



DIRO  
IFT 6150

# TRAITEMENT D'IMAGES TRANSFORMÉE DE FOURIER DISCRÈTE

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# TF DISCRÈTE

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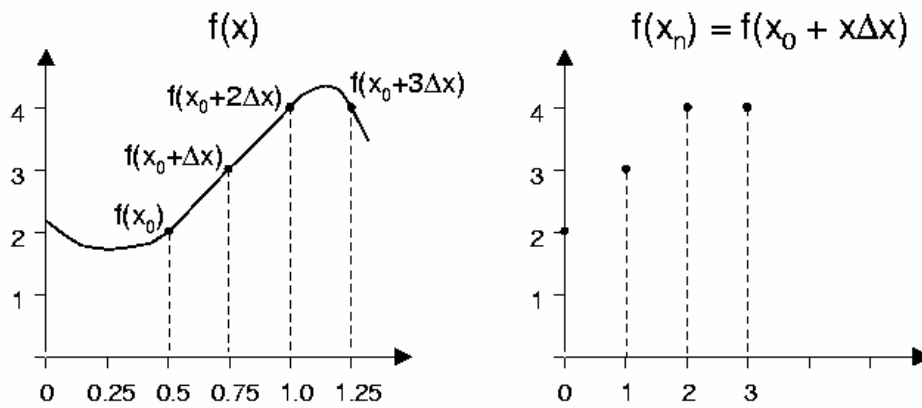
# TF DISCRÈTE

## TF DISCRÈTE 1D (1)

### CAS CONTINU (RAPPEL)

$$\mathcal{F}[f(x)] = F(u) = \int_{-\infty}^{+\infty} f(x) \exp(-2\pi j u x) dx \quad u \in \mathbb{R}.$$

### CAS DISCRET



$$f(x) = f(x_0 + x\Delta x) \quad x = 0, 1, 2, \dots, N-1$$

### TFD et TFD inverse

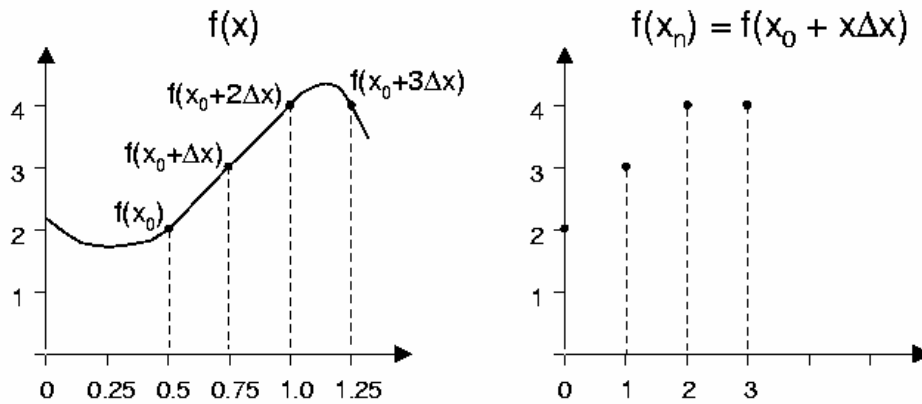
$$F(u) = \frac{1}{N} \sum_{x=0}^{N-1} f(x) \exp\left(\frac{-2\pi j u x}{N}\right) \quad u = 0, 1, 2, \dots, N-1$$

$$f(x) = \sum_{u=0}^{N-1} F(u) \exp\left(\frac{2\pi j u x}{N}\right) \quad x = 0, 1, 2, \dots, N-1$$

$$\Delta u = \frac{1}{N\Delta x}$$

# TF DISCRÈTE

## TF DISCRÈTE 1D -EXEMPLE- (2)



$$\begin{aligned}
 F(0) &= \frac{1}{4} \sum_{x=0}^3 f(x) \exp\left(\frac{-2\pi j 0 x}{4}\right) = \frac{1}{4} \sum_{x=0}^3 f(x) \cdot 1 \\
 &= \frac{1}{4} (f(0) + f(1) + f(2) + f(3)) = \frac{1}{4} (2 + 3 + 4 + 4) = 3.25
 \end{aligned}$$

