty}^{+\infty} f(x) \exp(-2\pi j u x) dx \quad u \in \mathbb{R}.$$

CAS DISCRET



$$f(x) = f(x_0 + x \Delta x) \quad x = 0, 1, 2, \dots, N - 1$$

TFD et TFD inverse

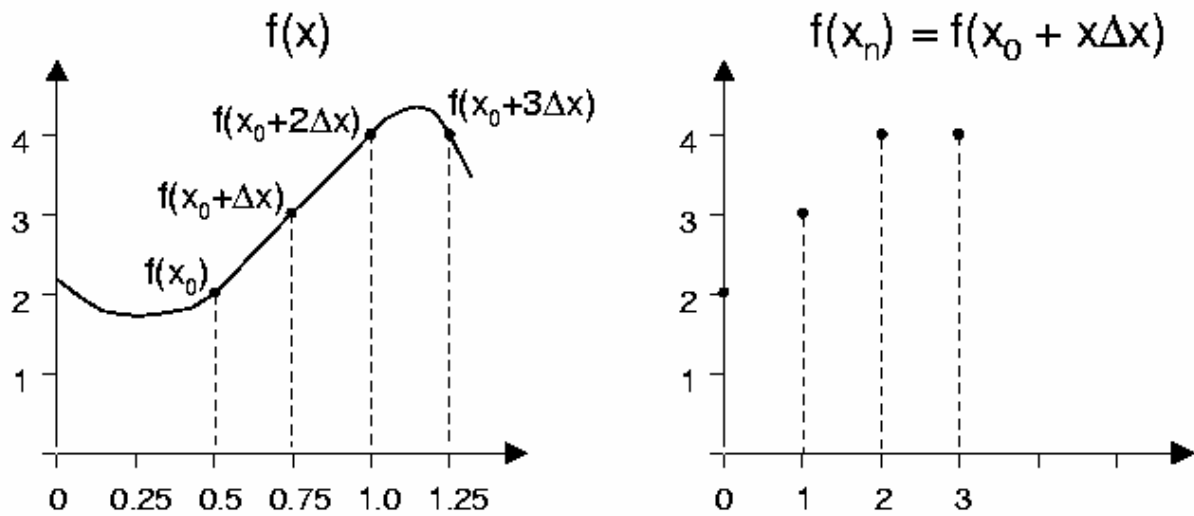
$$F(u) = \frac{1}{N} \sum_{x=0}^{N-1} f(x) \exp\left(\frac{-2\pi j u x}{N}\right) \quad u = 0, 1, \dots, N - 1$$

$$f(x) = \sum_{u=0}^{N-1} F(u) \exp\left(\frac{2\pi j u x}{N}\right) \quad x = 0, 1, \dots, N - 1$$

$$\Delta u = \frac{1}{N \Delta x}$$

TF DISCRÈTE

TF DISCRÈTE 1D -EXEMPLE- (2)



$$\begin{aligned}
 F(0) &= \frac{1}{4} \sum_{x=0}^3 f(x) \exp\left(\frac{-2\pi j 0 x}{4}\right) = \frac{1}{4} \sum_{x=0}^3 f(x) \cdot 1 \\
 &= \frac{1}{4} (f(0) + f(1) + f(2) + f(3)) = \frac{1}{4} (2 + 3 + 4 + 4) = 3.25
 \end{aligned}$$

$$\begin{aligned}
 F(1) &= \frac{1}{4} \sum_{x=0}^3 f(x) \exp\left(\frac{-2\pi j x}{4}\right) \\
 &= \frac{1}{4} (2 \exp(0) + 3 \exp(-j\pi/2) + 4 \exp(-j\pi) + 4 \exp(-3j\pi/2)) \\
 &= \frac{1}{4} (-2 + j)
 \end{aligned}$$

$$F(2) = \frac{1}{4} \sum_{x=0}^3 f(x) \exp\left(\frac{-4\pi j x}{4}\right) = \dots = -1/4$$

$$F(3) = \frac{1}{4} \sum_{x=0}^3 f(x) \exp\left(\frac{-6\pi j 0 x}{4}\right) = \dots = 1/4(-2 - j)$$

Spectre d'amplitude

$$|F(0)| = 3.25 \quad |F(1)| = \sqrt{5}/4 \quad |F(2)| = 1/4 \quad |F(3)| = \sqrt{5}/4$$

TF DISCRÈTE

TF DISCRÈTE 2D (1)

TFD 2D et TFD 2D inverse

$$F(u, \nu) = \frac{1}{MN} \sum_{x=0}^{N-1} \sum_{y=0}^{M-1} f(x, y) \exp \left(-2\pi j \left(\frac{ux}{N} + \frac{\nu y}{M} \right) \right)$$

$$u = 0, 1, 2, \dots, N - 1$$

$$\nu = 0, 1, 2, \dots, M - 1$$

$$f(x, y) = \sum_{u=0}^{N-1} \sum_{\nu=0}^{M-1} F(u, \nu) \exp \left(2\pi j \left(\frac{ux}{N} + \frac{\nu y}{M} \right) \right)$$

$$x = 0, 1, 2, \dots, N - 1$$

$$y = 0, 1, 2, \dots, M - 1$$

Le plus souvent l'image à traiter est carrée $\blacktriangleleft \blacktriangleright M = N$

PROPRIÉTÉS

- Les mêmes que ceux énoncés pour la TF continue
- Cyclique (périodique)

