



DIRO
IFT 6150

TRAITEMENT D'IMAGES

TRANSFORMÉE DE FOURIER DISCRÈTE

Max Mignotte

Département d'Informatique et de Recherche Opérationnelle.

Http : [//www.iro.umontreal.ca/~mignotte/ift6150](http://www.iro.umontreal.ca/~mignotte/ift6150)

E-mail : mignotte@iro.umontreal.ca

TF DISCRÈTE

SOMMAIRE

TFD 1D	2
TFD 2D	4
Convolution Discrète 1D & 2D	14
Théorème de Shannon	18
Interpolation Optimale	19
FFT	21

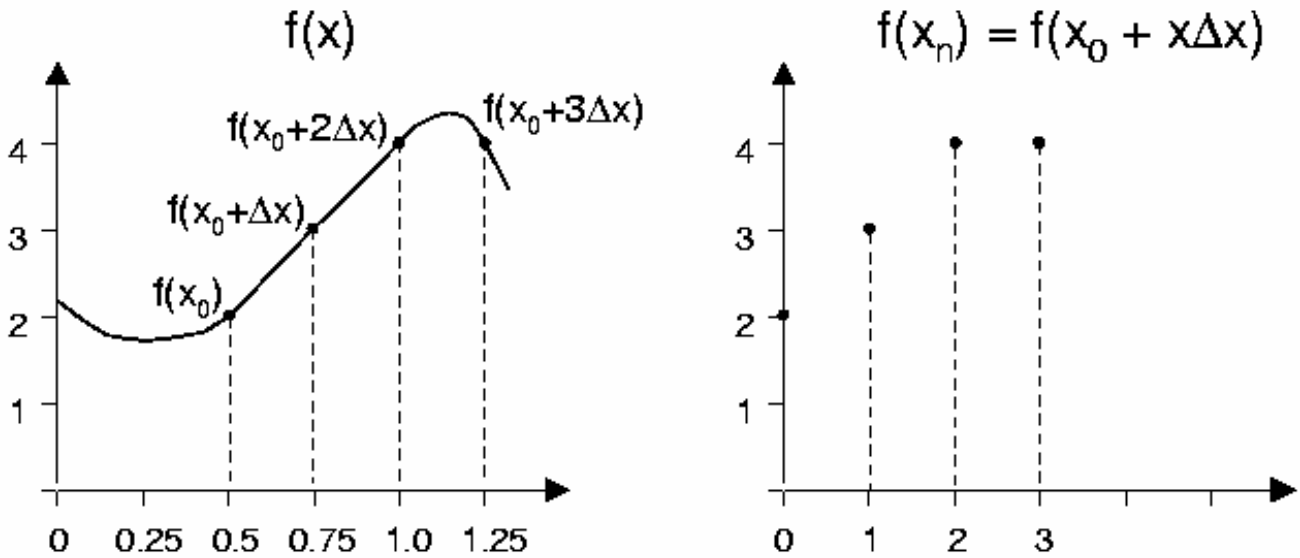
TF DISCRÈTE

TF DISCRÈTE 1D (1)

CAS CONTINU (RAPPEL)

$$\mathcal{F}[f(x)] = F(u) = \int_{-\infty}^{+\infty} f(x) \exp(-2\pi j u x) dx \quad u \in \mathbb{R}.$$

CAS DISCRET



$$f(x) = f(x_0 + x\Delta x) \quad x = 0, 1, 2, \dots, N - 1$$

TFD et TFD inverse

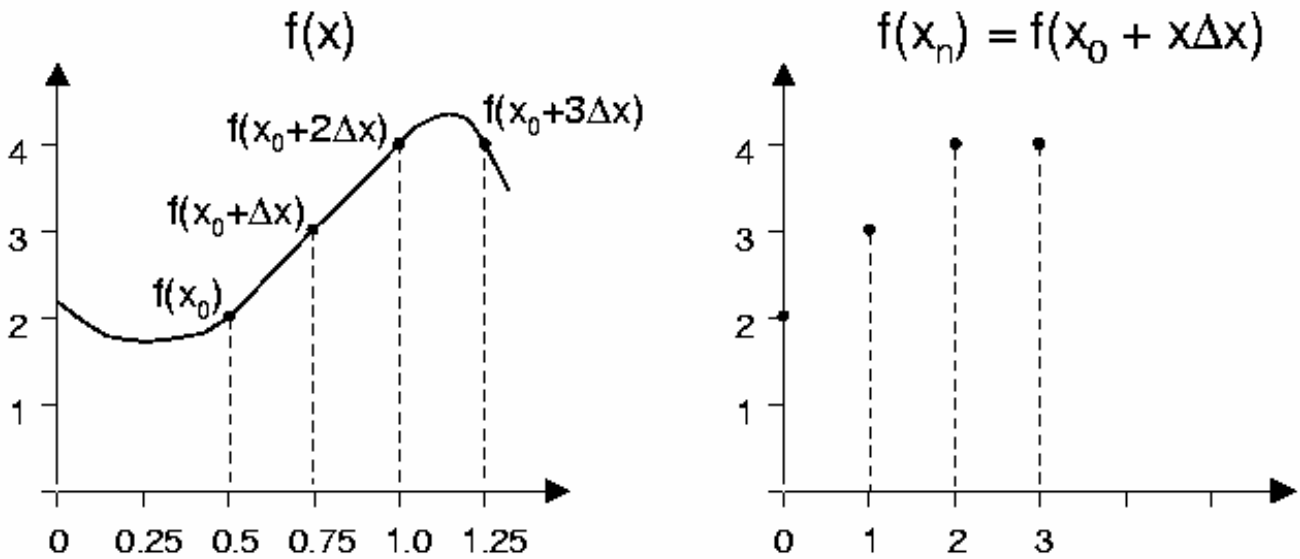
$$F(u) = \frac{1}{N} \sum_{x=0}^{N-1} f(x) \exp\left(\frac{-2\pi j u x}{N}\right) \quad u = 0, 1, 2, \dots, N - 1$$

$$f(x) = \sum_{u=0}^{N-1} F(u) \exp\left(\frac{2\pi j u x}{N}\right) \quad x = 0, 1, 2, \dots, N - 1$$

$$\Delta u = \frac{1}{N\Delta x}$$

TF DISCRÈTE

TF DISCRÈTE 1D -EXEMPLE- (2)



$$\begin{aligned}
 F(0) &= \frac{1}{4} \sum_{x=0}^3 f(x) \exp\left(\frac{-2\pi j 0 x}{4}\right) = \frac{1}{4} \sum_{x=0}^3 f(x) \cdot 1 \\
 &= \frac{1}{4} (f(0) + f(1) + f(2) + f(3)) = \frac{1}{4} (2 + 3 + 4 + 4) = 3.25
 \end{aligned}$$

$$\begin{aligned}
 F(1) &= \frac{1}{4} \sum_{x=0}^3 f(x) \exp\left(\frac{-2\pi j x}{4}\right) \\
 &= \frac{1}{4} (2 \exp(0) + 3 \exp(-j\pi/2) + 4 \exp(-j\pi) + 4 \exp(-3j\pi/2)) \\
 &= \frac{1}{4} (-2 + j)
 \end{aligned}$$

$$F(2) = \frac{1}{4} \sum_{x=0}^3 f(x) \exp\left(\frac{-4\pi j x}{4}\right) = \dots = -1/4$$

$$F(3) = \frac{1}{4} \sum_{x=0}^3 f(x) \exp\left(\frac{-6\pi j 0 x}{4}\right) = \dots = 1/4(-2 - j)$$

Spectre d'amplitude

$$|F(0)| = 3.25 \quad |F(1)| = \sqrt{5}/4 \quad |F(2)| = 1/4 \quad |F(3)| = \sqrt{5}/4$$

TF DISCRÈTE

TF DISCRÈTE 2D (1)

TFD 2D et TFD 2D inverse

$$F(u, \nu) = \frac{1}{MN} \sum_{x=0}^{N-1} \sum_{y=0}^{M-1} f(x, y) \exp\left(-2\pi j\left(\frac{ux}{N} + \frac{\nu y}{M}\right)\right)$$
$$u = 0, 1, 2, \dots, N - 1$$
$$\nu = 0, 1, 2, \dots, M - 1$$

$$f(x, y) = \sum_{u=0}^{N-1} \sum_{\nu=0}^{M-1} F(u, \nu) \exp\left(2\pi j\left(\frac{ux}{N} + \frac{\nu y}{M}\right)\right)$$
$$x = 0, 1, 2, \dots, N - 1$$
$$y = 0, 1, 2, \dots, M - 1$$

Le plus souvent, l'image à traiter est carrée $\blacktriangleleft \blacktriangleright M = N$