$$\begin{aligned}
 F(1) &= \frac{1}{4} \sum_{x=0}^3 f(x) \exp\left(\frac{-2\pi j x}{4}\right) \\
 &= \frac{1}{4} (2 \exp(0) + 3 \exp(-j\pi/2) + 4 \exp(-j\pi) + 4 \exp(-3j\pi/2)) \\
 &= \frac{1}{4} (-2 + j)
 \end{aligned}$$

$$F(2) = \frac{1}{4} \sum_{x=0}^3 f(x) \exp\left(\frac{-4\pi j x}{4}\right) = \dots = -1/4$$

$$F(3) = \frac{1}{4} \sum_{x=0}^3 f(x) \exp\left(\frac{-6\pi j 0 x}{4}\right) = \dots = 1/4(-2 - j)$$

### Spectre d'amplitude

$$|F(0)| = 3.25 \quad |F(1)| = \sqrt{5}/4 \quad |F(2)| = 1/4 \quad |F(3)| = \sqrt{5}/4$$

# TF DISCRÈTE

## TF DISCRÈTE 2D (1)

### TFD 2D et TFD 2D inverse

$$F(u, \nu) = \frac{1}{MN} \sum_{x=0}^{N-1} \sum_{y=0}^{M-1} f(x, y) \exp\left(-2\pi j\left(\frac{ux}{N} + \frac{\nu y}{M}\right)\right)$$
$$u = 0, 1, 2, \dots, N - 1$$
$$\nu = 0, 1, 2, \dots, M - 1$$

$$f(x, y) = \sum_{u=0}^{N-1} \sum_{\nu=0}^{M-1} F(u, \nu) \exp\left(2\pi j\left(\frac{ux}{N} + \frac{\nu y}{M}\right)\right)$$
$$x = 0, 1, 2, \dots, N - 1$$
$$y = 0, 1, 2, \dots, M - 1$$

Le plus souvent, l'image à traiter est carrée  $\blacktriangleleft M = N$

### PROPRIÉTÉS

- Les mêmes que ceux énoncés pour la TF continue
- Cyclique (périodique)

$$F(u, \nu) = F(u + N, \nu) = F(u, \nu + N) = F(u + N, \nu + N)$$

$$f(x, y) = f(x + N, y + N)$$

car

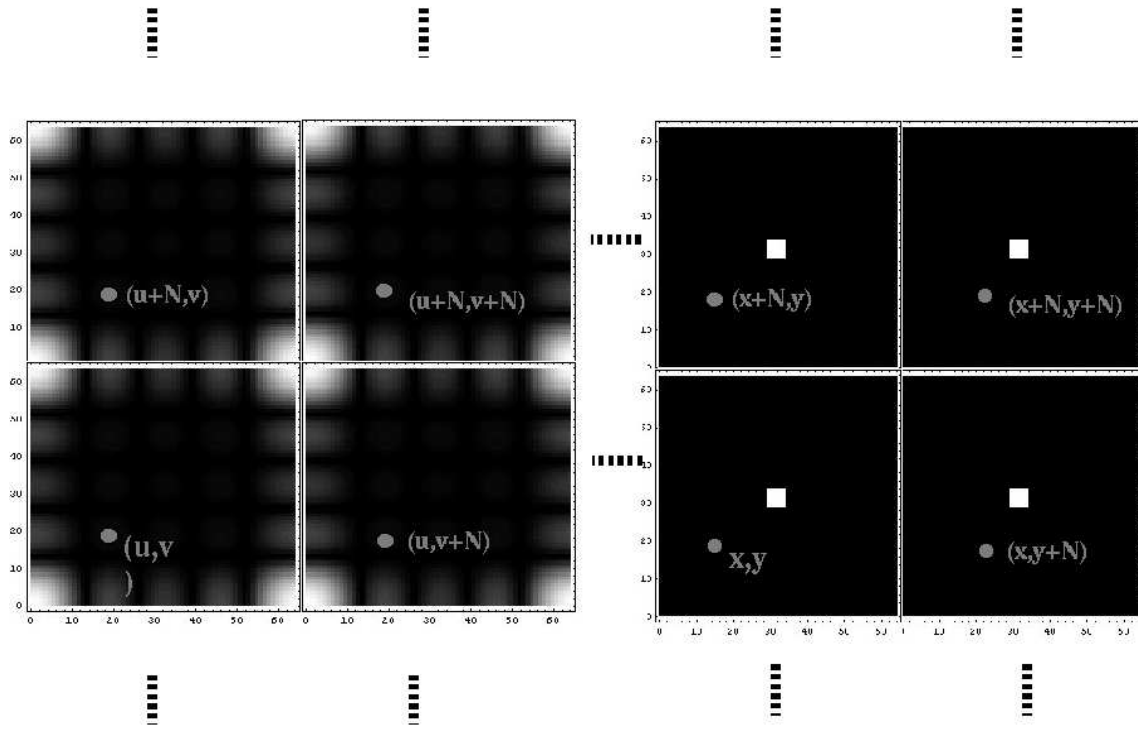
Signal périodique  $-\mathcal{F} \rightarrow$  Spectre de raies