$$F(u, \nu) = F(u + N, \nu) = F(u, \nu + N) = F(u + N, \nu + N)$$

$$f(x, y) = f(x + N, y + N)$$

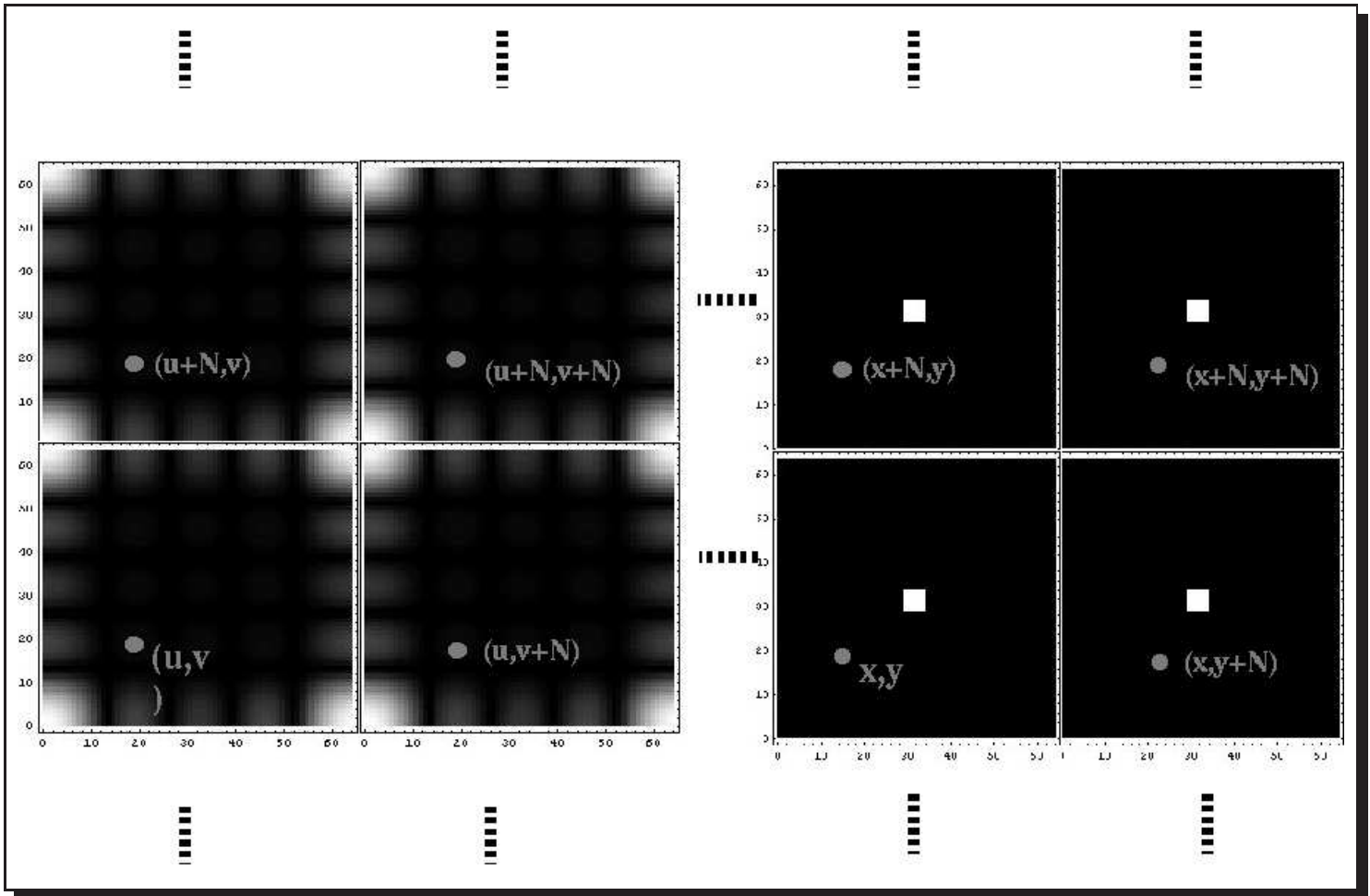
car

Signal périodique	$\xLeftrightarrow{\mathcal{F}}$	Spectre de raies
Signal échantillonné	$\xLeftrightarrow{\mathcal{F}}$	Spectre périodique

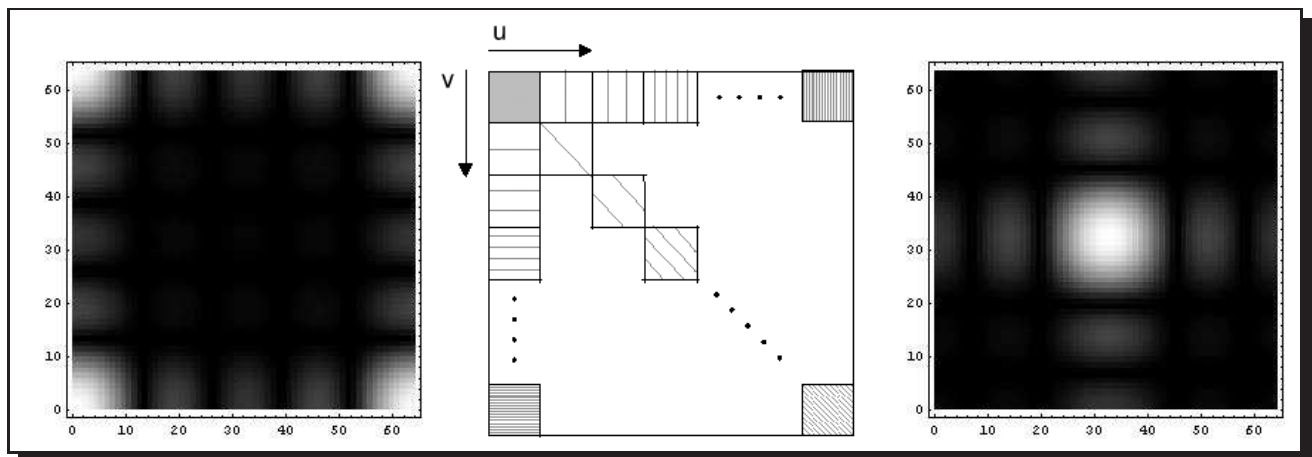
TF DISCRÈTE

TF DISCRÈTE 2D (2)

• Périodicité



• Nota



TF DISCRÈTE

TF DISCRÈTE 2D (3)

Donc Puisque

Signal périodique $\xLeftrightarrow{\mathcal{F}}$ Spectre discret

$\Leftrightarrow$

TFD $\left(\begin{array}{c} \text{Portrait of Fourier} \end{array} \right)$

=

TF $\left(\begin{array}{c} \vdots \vdots \vdots \\ \dots \text{Portrait of Fourier} \text{Portrait of Fourier} \text{Portrait of Fourier} \text{Portrait of Fourier} \dots \\ \dots \text{Portrait of Fourier} \text{Portrait of Fourier} \text{Portrait of Fourier} \text{Portrait of Fourier} \dots \\ \dots \text{Portrait of Fourier} \text{Portrait of Fourier} \text{Portrait of Fourier} \text{Portrait of Fourier} \dots \\ \vdots \vdots \vdots \end{array} \right)$

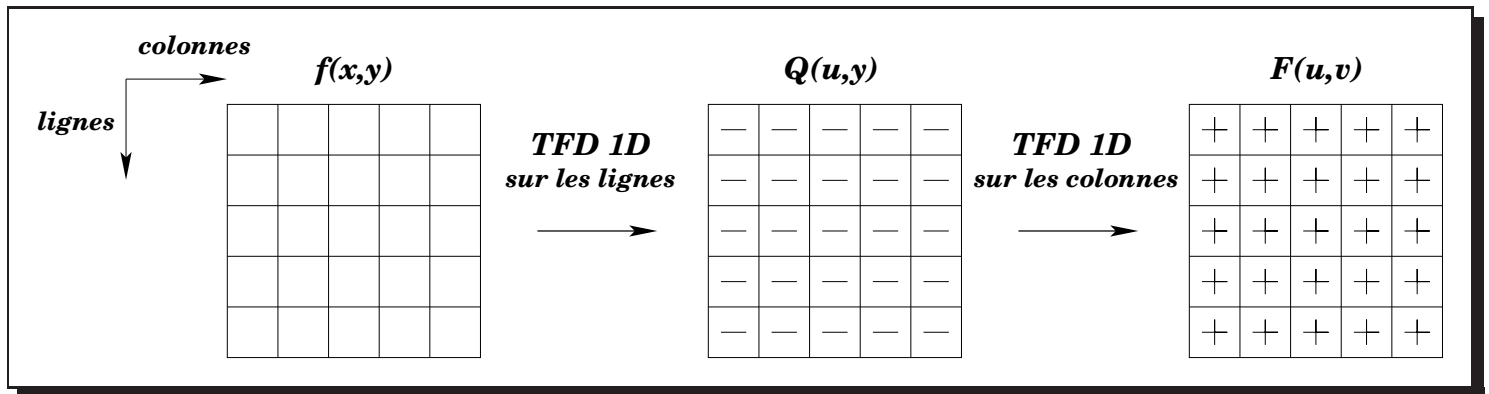
TF DISCRÈTE

TF DISCRÈTE 2D (4)

• Séparabilité

Pour une image carrée ◀▶ $M = N$

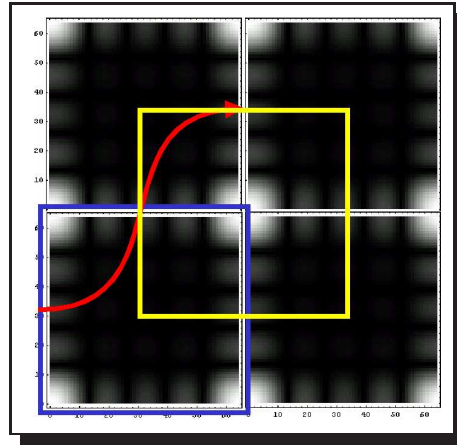
$$\begin{aligned}
 F(u, \nu) &= \frac{1}{N^2} \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} f(x, y) \exp \left(-2\pi j \left(\frac{ux + \nu y}{N} \right) \right) \\
 &= \frac{1}{N} \sum_{x=0}^{N-1} \left(\frac{1}{N} \sum_{y=0}^{N-1} f(x, y) \exp \left(-2\pi j \frac{\nu y}{N} \right) \right) \exp \left(-2\pi j \frac{ux}{N} \right) \\
 &= \underbrace{\frac{1}{N} \sum_{x=0}^{N-1} [\text{TF 1D de la } x \text{ ième colonne}]}_{\text{TF 1D de la } y \text{ ième ligne}} \exp \left(-2\pi j \frac{ux}{N} \right)
 \end{aligned}$$



TF DISCRÈTE

TF DISCRÈTE 2D (4)