PROPRIÉTÉS

- Les mêmes que ceux énoncés pour la TF continue
- Cyclique (périodique)

$$F(u, \nu) = F(u + N, \nu) = F(u, \nu + N) = F(u + N, \nu + N)$$

$$f(x, y) = f(x + N, y + N)$$

car

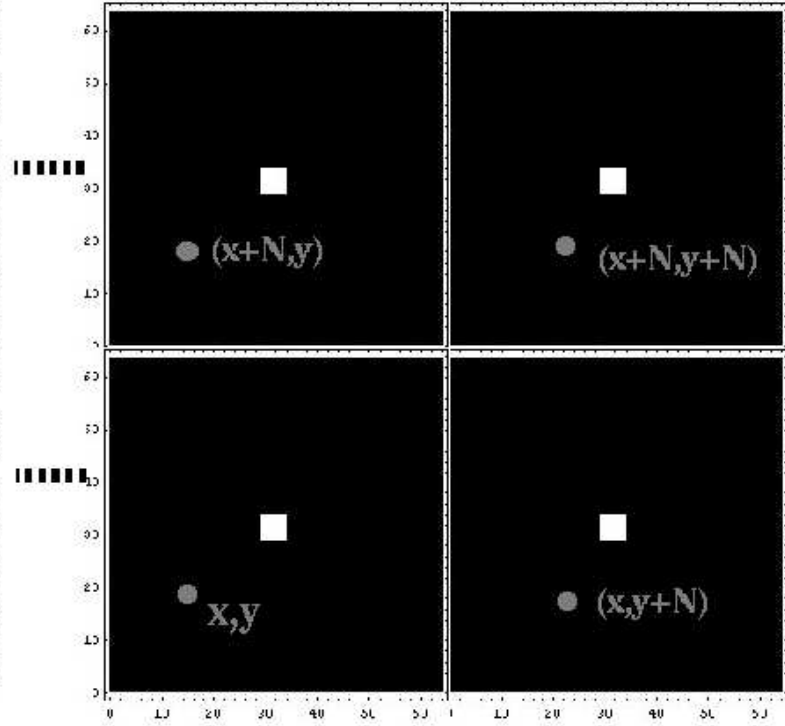
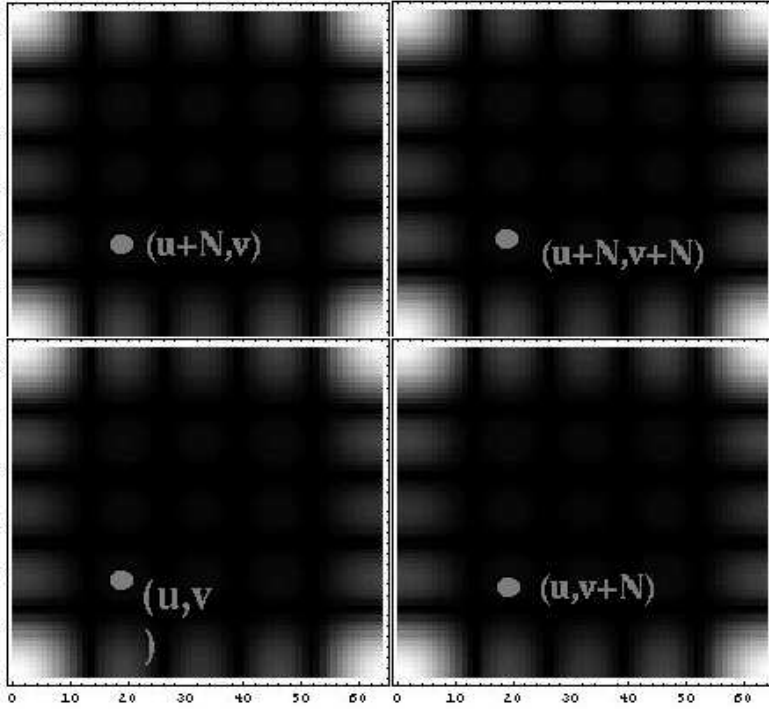
Signal périodique $-\mathcal{F} \rightarrow$ Spectre de raies

Signal échantillonné $-\mathcal{F} \rightarrow$ Spectre périodique

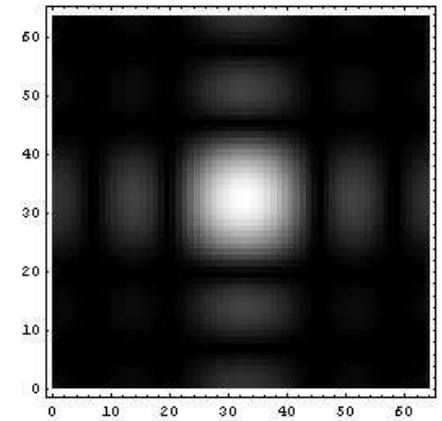
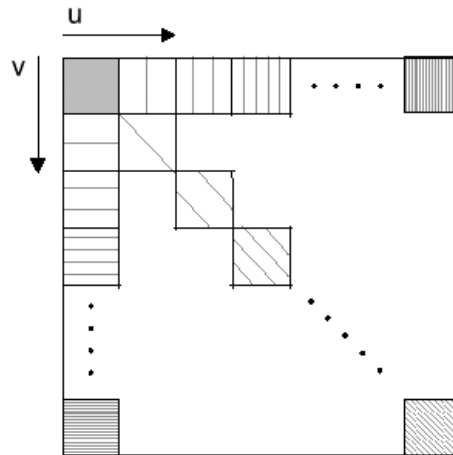
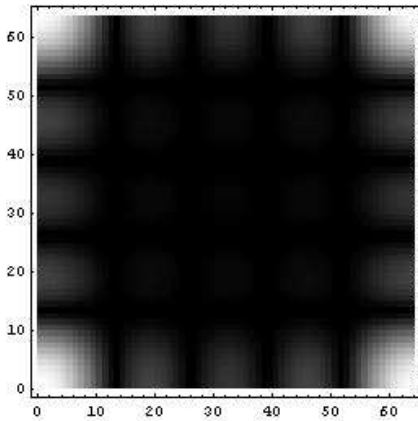
TF DISCRÈTE

TF DISCRÈTE 2D (4)

- Périodicité



- Nota



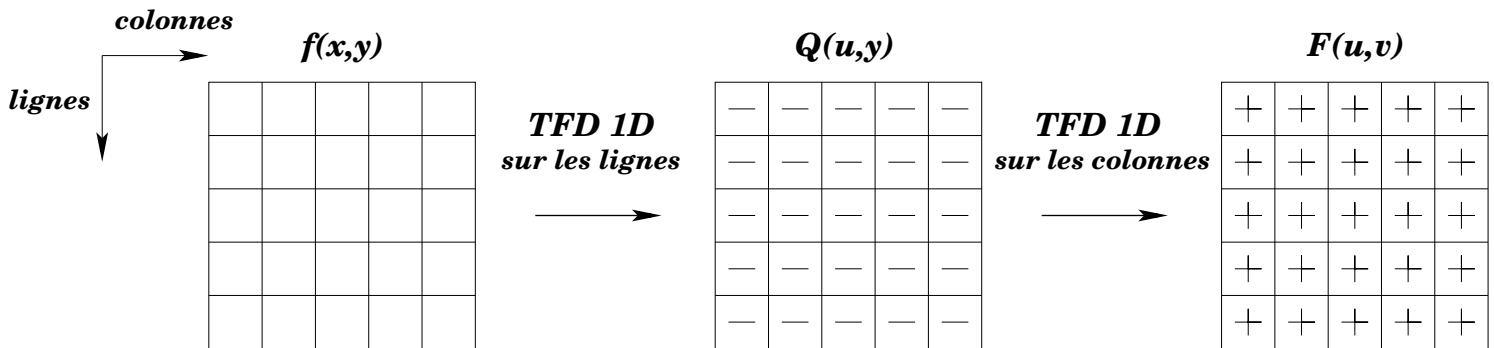
TF DISCRÈTE

TF DISCRÈTE 2D (5)

- Séparabilité

Pour une image carrée ◀▶ $M = N$

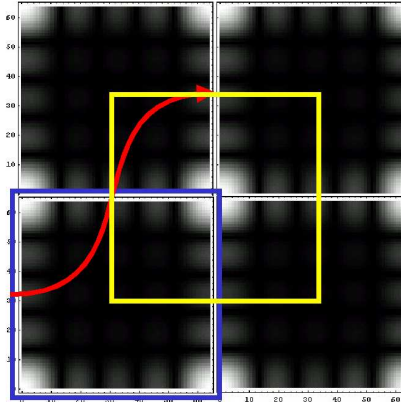
$$\begin{aligned}
 F(u, \nu) &= \frac{1}{N^2} \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} f(x, y) \exp \left(-2\pi j \left(\frac{ux + \nu y}{N} \right) \right) \\
 &= \frac{1}{N} \sum_{x=0}^{N-1} \left(\frac{1}{N} \sum_{y=0}^{N-1} f(x, y) \exp \left(-2\pi j \frac{\nu y}{N} \right) \right) \exp \left(-2\pi j \frac{ux}{N} \right) \\
 &= \frac{1}{N} \sum_{x=0}^{N-1} \underbrace{\left[\text{TF 1D de la } x \text{ ième colonne} \right]}_{\text{TF 1D de la } y \text{ ième ligne}} \exp \left(-2\pi j \frac{ux}{N} \right)
 \end{aligned}$$