Signal échantillonné  $-\mathcal{F} \rightarrow$  Spectre périodique

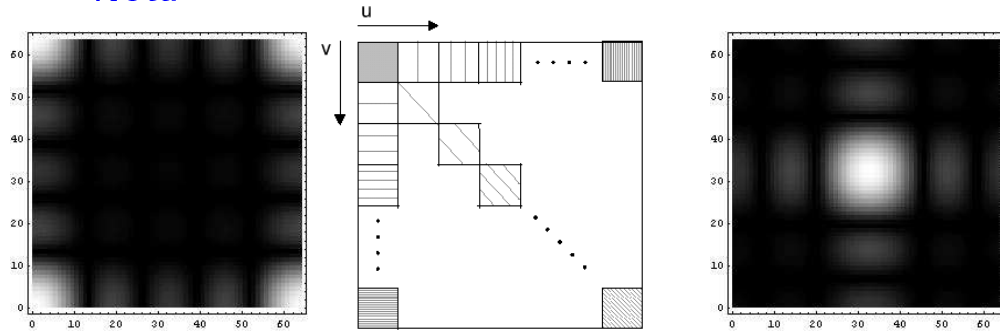
# TF DISCRÈTE

## TF DISCRÈTE 2D (4)

- Périodicité



- Nota



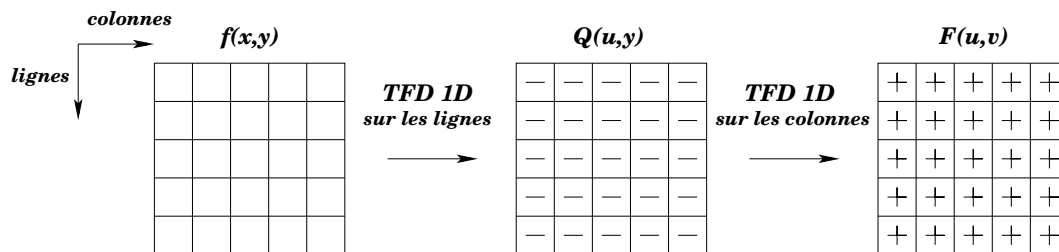
# TF DISCRÈTE

## TF DISCRÈTE 2D (5)

- Séparabilité

Pour une image carrée  $\blacktriangleleft M = N$

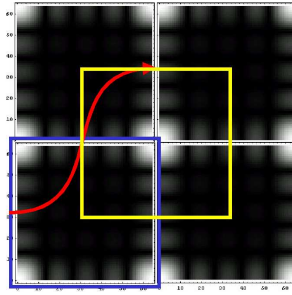
$$\begin{aligned}
 F(u, \nu) &= \frac{1}{N^2} \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} f(x, y) \exp\left(-2\pi j \left(\frac{ux + \nu y}{N}\right)\right) \\
 &= \frac{1}{N} \sum_{x=0}^{N-1} \left( \frac{1}{N} \sum_{y=0}^{N-1} f(x, y) \exp\left(-2\pi j \frac{\nu y}{N}\right) \right) \exp\left(-2\pi j \frac{ux}{N}\right) \\
 &= \frac{1}{N} \sum_{x=0}^{N-1} \underbrace{\left[ \text{TF 1D de la } x \text{ ième colonne} \right]}_{\text{TF 1D de la } y \text{ ième ligne}} \exp\left(-2\pi j \frac{ux}{N}\right)
 \end{aligned}$$