Recalage Cyclique

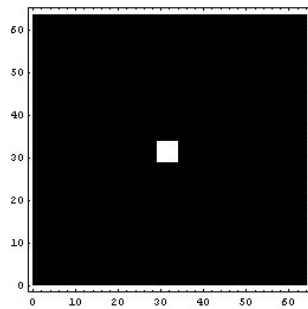


Translation –Rappel–

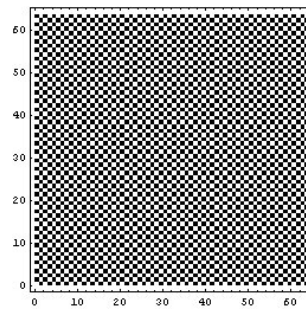
$$f(x - x_0, y - y_0) \xrightarrow{\mathcal{F}} F(u, \nu) \exp \left(\frac{-2\pi j (ux_0 + \nu y_0)}{N} \right)$$

$$f(x, y) \exp \left(\frac{2\pi j (u_0 x + \nu_0 y)}{N} \right) \xrightarrow{\mathcal{F}} F(u - u_0, \nu - \nu_0)$$

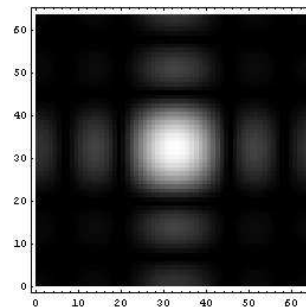
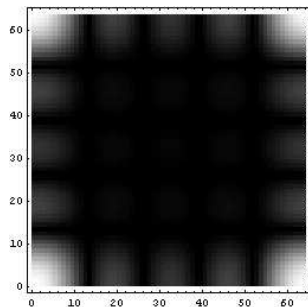
$$u_0 = \nu_0 = \frac{N}{2} \triangleright \exp \left(\frac{2\pi j (u_0 x + \nu_0 y)}{N} \right) = \exp(\pi j (x + y)) = (-1)^{(x+y)}$$



$$\times (-1)^{(x+y)}$$



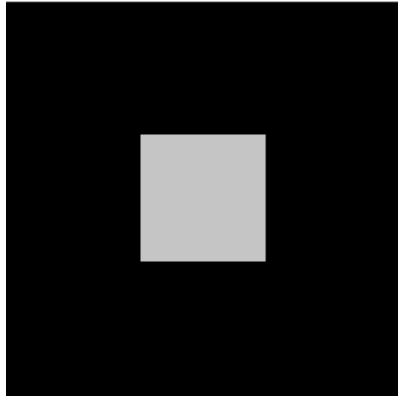
$$\Downarrow \mathcal{F}$$



TF DISCRÈTE

TF DISCRÈTE 2D -VISUALISATION- (5)

Image originale



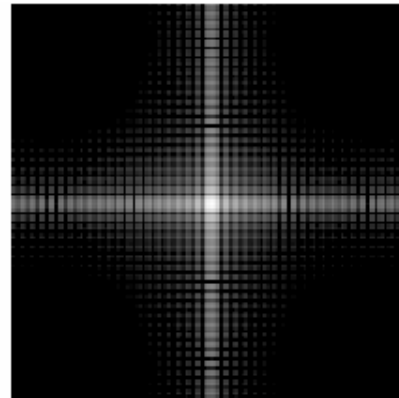
$|F(u, \nu)|$



$|F(0,0)|$ au centre



$k \log(1 + |F(u, \nu)|)$

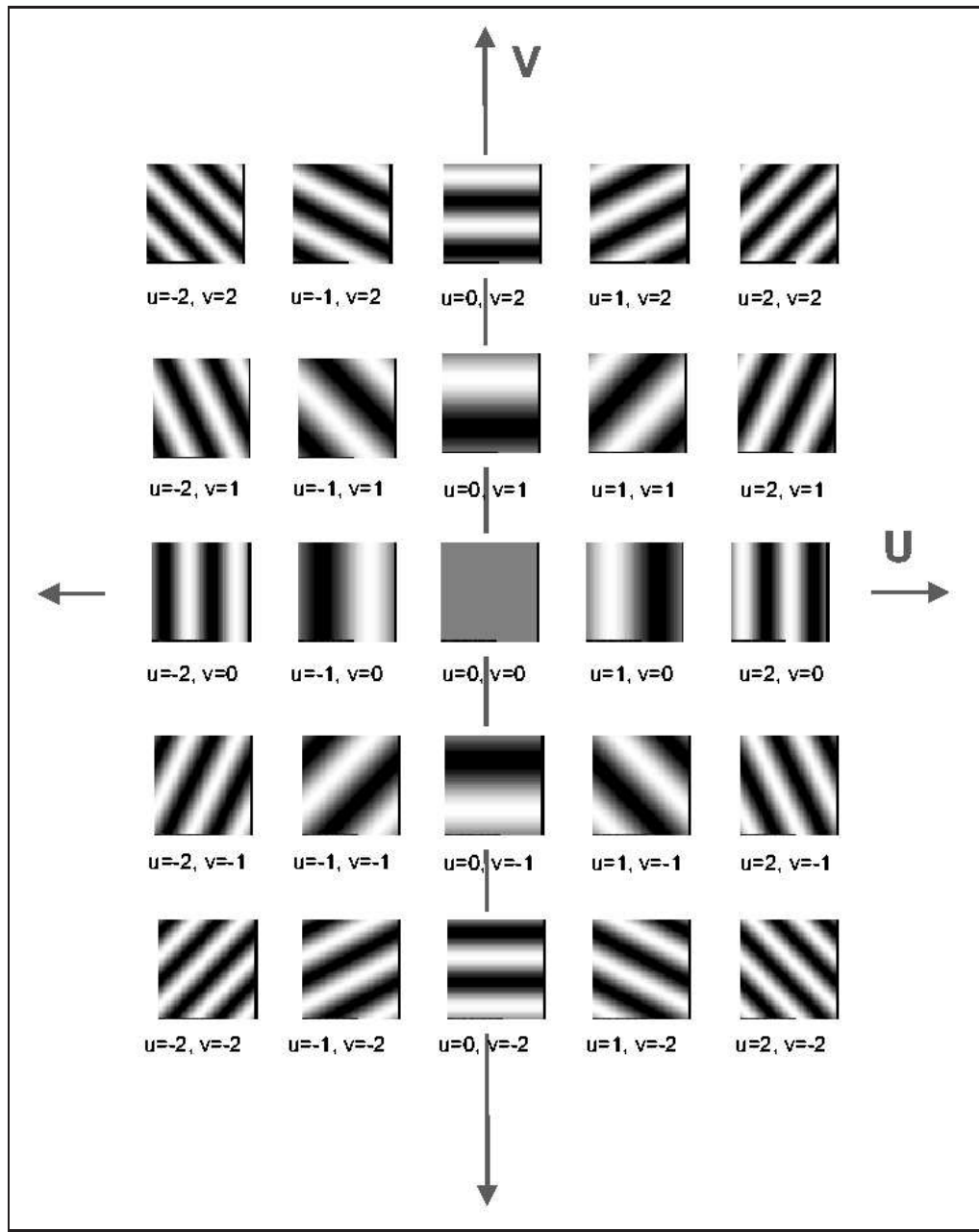
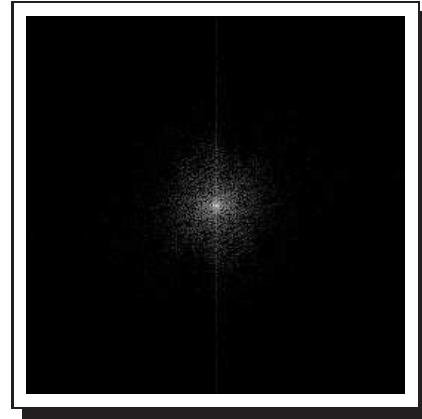


—Nota—

- On affiche généralement $|F(u, \nu)|$ ($F(u, \nu)$ complexe)
- Les fréquences élevées sont bcp plus faible que les fréquences plus basses. On affiche donc plutôt $k \log(1 + |F(u, \nu)|)$ (k est une constante de normalisation pour recaler les niveau de gris dans $[0, 255]$)
- On met l'origine au centre de l'image en effectuant un décalage cyclique

TF DISCRÈTE

TF DISCRÈTE 2D -INTERPRÉTATION- (6)