TF DISCRÈTE

TF DISCRÈTE 2D (6)

Recalage Cyclique

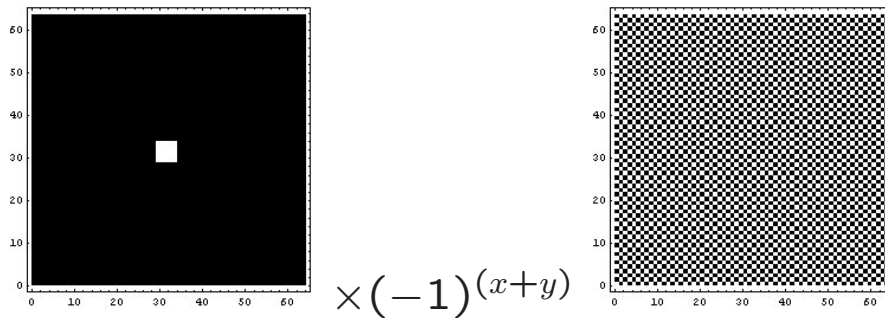


Translation -Rappel-

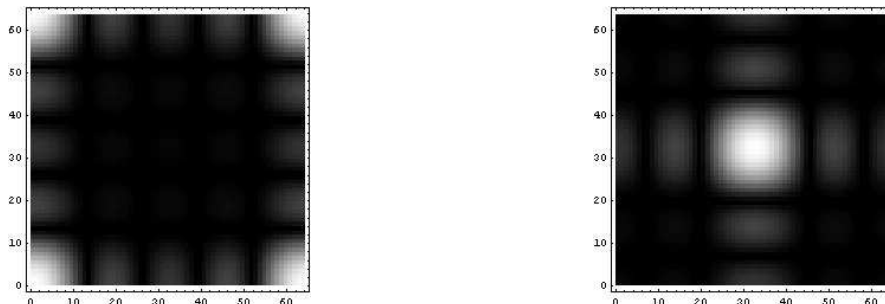
$$f(x - x_0, y - y_0) \quad \langle \mathcal{F} \rangle \quad F(u, \nu) \exp\left(\frac{-2\pi j(ux_0 + \nu y_0)}{N}\right)$$

$$f(x, y) \exp\left(\frac{2\pi j(u_0x + \nu_0y)}{N}\right) \quad \langle \mathcal{F} \rangle \quad F(u - u_0, \nu - \nu_0)$$

$$u_0 = \nu_0 = \frac{N}{2} \quad \blacktriangleright \quad \exp\left(\frac{2\pi j(u_0x + \nu_0y)}{N}\right) = \exp(\pi j(x + y)) = (-1)^{(x+y)}$$



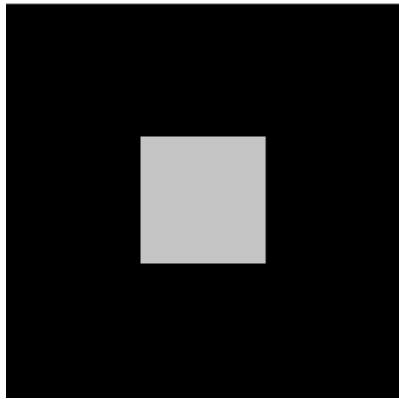
\mathcal{F}



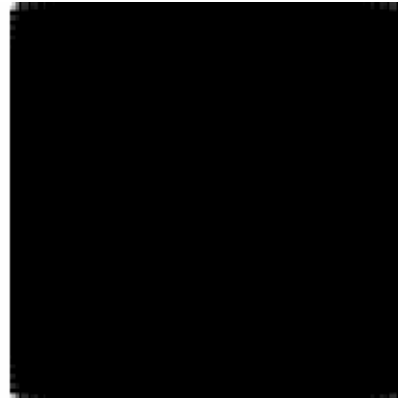
TF DISCRÈTE

TF DISCRÈTE 2D -VISUALISATION- (7)

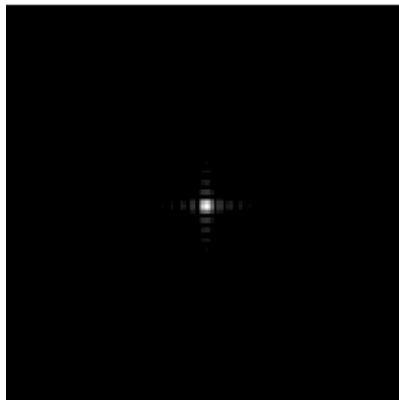
Image originale



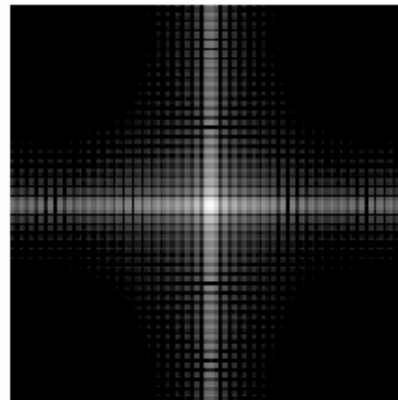
$|F(u, \nu)|$



$|F(0,0)|$ au centre



$k \log(1 + |F(u, \nu)|)$

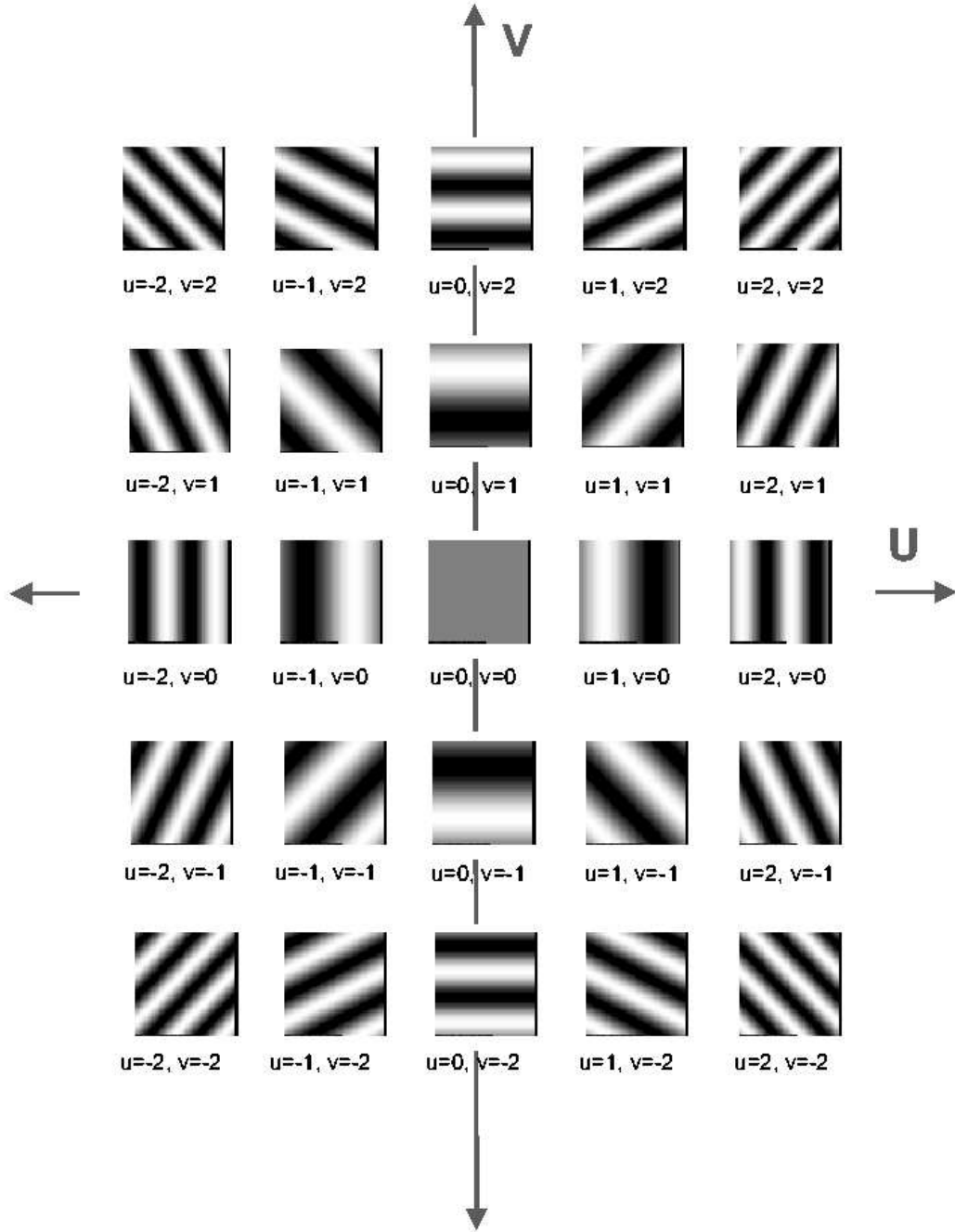
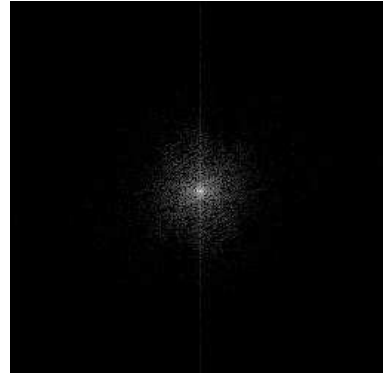