# TF DISCRÈTE

## TF DISCRÈTE 2D (6)

### Recalage Cyclique

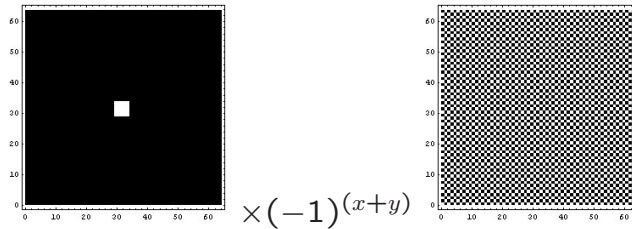


### Translation -Rappel-

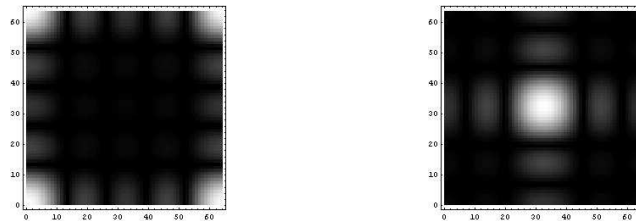
$$f(x - x_0, y - y_0) \quad \left\langle \mathcal{F} \right\rangle \quad F(u, \nu) \exp\left(\frac{-2\pi j(ux_0 + \nu y_0)}{N}\right)$$

$$f(x, y) \exp\left(\frac{2\pi j(u_0x + \nu_0y)}{N}\right) \quad \left\langle \mathcal{F} \right\rangle \quad F(u - u_0, \nu - \nu_0)$$

$$u_0 = \nu_0 = \frac{N}{2} \quad \blacktriangleright \quad \exp\left(\frac{2\pi j(u_0x + \nu_0y)}{N}\right) = \exp(\pi j(x + y)) = (-1)^{(x+y)}$$