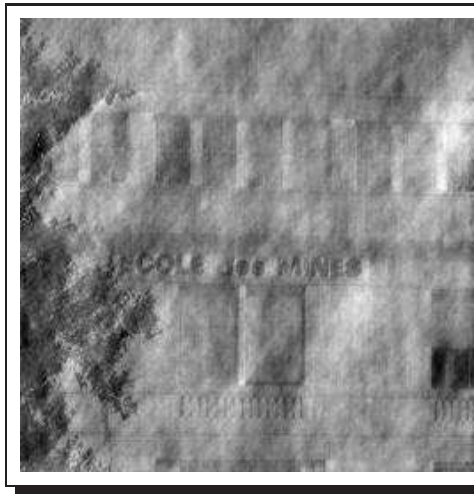
TF DISCRÈTE

TF DISCRÈTE 2D -INTERPRÉTATION- (7)

Expérience de Oppenheim et Lim (1981)



Reconstruction obtenue avec le module de Batiment et la phase Lena et inversement pour l'autre image



- **Module** $\Leftrightarrow$ puissance/intensité de la contribution des différentes composantes fréquentielles

$\approx$ intensité des contours dans l'image

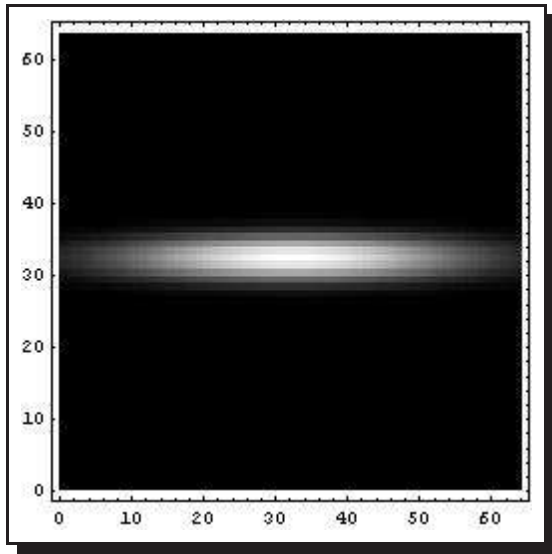
- **Phase** $\Leftrightarrow$ position des composantes fréquentielles à reconstruire

$\approx$ aspect/texture de l'image

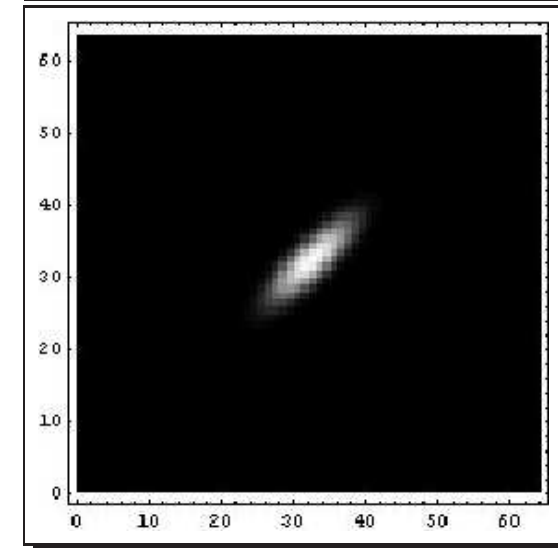
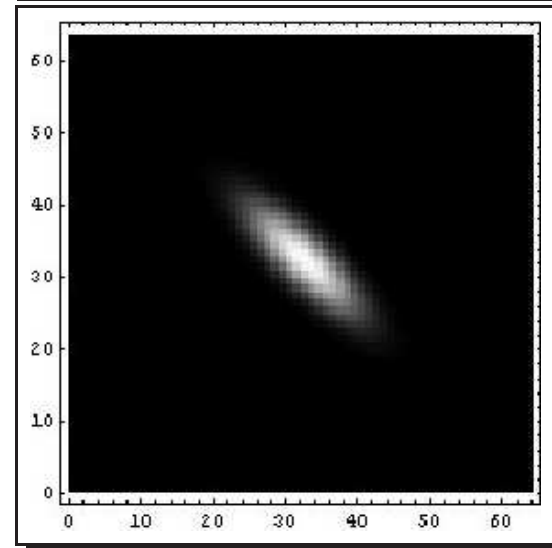
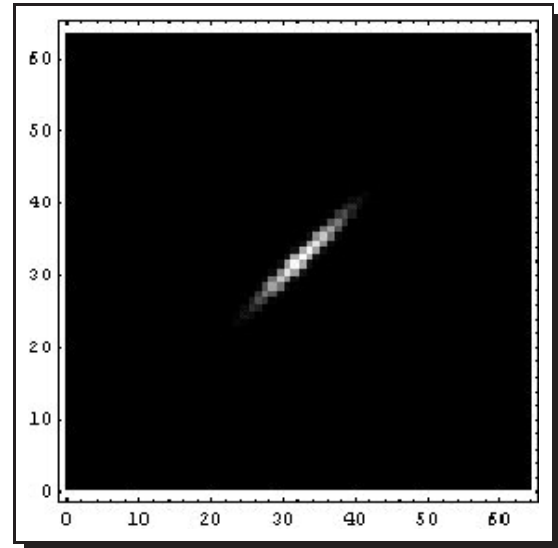
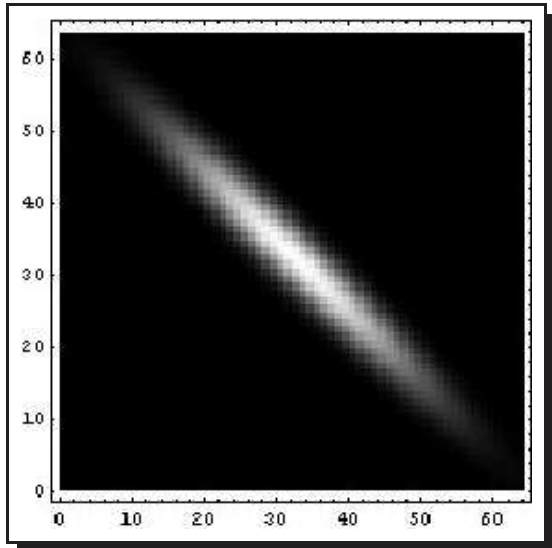
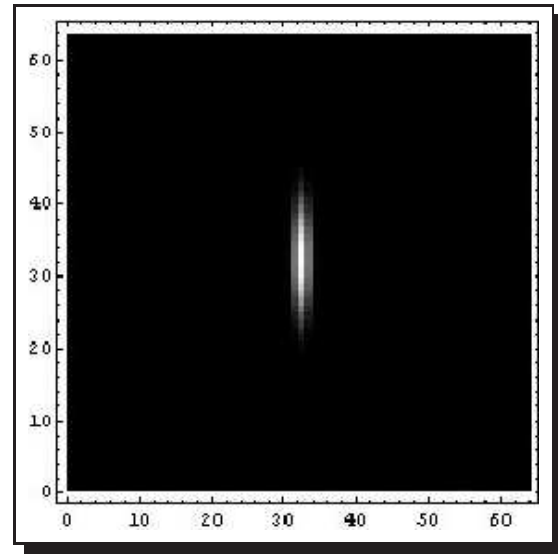
TF DISCRÈTE

TF DISCRÈTE 2D -EXEMPLES- (8)