Nota

- On affiche généralement $|F(u, \nu)|$ ($F(u, \nu)$ complexe)
- Les fréquences élevées sont bcp plus faible que les fréquences plus basses. On affiche donc plutot $k \log(1 + |F(u, \nu)|)$ (k est une constante de normalisation pour recalcr les niveau de gris dans $[0, 255]$)
- On met l'origine au centre de l'image en effectuant un décalage cyclique

TF DISCRÈTE

TF DISCRÈTE 2D -INTERPRÉTATION- (8)

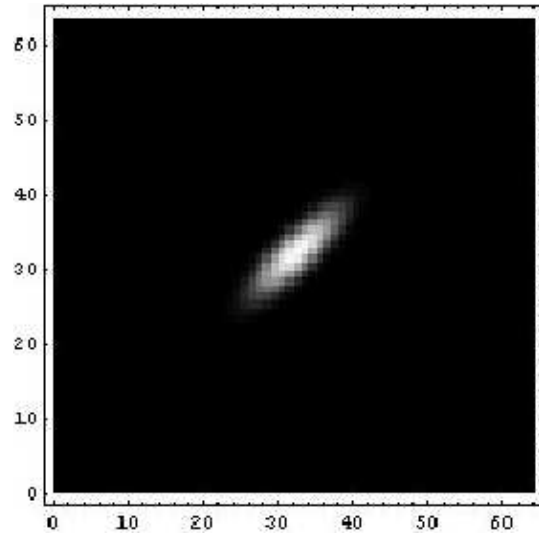
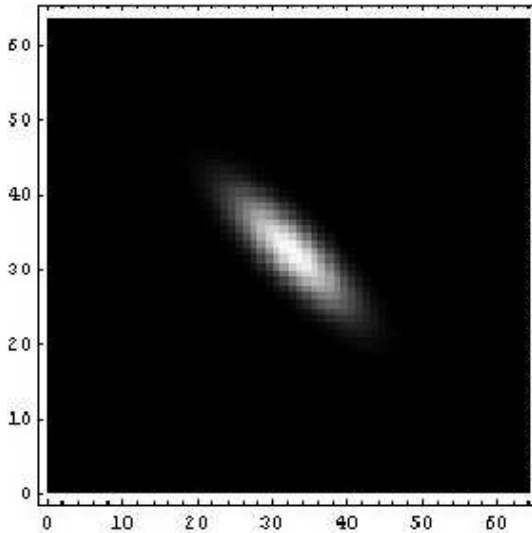
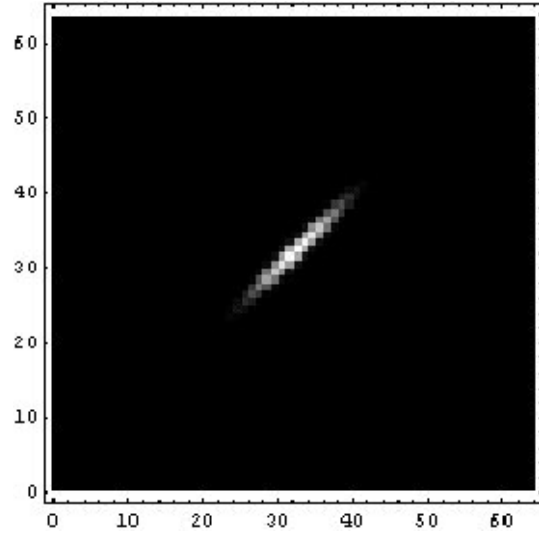
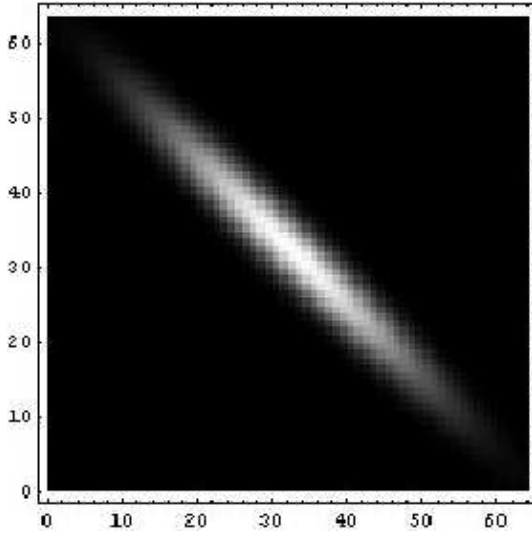
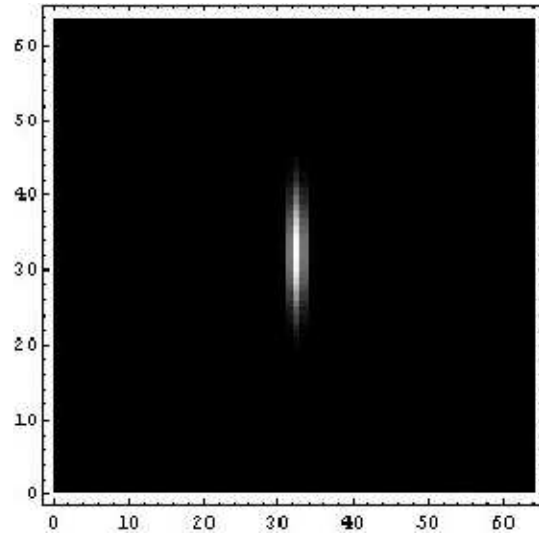
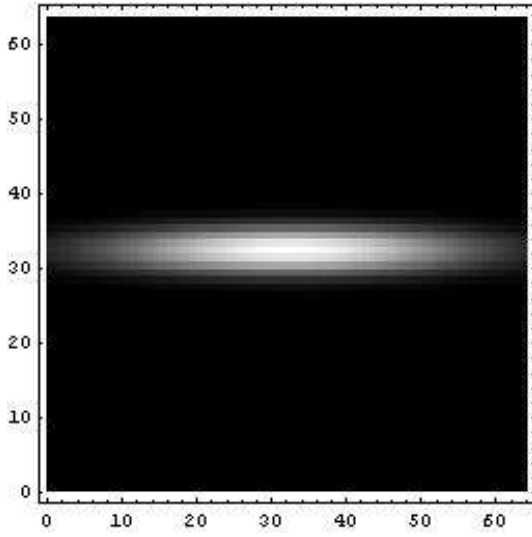


TF DISCRÈTE

TF DISCRÈTE 2D -EXEMPLES- (9)