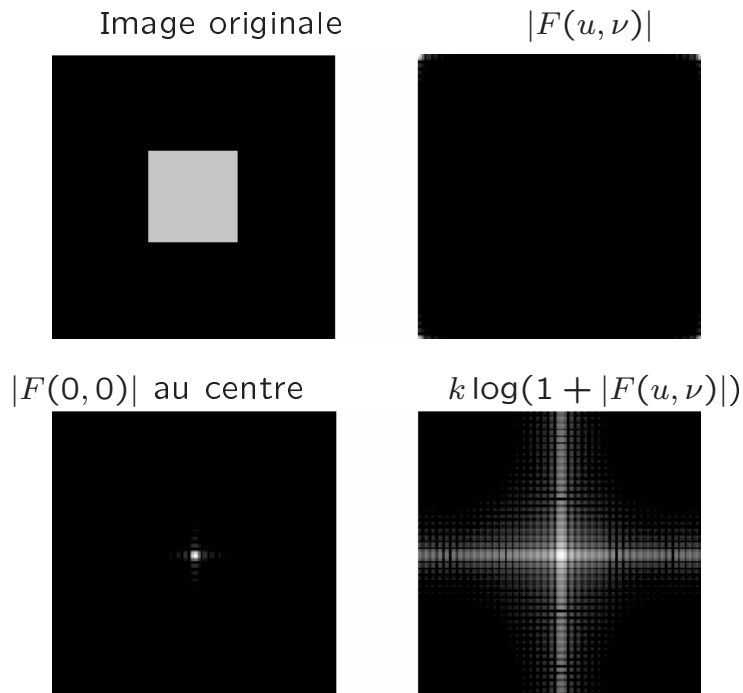
$\mathcal{F}$





# TF DISCRÈTE

## TF DISCRÈTE 2D -VISUALISATION- (7)

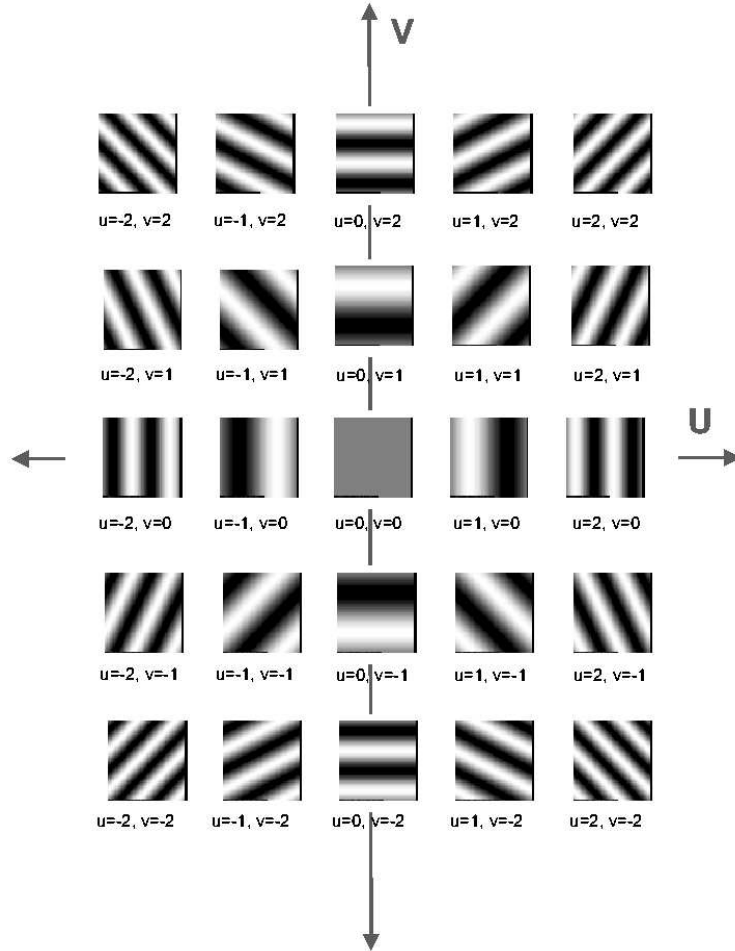


### Nota

- On affiche généralement  $|F(u, \nu)|$  ( $F(u, \nu)$  complexe)
- Les fréquences élevées sont bcp plus faible que les fréquences plus basses. On affiche donc plutôt  $k \log(1 + |F(u, \nu)|)$  ( $k$  est une constante de normalisation pour recalculer les niveaux de gris dans  $[0, 255]$ )
- On met l'origine au centre de l'image en effectuant un décalage cyclique