Image



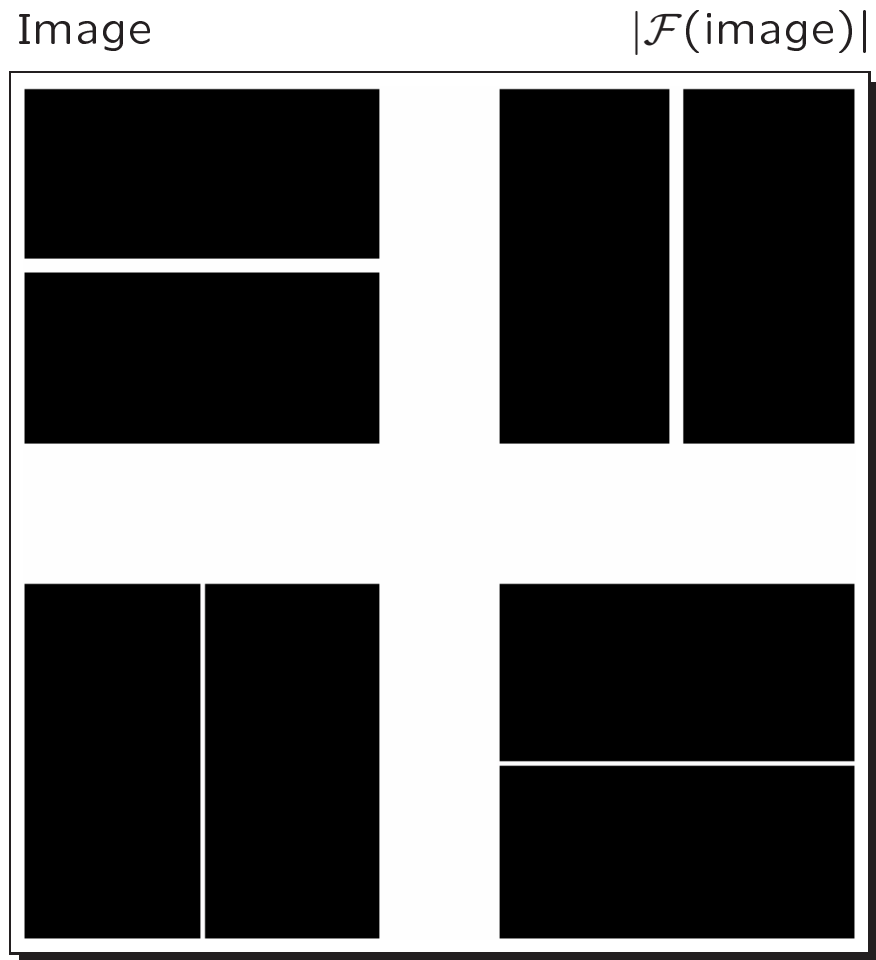
$|\mathcal{F}(\text{image})|$



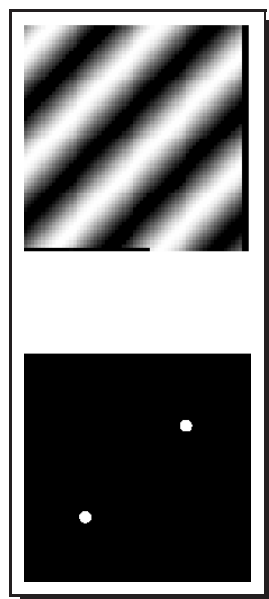
TF DISCRÈTE

TF DISCRÈTE 2D -EXEMPLES- (9)

- Rotation



- Périodicité

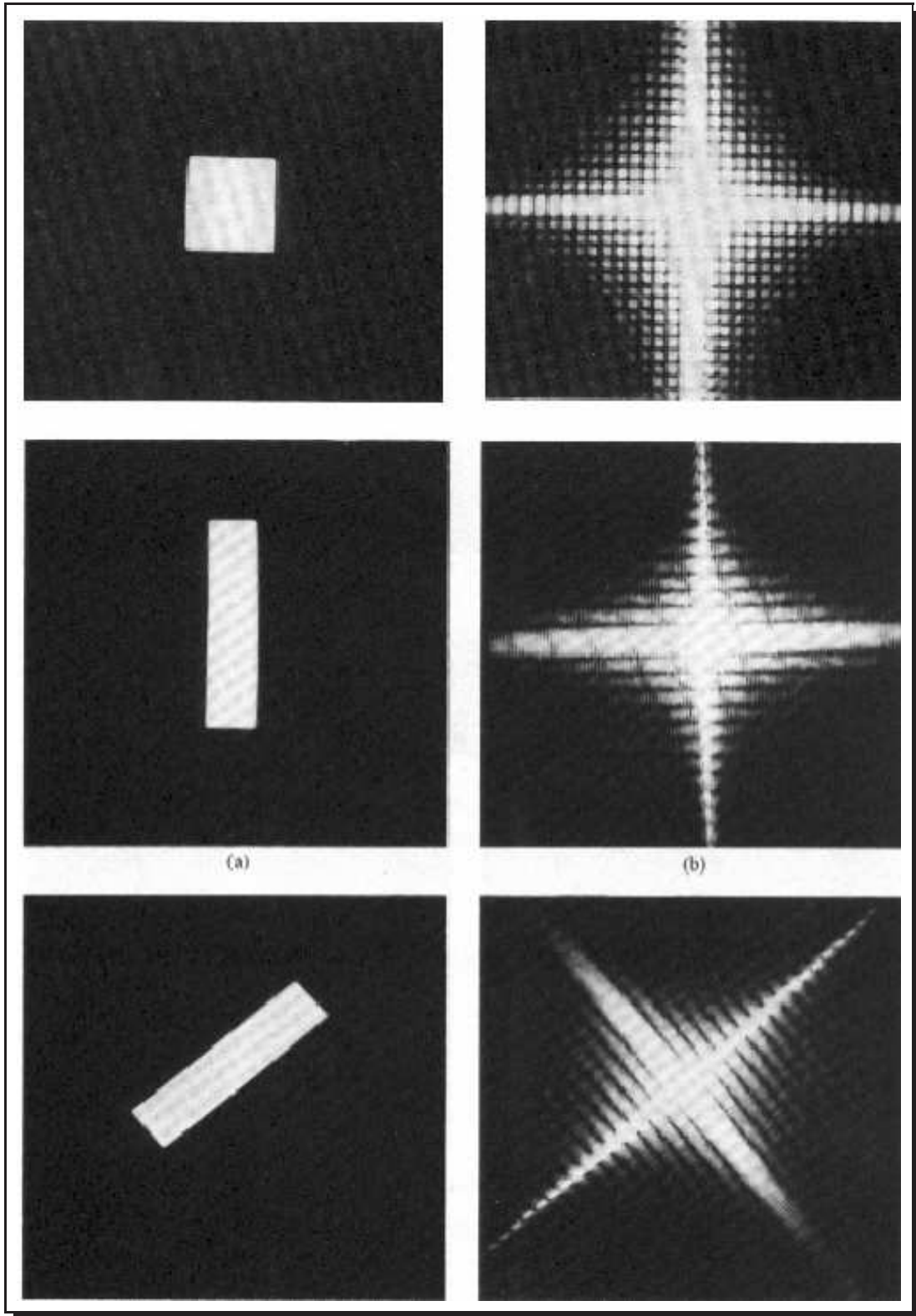


TF DISCRÈTE

TF DISCRÈTE 2D -EXEMPLES- (10)