Image

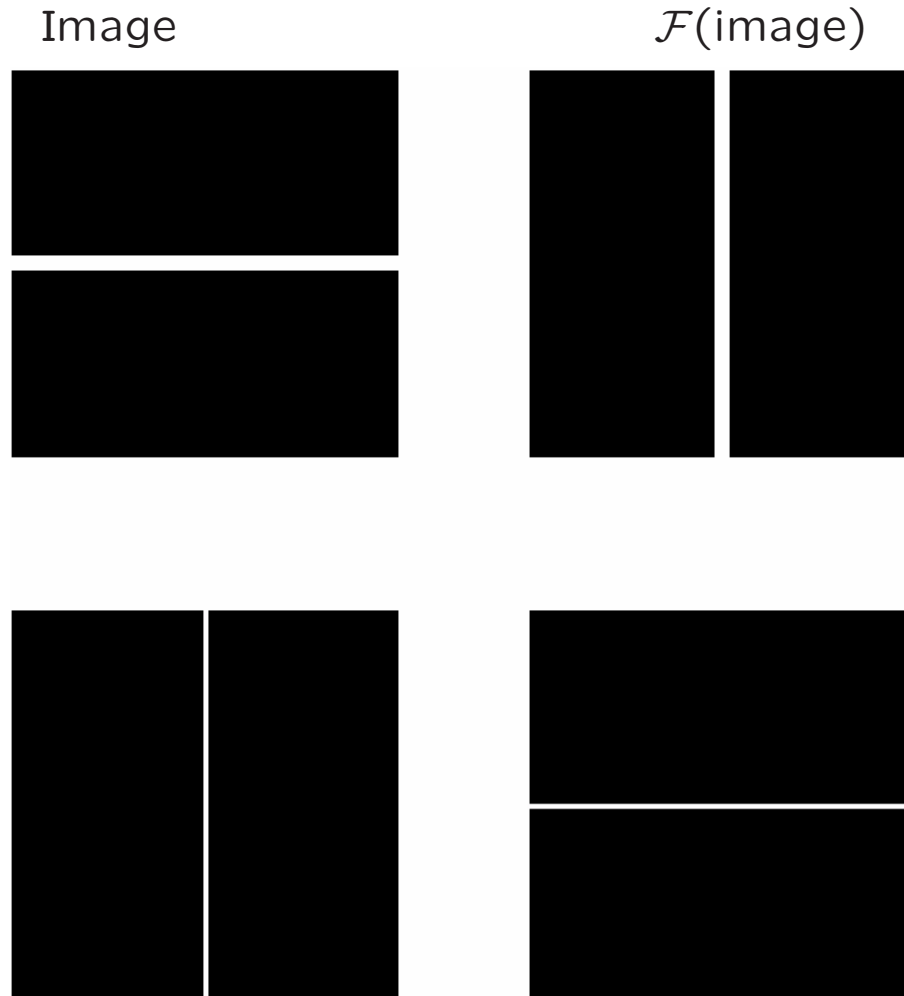
$\mathcal{F}(\text{image})$



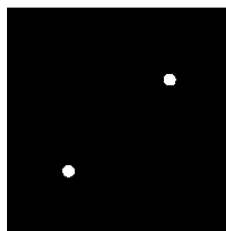
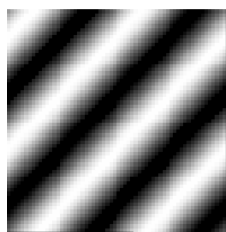
TF DISCRÈTE

TF DISCRÈTE 2D -EXEMPLES- (10)

- Rotation



- Périodicité

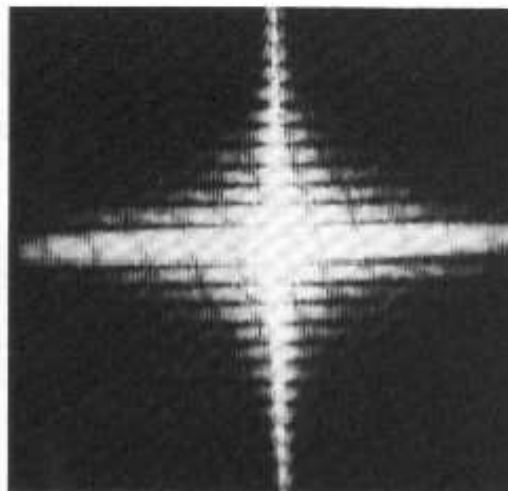
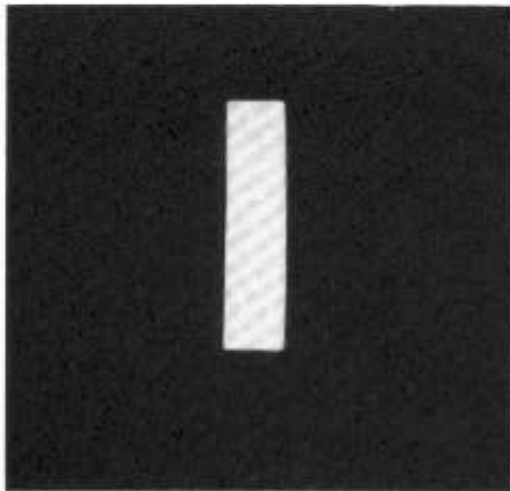
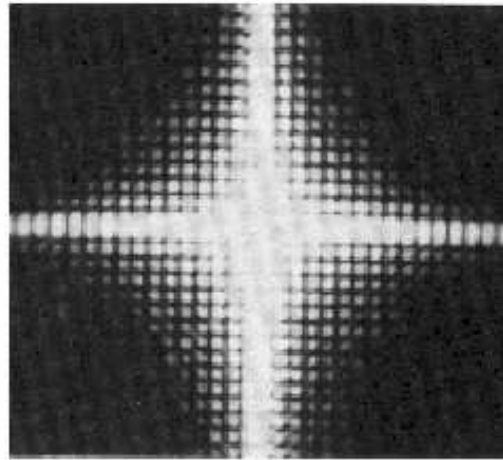
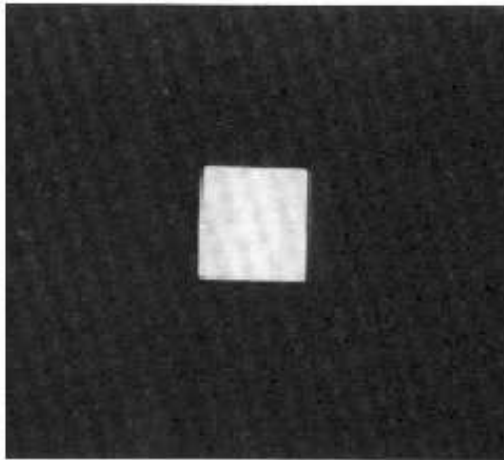


TF DISCRÈTE

TF DISCRÈTE 2D -EXEMPLES- (11)

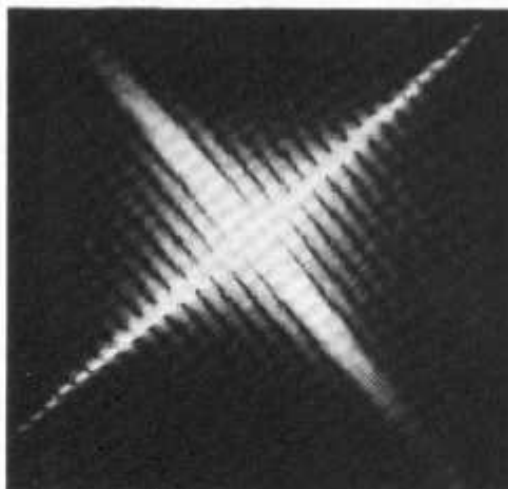
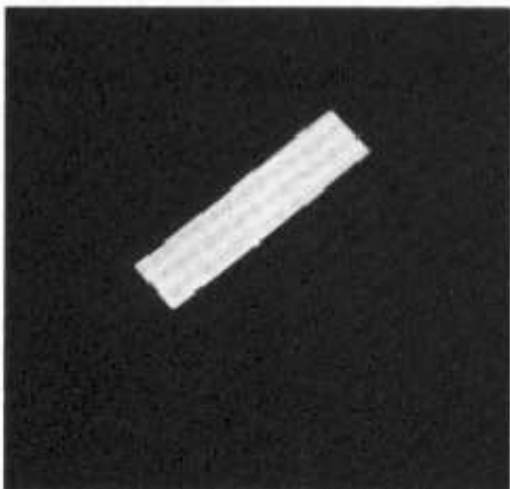
Image

$\mathcal{F}(\text{image})$



(a)

(b)

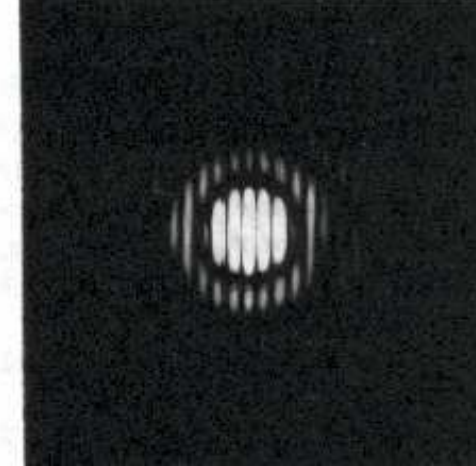
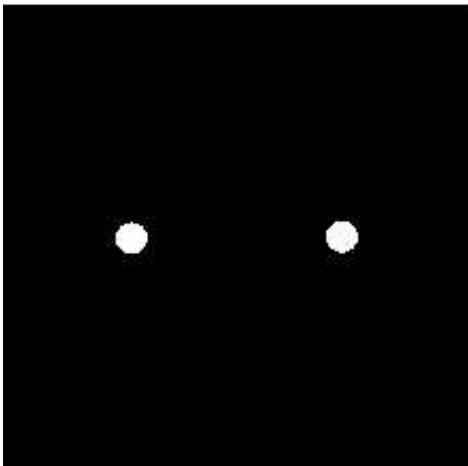
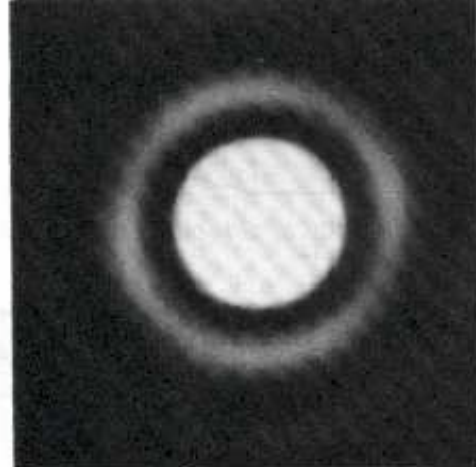
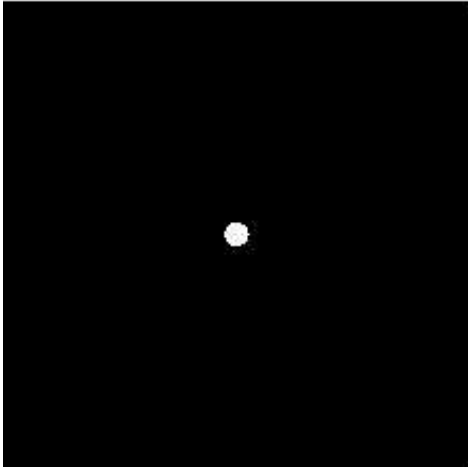