# TF DISCRÈTE

## TF DISCRÈTE 2D -INTERPRÉTATION- (8)

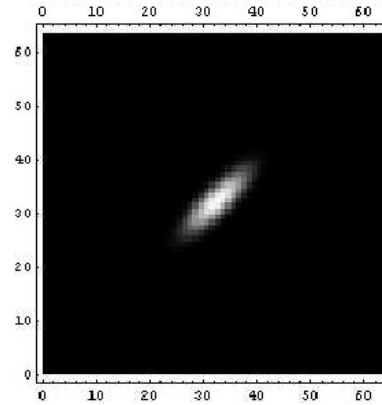
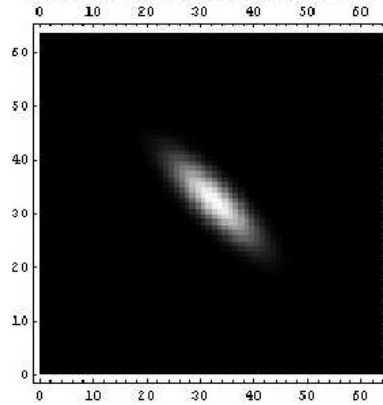
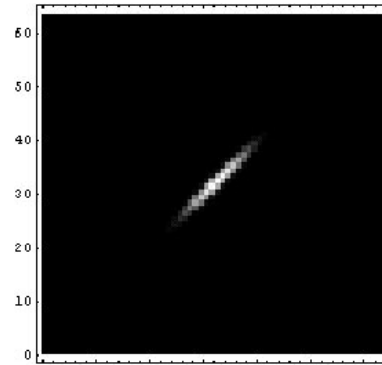
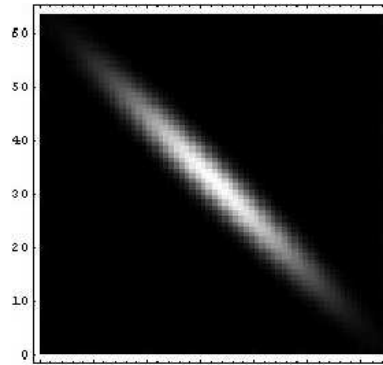
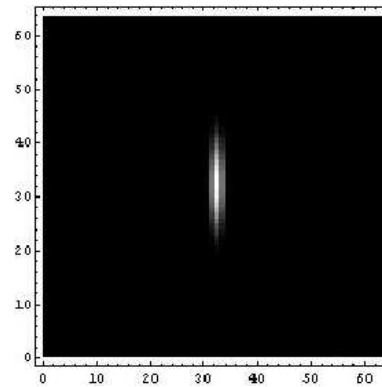
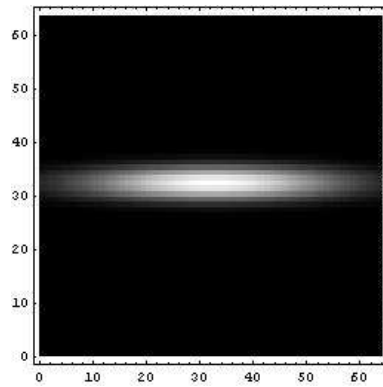


# TF DISCRÈTE

## TF DISCRÈTE 2D -EXEMPLES- (9)

Image

$\mathcal{F}(\text{image})$

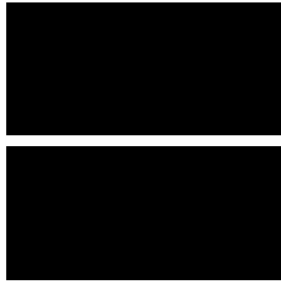


# TF DISCRÈTE

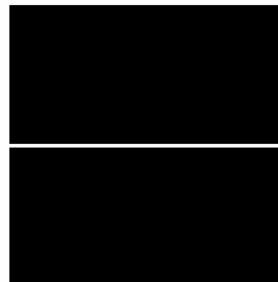
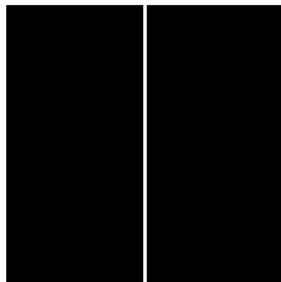
## TF DISCRÈTE 2D -EXEMPLES- (10)

- Rotation

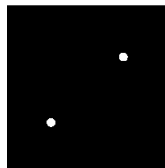
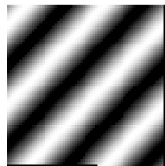
Image