Image

$|\mathcal{F}(\text{image})|$

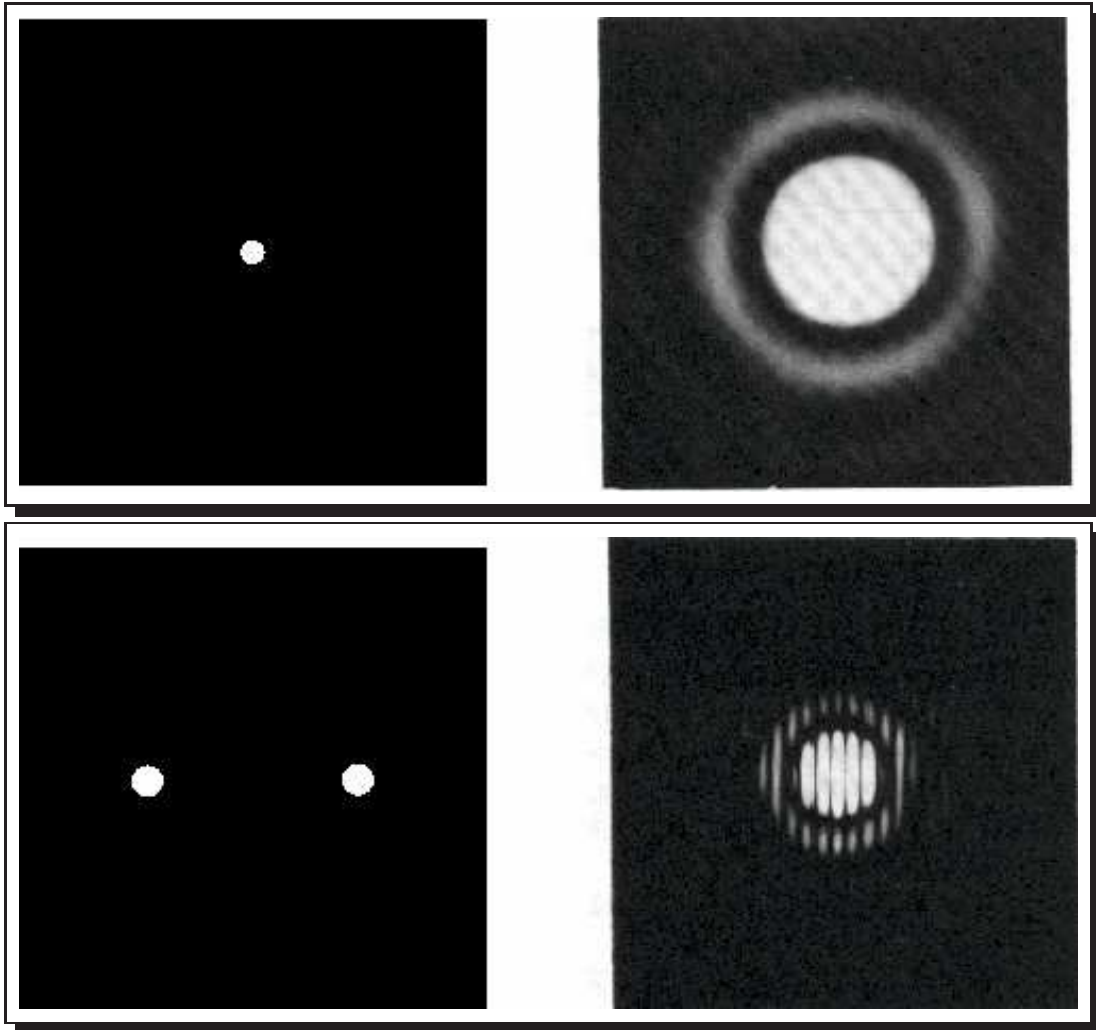


TF DISCRÈTE

TF DISCRÈTE 2D -EXEMPLES- (11)

Image

$|\mathcal{F}(\text{image})|$



TF DISCRÈTE

CONVOLUTION DISCRÈTE 1D & 2D (1)

Théorème de convolution 2D

$$f(x, y) * g(x, y) \xrightarrow{\mathcal{F}} F(u, \nu) \cdot G(u, \nu)$$

$$f(x, y) \cdot g(x, y) \xrightarrow{\mathcal{F}} F(u, \nu) * G(u, \nu)$$

Convolution 2D -Version continue-

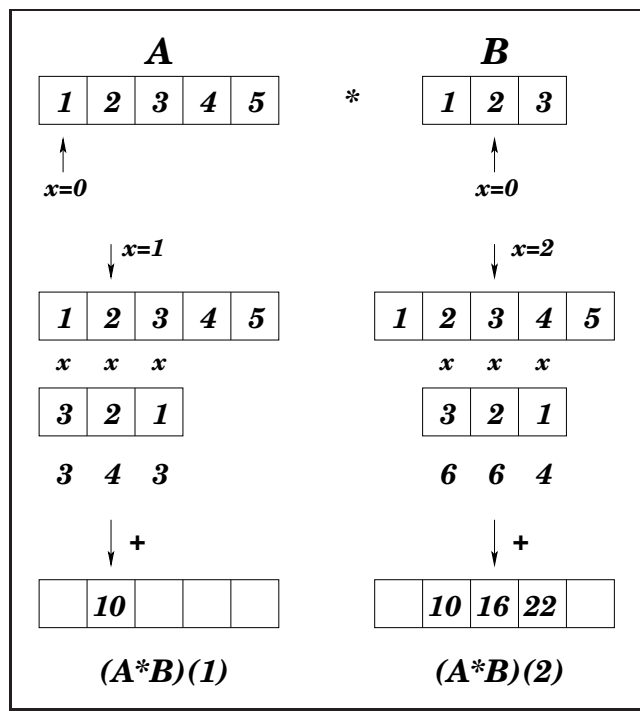
$$(f * g)(x, y) = \int_0^\infty \int_0^\infty f(\alpha, \beta) g(x - \alpha, y - \beta) d\alpha d\beta$$

Convolution 1D -Version discrète-

$$\begin{array}{|c|c|c|c|c|} \hline & & \mathbf{A} & & \\ \hline \end{array} * \begin{array}{|c|c|c|} \hline & \mathbf{B} & \\ \hline \end{array}$$

$$(A * B)(x) = \sum_i A(i) B(x - i)$$

-Exemple-



TF DISCRÈTE

CONVOLUTION DISCRÈTE 1D & 2D (2)

Convolution des bords

A) Ajout de zéro

0 0

1	2	3	4	5
---	---	---	---	---

 0 0

4	10	16	22	22
---	----	----	----	----

B) Enroulement

4 5

1	2	3	4	5
---	---	---	---	---

 1 2

19	10	16	22	23
----	----	----	----	----

C) Réflexion

3 2

1	2	3	4	5
---	---	---	---	---

 4 3

10	10	16	22	26
----	----	----	----	----

D) Défaut

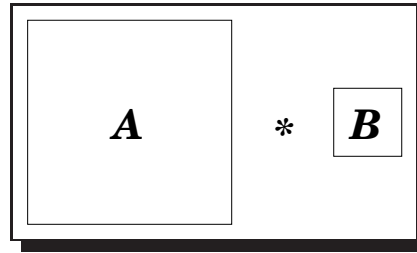
1	2	3	4	5
---	---	---	---	---

X	10	16	22	X
---	----	----	----	---

TF DISCRÈTE

CONVOLUTION DISCRÈTE 1D & 2D (3)

Convolution 2D -Version discrète-



$$(A * B)(x, y) = \sum_i \sum_j A(i, j) B(x - i, y - j)$$

-Exemple-

<table style="border-collapse: collapse;"> <tr><td>10</td><td>5</td><td>20</td><td>20</td><td>20</td></tr> <tr><td>10</td><td>5</td><td>20</td><td>20</td><td>20</td></tr> <tr><td>10</td><td>5</td><td>20</td><td>20</td><td>20</td></tr> <tr><td>10</td><td>5</td><td>20</td><td>20</td><td>20</td></tr> <tr><td>10</td><td>5</td><td>20</td><td>20</td><td>20</td></tr> </table>	10	5	20	20	20	10	5	20	20	20	10	5	20	20	20	10	5	20	20	20	10	5	20	20	20	*	<table style="border-collapse: collapse;"> <tr><td style="border: 1px solid black; border-radius: 50%;">-1</td><td>1</td></tr> <tr><td>0</td><td>1</td></tr> </table>	-1	1	0	1	<table style="border-collapse: collapse;"> <tr><td>-10</td><td>5</td><td>-15</td><td>0</td><td>0</td></tr> <tr><td>-10</td><td>15</td><td>-10</td><td>20</td><td>20</td></tr> <tr><td>-10</td><td>15</td><td>-10</td><td>20</td><td>20</td></tr> <tr><td>-10</td><td>15</td><td>-10</td><td>20</td><td>20</td></tr> <tr><td>-10</td><td>15</td><td>-10</td><td>20</td><td>20</td></tr> </table>	-10	5	-15	0	0	-10	15	-10	20	20	-10	15	-10	20	20	-10	15	-10	20	20	-10	15	-10	20	20
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– Propriétés –

- Commutatif ▶ $f_1 * f_2 * f = f_2 * f_1 * f$
- Associatif ▶ $(f_1 * f_2) * f = f_1 * (f_2 * f)$
- Distributif ▶ $(f_1 + f_2) * f = f_1 * f + f_2 * f$
 ▶ $f * (f_1 + f_2) = f * f_1 + f * f_2$

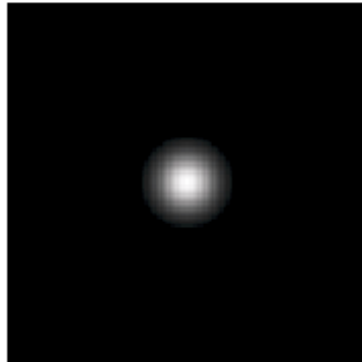
TF DISCRÈTE

CONVOLUTION DISCRÈTE 1D & 2D (4)