TF DISCRÈTE

TF DISCRÈTE 2D -EXEMPLES- (12)

Image

$\mathcal{F}(\text{image})$



TF DISCRÈTE

CONVOLUTION DISCRÈTE 1D & 2D (1)

Théorème de convolution 2D

$$\begin{aligned}
 f(x, y) * g(x, y) & \quad -\mathcal{F}- > \quad F(u, \nu) \cdot G(u, \nu) \\
 f(x, y) \cdot g(x, y) & \quad -\mathcal{F}- > \quad F(u, \nu) * G(u, \nu)
 \end{aligned}$$

Convolution 2D -Version continue-

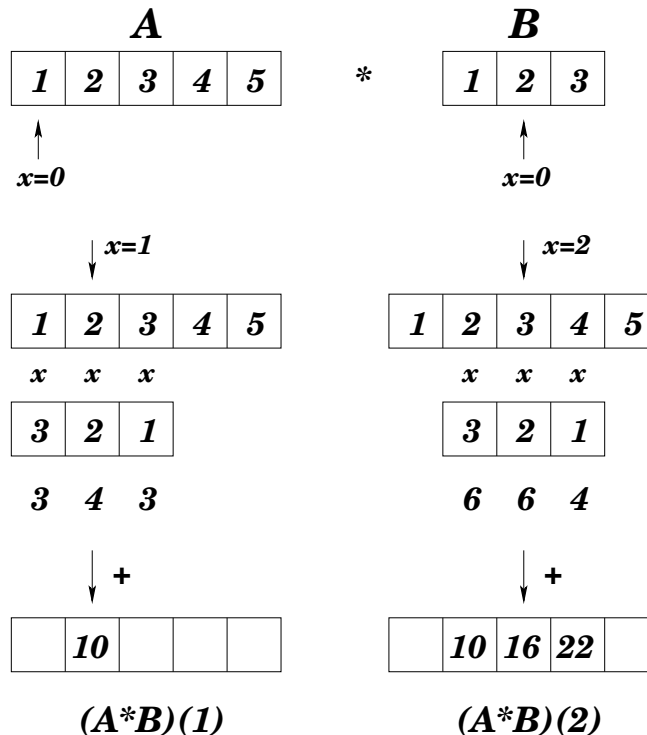
$$(f * g)(x, y) = \int_0^\infty \int_0^\infty f(\alpha, \beta) g(x - \alpha, y - \beta) d\alpha d\beta$$

Convolution 1D -Version discrète-



$$(A * B)(x) = \sum_i A(i) B(x - i)$$

-Exemple-



TF DISCRÈTE

CONVOLUTION DISCRÈTE 1D & 2D (2)

Convolution des bords

A) Ajout de zéro

0 0

1	2	3	4	5
---	---	---	---	---

 0 0

4	10	16	22	22
---	----	----	----	----

B) Enroulement

4 5

1	2	3	4	5
---	---	---	---	---

 1 2

19	10	16	22	23
----	----	----	----	----

C) Réflexion

3 2

1	2	3	4	5
---	---	---	---	---

 4 3

10	10	16	22	26
----	----	----	----	----

D) Défaut

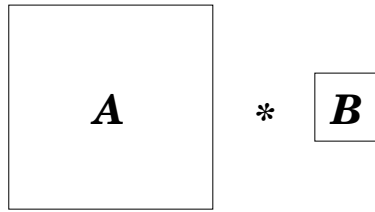
1	2	3	4	5
---	---	---	---	---

⊗	10	16	22	⊗
---	----	----	----	---

TF DISCRÈTE

CONVOLUTION DISCRÈTE 1D & 2D (3)

Convolution 2D -Version discrète-



$$(A * B)(x, y) = \sum_i \sum_j A(i, j) B(x - i, y - j)$$

-Exemple-

10	5	20	20	20
10	5	20	20	20
10	5	20	20	20
10	5	20	20	20
10	5	20	20	20

*

-1	1
0	1

-10	5	-15	0	0
-10	15	-10	20	20
-10	15	-10	20	20
-10	15	-10	20	20
-10	15	-10	20	20

(Ajout de zéro)

-1	1
0	1

1	-1
1	0

1	0
1	-1

Original

x reflexion

y reflexion

PROPRIÉTÉS

- Commutatif ▶ $f_1 * f_2 * f = f_2 * f_1 * f$
- Associatif ▶ $(f_1 * f_2) * f = f_1 * (f_2 * f)$
- Distributif ▶ $(f_1 + f_2) * f = f_1 * f + f_2 * f$
 ▶ $f * (f_1 + f_2) = f * f_1 + f * f_2$

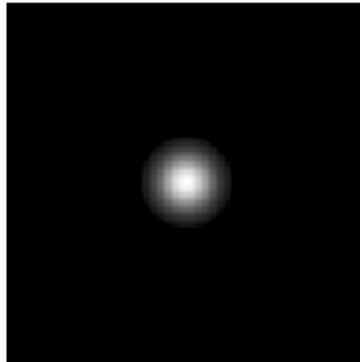
TF DISCRÈTE

CONVOLUTION DISCRÈTE 1D & 2D (4)

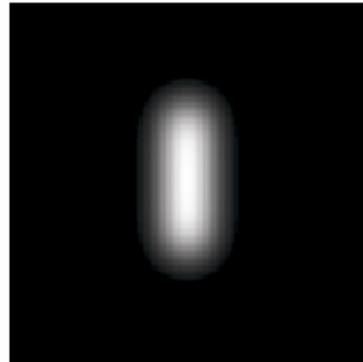
$f(x,y)$



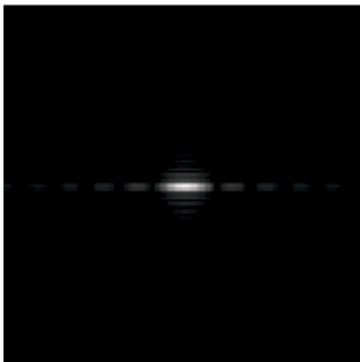
$g(x,y)$



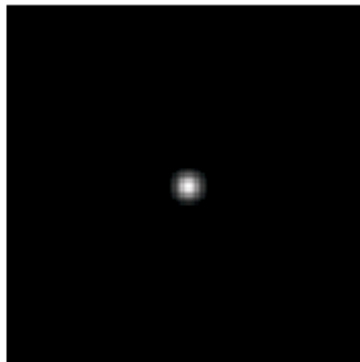
$f * g(x,y)$



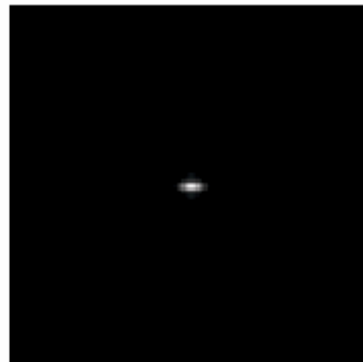
$F(u,v)$



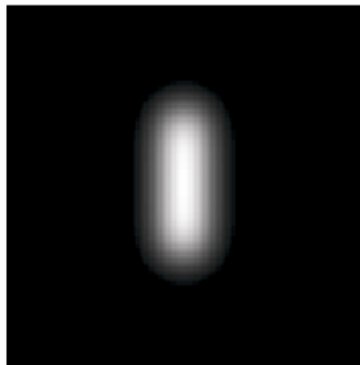
$G(u,v)$



$F(u,v) \cdot G(u,v)$

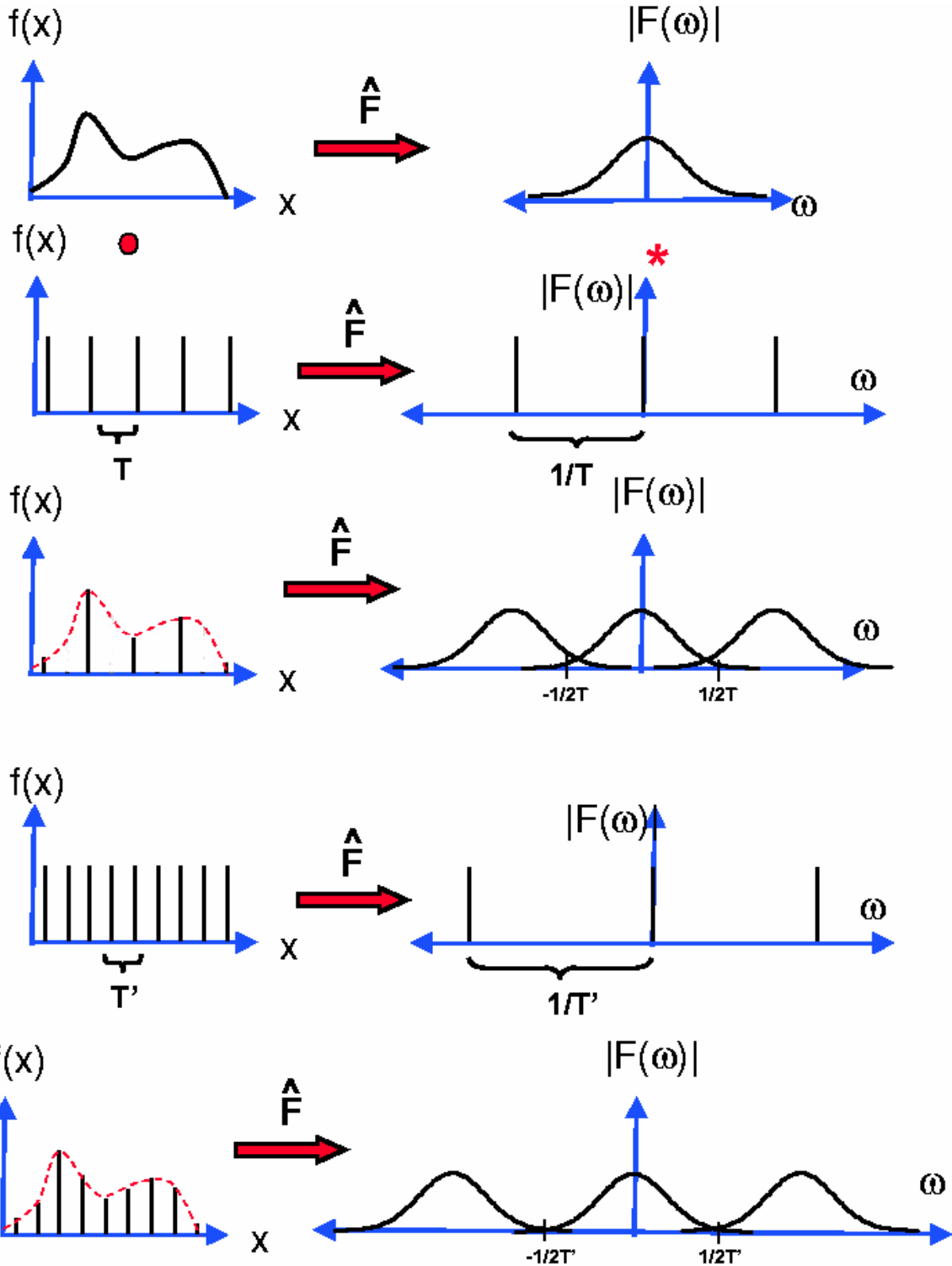


$\mathcal{F}^{-1}[F(u,v) \cdot G(u,v)]$



TF DISCRÈTE

THÉORÈME DE SCHANNON

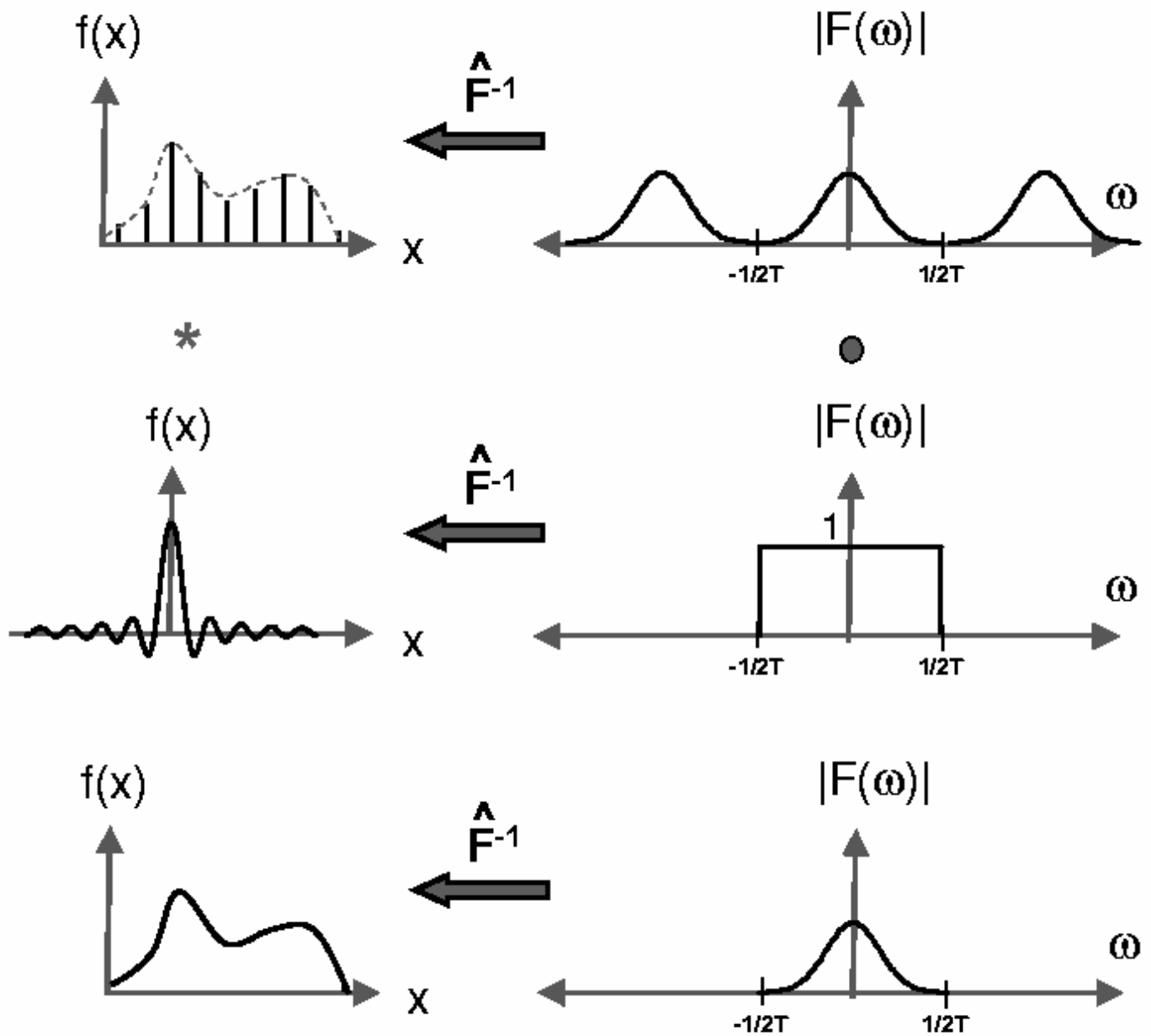


$$T_e \leq \frac{1}{2\nu_0}$$

► sinon phénomène d'aliasing

TF DISCRÈTE

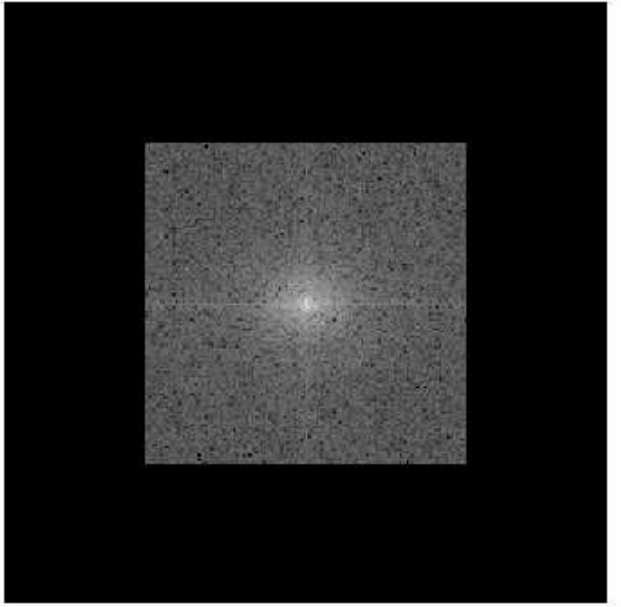
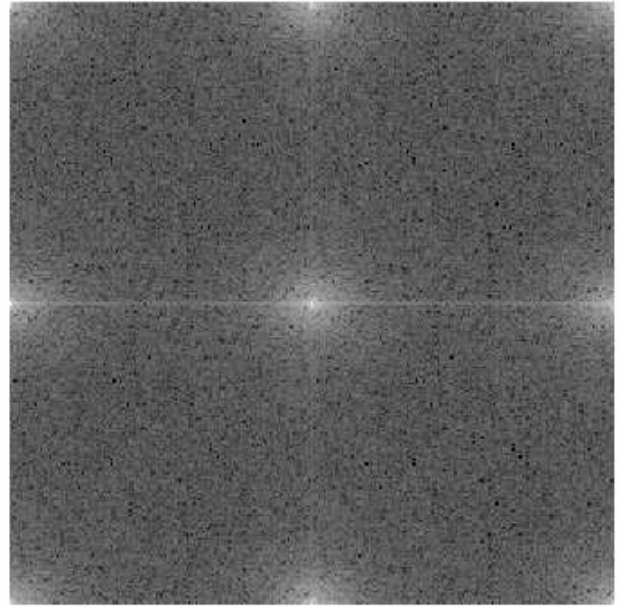
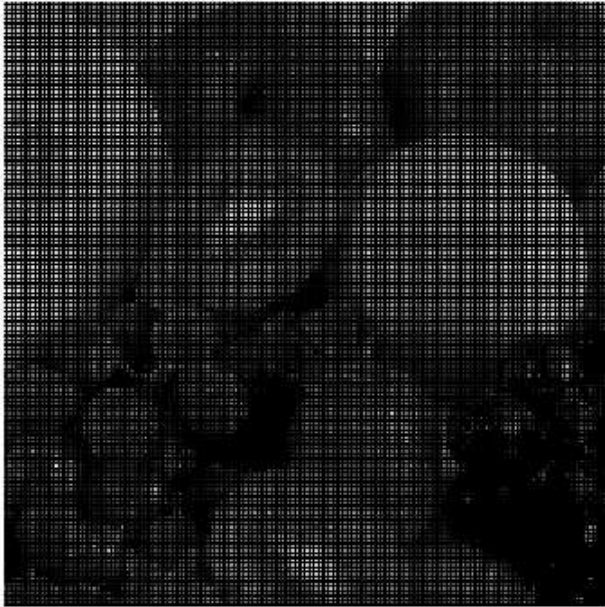
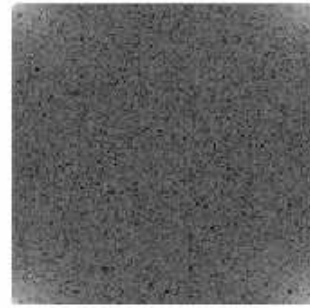
INTERPOLATION OPTIMALE (1)



► Possibilité de reconstruire totalement le signal 2D à partir de ses échantillons

TF DISCRÈTE

INTERPOLATION OPTIMALE (2)



TF DISCRÈTE

FFT (1)

$$F(u) = \frac{1}{N} \sum_{x=0}^{N-1} f(x) \exp\left(\frac{-2\pi j u x}{N}\right) \quad u = 0, 1, 2, \dots, N-1$$

$O(N^2)$ opérations

$$F(u) = \underbrace{\frac{1}{N} \sum_{x=0}^{N/2-1} f(2x) \exp\left(\frac{-2\pi j u 2x}{N}\right)}_{x \text{ pair}} + \underbrace{\frac{1}{N} \sum_{x=0}^{N/2-1} f(2x+1) \exp\left(\frac{-2\pi j u (2x+1)}{N}\right)}_{x \text{ impair}}$$

$$= \frac{1}{2} \left[\underbrace{\frac{1}{N/2} \sum_{x=0}^{N/2-1} f(2x) \exp\left(\frac{-2\pi j u x}{N/2}\right)}_{\text{TF de la partie paire}} + \exp\left(\frac{-2\pi j u}{N}\right) \underbrace{\frac{1}{N/2} \sum_{x=0}^{N/2-1} f(2x+1) \exp\left(\frac{-2\pi j u x}{N/2}\right)}_{\text{TF de la partie impaire}} \right]$$