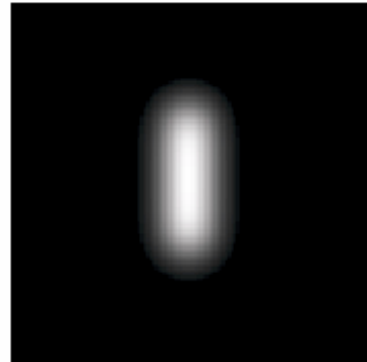
$f(x,y)$



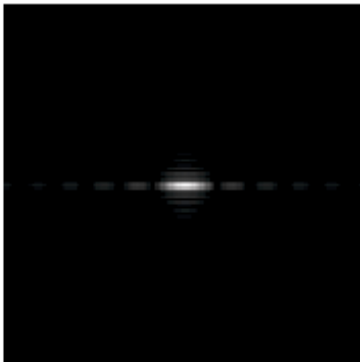
$g(x,y)$



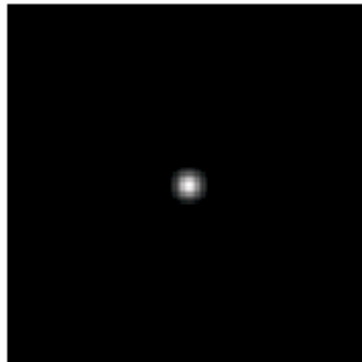
$f * g(x,y)$



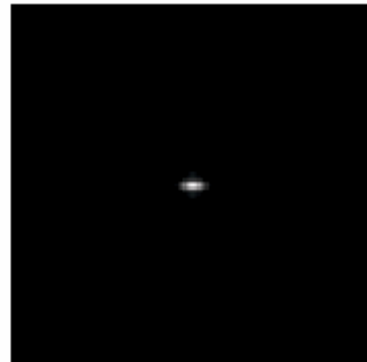
$F(u,v)$



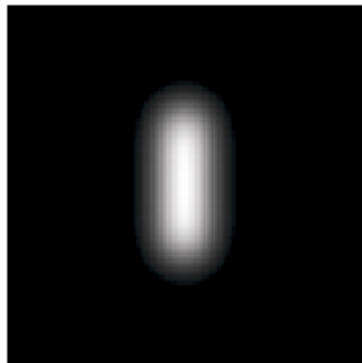
$G(u,v)$



$F(u,v) \cdot G(u,v)$

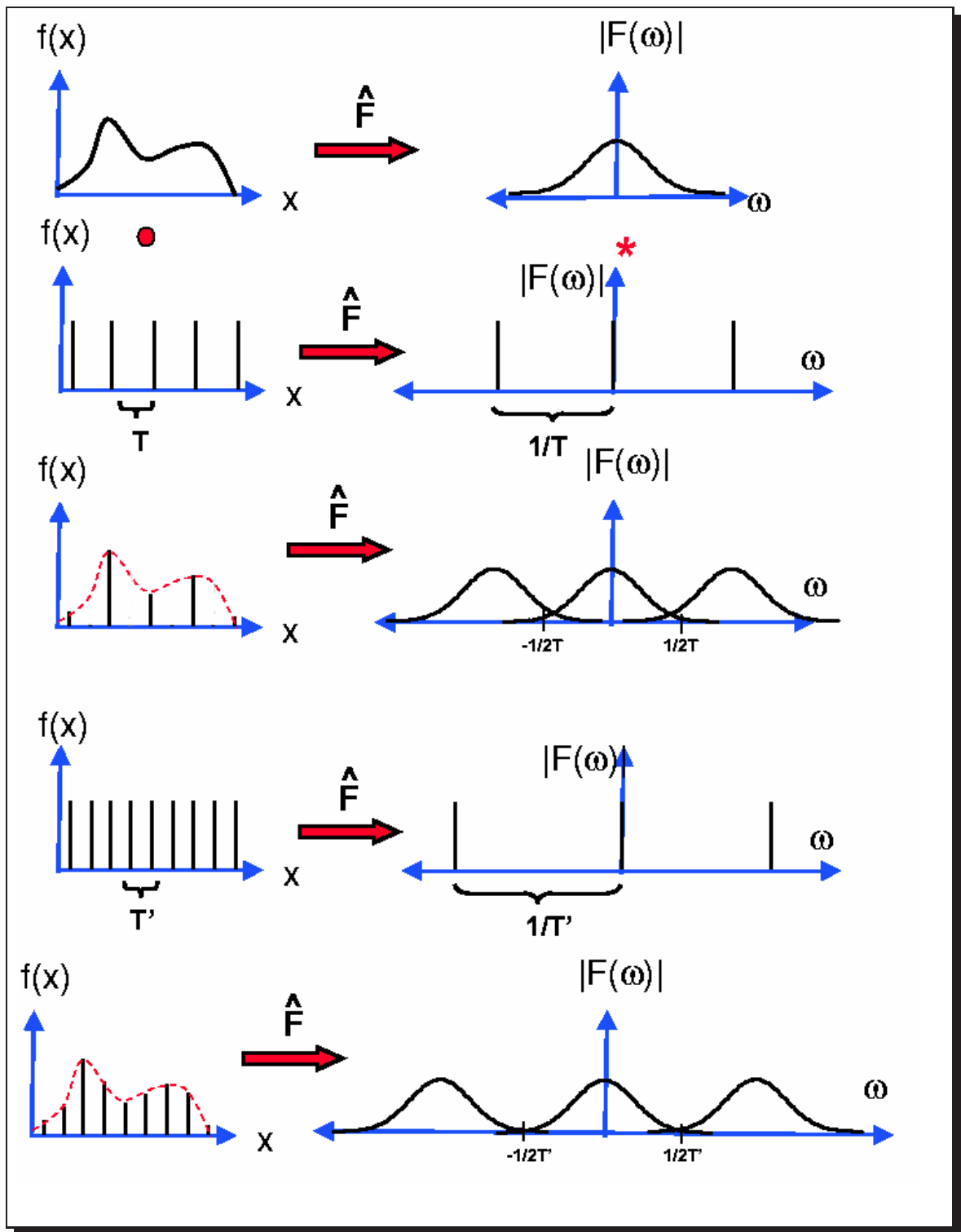


$\mathcal{F}^{-1}[F(u,v) \cdot G(u,v)]$



TF DISCRÈTE

THÉORÈME DE SCHANNON

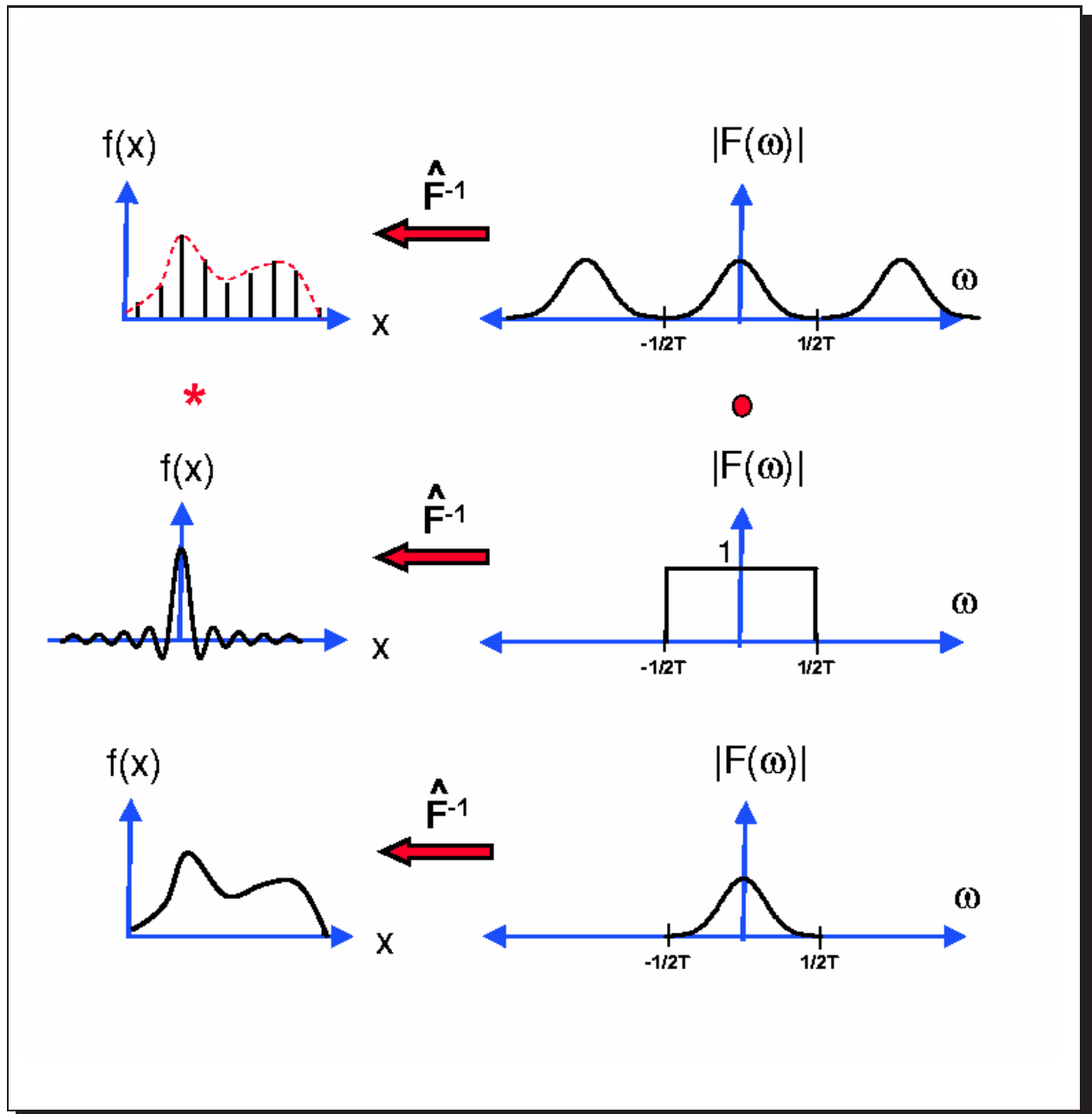


$$T_e \leq \frac{1}{2\nu_0}$$

► sinon phénomène d'aliasing

TF DISCRÈTE

INTERPOLATION OPTIMALE



- Possibilité de reconstruire totalement le signal 2D à partir de ses échantillons

TF DISCRÈTE

FFT (1)

$$F(u) = \frac{1}{N} \sum_{x=0}^{N-1} f(x) \exp\left(\frac{-2\pi j u x}{N}\right) \quad u = 0, 1, 2, \dots, N-1$$

$O(N^2)$ opérations

$$F(u) = \underbrace{\frac{1}{N} \sum_{x=0}^{N/2-1} f(2x) \exp\left(\frac{-2\pi j u 2x}{N}\right)}_{\text{x pair}} + \underbrace{\frac{1}{N} \sum_{x=0}^{N/2-1} f(2x+1) \exp\left(\frac{-2\pi j u (2x+1)}{N}\right)}_{\text{x impair}}$$

$$= \frac{1}{2} \left[\underbrace{\frac{1}{N/2} \sum_{x=0}^{N/2-1} f(2x) \exp\left(\frac{-2\pi j u x}{N/2}\right)}_{\text{TF de la partie paire}} + \exp\left(\frac{-2\pi j u}{N}\right) \underbrace{\frac{1}{N/2} \sum_{x=0}^{N/2-1} f(2x+1) \exp\left(\frac{-2\pi j u x}{N/2}\right)}_{\text{TF de la partie impaire}} \right]$$

- TF [N éléments] = 2 TF [$N/2$ éléments]