- TF [N éléments] = 2 TF [$N/2$ éléments] + $N/2$ multiplications et $N/2$ additions complexes (en fait $N/4$ pour les multiplications)
- $Op(N) = 2Op(N/2) + N$

$$F_N(u) = \frac{1}{2} \left[F_{N/2}^{\text{paire}}(u) + \exp\left(-\frac{2\pi j u}{N}\right) F_{N/2}^{\text{impaire}}(u) \right]$$

Pour $u' = u + \frac{N}{2}$

$$\exp\left(-\frac{2\pi j u'}{N}\right) = \exp\left(-\frac{2\pi j (u + \frac{N}{2})}{N}\right) = \exp\left(-\frac{2\pi j u}{N}\right) \exp(-\pi j) = -\exp\left(-\frac{2\pi j u}{N}\right)$$

TF DISCRÈTE

FFT (2)

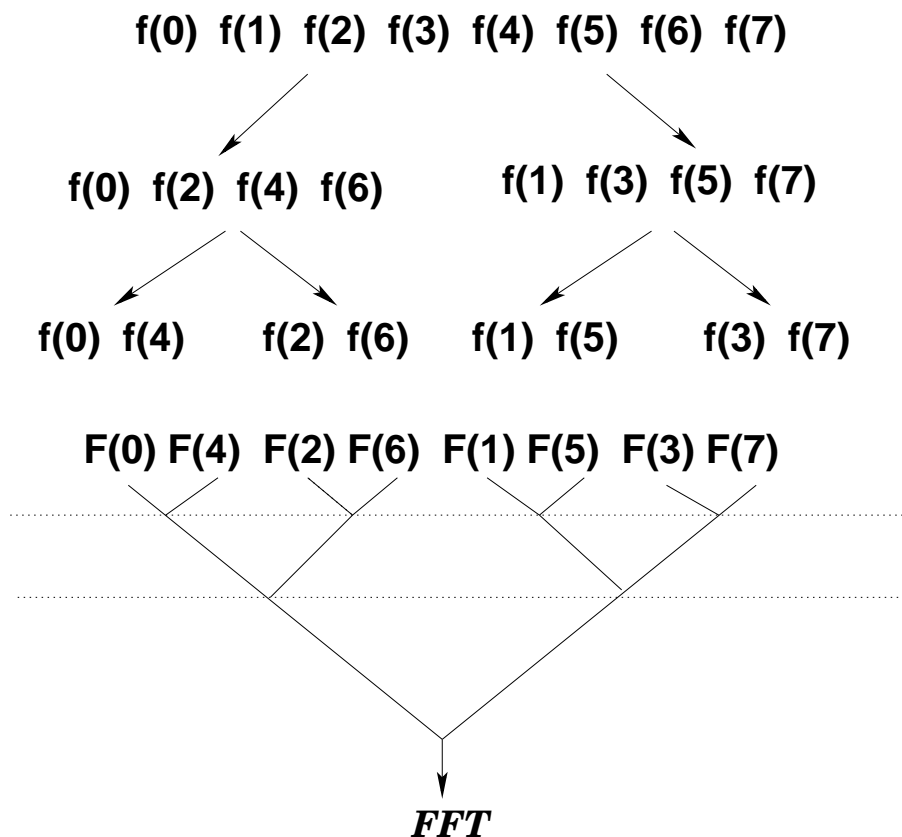
Ainsi

$$F_N(u) = \frac{1}{2} \left[F_{N/2}^{\text{paire}}(u) + \exp\left(-\frac{2\pi ju}{N}\right) F_{N/2}^{\text{impaire}}(u) \right]$$

$$F_N\left(u + \frac{N}{2}\right) = \frac{1}{2} \left[F_{N/2}^{\text{paire}}(u) - \exp\left(-\frac{2\pi ju}{N}\right) F_{N/2}^{\text{impaire}}(u) \right]$$

Pour $u = 0, 1, 2, \dots, N/2 - 1$

1 multiplication est nécessaire pour deux termes



FFT 1D ► $O(N \log N)$
 FFT 2D ► $O(N^2 \log N)$

TF DISCRÈTE

FFT (3)

Vérifions que $O(N \log N)$

(Par induction, sachant que $Op(N) = 2 Op(N/2) + N$)

- Vrai pour $N = 1$? $\blacktriangleright Op(1) = 1 \log 1 = 0 \blacktriangleright$ oui
- Vrai pour $N = 2$? $\blacktriangleright Op(2) = 2 \log 2 = 2 \blacktriangleright$ oui
- Vrai pour $N \log N$?
 - $Op(N) = N \log N$?
 - $Op(N) = 2N/2 \log(2N/2)$
 - $Op(N) = 2N/2(\log(N/2) + \log 2)$
 - $Op(N) = 2N/2 \log N/2 + 2N/2 \log 2$
 - $Op(N) = 2 Op(N/2) + N$

C.Q.F.D.

Rehaussement d'image