+ $N/2$ multiplications et $N/2$ additions complexes (en fait $N/4$ pour les multiplications)

- $Op(N) = 2Op(N/2) + N$

$$F_N(u) = \frac{1}{2} \left[F_{N/2}^{\text{paire}}(u) + \exp\left(-\frac{2\pi j u}{N}\right) F_{N/2}^{\text{impaire}}(u) \right]$$

Pour $u' = u + \frac{N}{2}$

$$\exp\left(-\frac{2\pi j u'}{N}\right) = \exp\left(-\frac{2\pi j (u + \frac{N}{2})}{N}\right) = \exp\left(-\frac{2\pi j u}{N}\right) \exp(-\pi j) = -\exp\left(-\frac{2\pi j u}{N}\right)$$

TF DISCRÈTE

FFT (2)

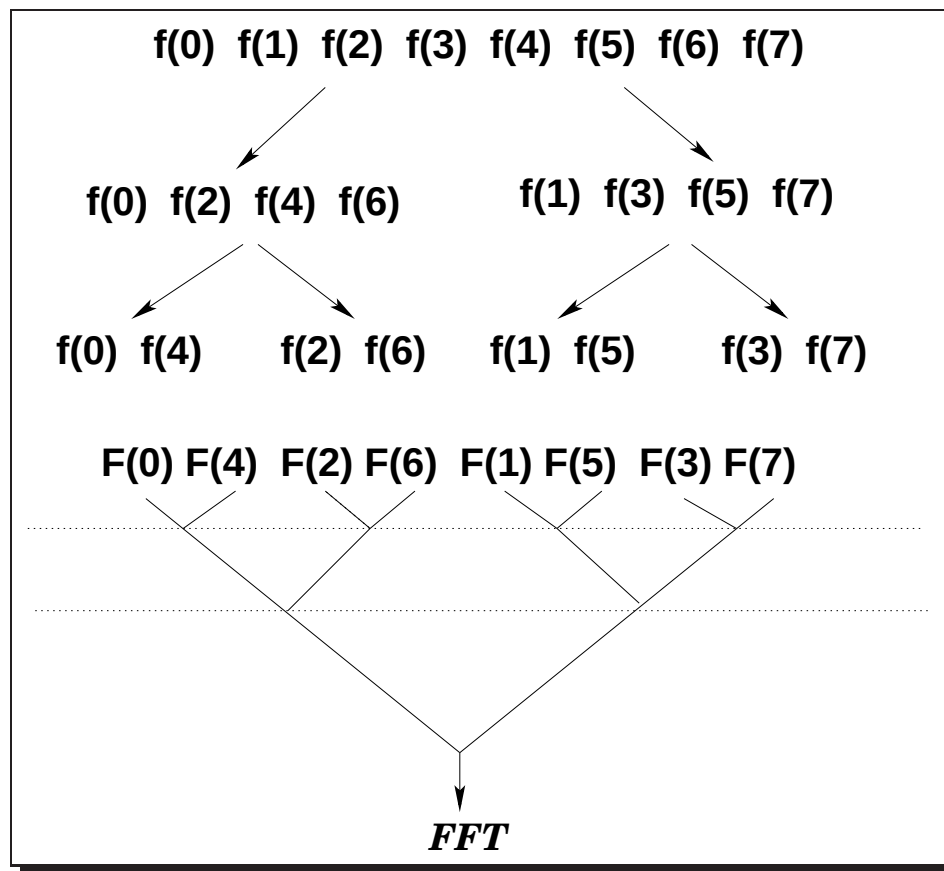
Ainsi

$$F_N(u) = \frac{1}{2} \left[F_{N/2}^{\text{paire}}(u) + \exp\left(-\frac{2\pi j u}{N}\right) F_{N/2}^{\text{impaire}}(u) \right]$$

$$F_N\left(u + \frac{N}{2}\right) = \frac{1}{2} \left[F_{N/2}^{\text{paire}}(u) - \exp\left(-\frac{2\pi j u}{N}\right) F_{N/2}^{\text{impaire}}(u) \right]$$

Pour $u = 0, 1, 2, \dots, N/2 - 1$

1 multiplication est nécessaire pour deux termes



FFT 1D ►

$O(N \log N)$

FFT 2D ►

$O(N^2 \log N)$

TF DISCRÈTE

FFT (3)

Vérifions que $O(N \log N)$

(Par induction, sachant que $Op(N) = 2 Op(N/2) + N$)

- Vrai pour $N = 1$? $\blacktriangleright Op(1) = 1 \log 1 = 0 \blacktriangleright$ oui
- Vrai pour $N = 2$? $\blacktriangleright Op(2) = 2 \log 2 = 2 \blacktriangleright$ oui
- Vrai pour $N \log N$?

- $Op(N) = N \log N$?
- $Op(N) = 2N/2 \log(2N/2)$
- $Op(N) = 2N/2(\log(N/2) + \log 2)$
- $Op(N) = 2N/2 \log N/2 + 2N/2 \log 2$
- $Op(N) = 2 Op(N/2) + N$

C.Q.F.D.

Principe du filtrage fréquentiel