$\mathcal{F}(\text{image})$



- Périodicité

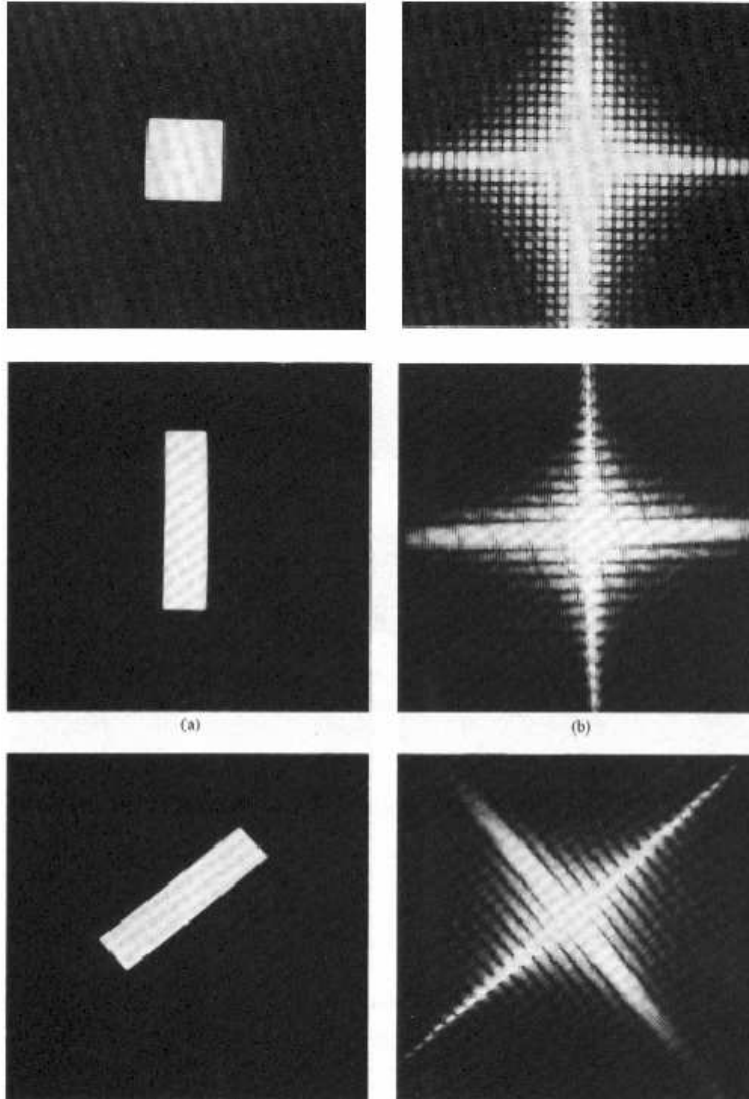


# TF DISCRÈTE

## TF DISCRÈTE 2D -EXEMPLES- (11)

Image

$\mathcal{F}(\text{image})$

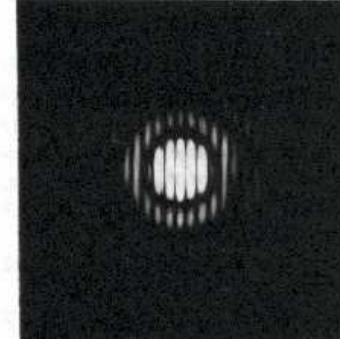
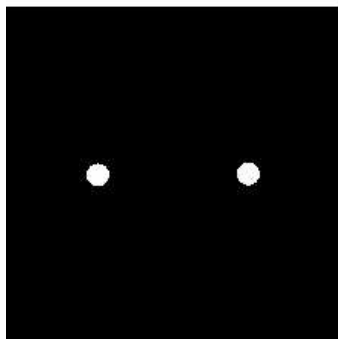
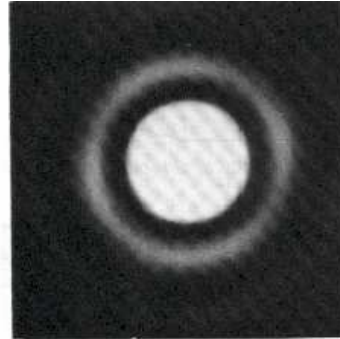
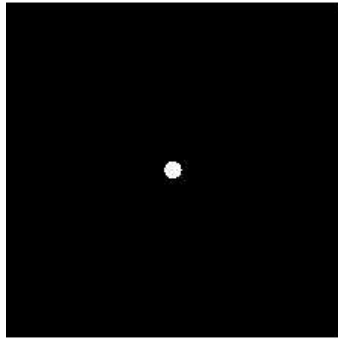


# TF DISCRÈTE

TF DISCRÈTE 2D -EXEMPLES- (12)

Image

$\mathcal{F}(\text{image})$



# TF DISCRÈTE

## CONVOLUTION DISCRÈTE 1D & 2D (1)

### Théorème de convolution 2D

$$f(x, y) * g(x, y) \xrightarrow{-\mathcal{F}-} F(u, \nu) \cdot G(u, \nu)$$

$$f(x, y) \cdot g(x, y) \xrightarrow{-\mathcal{F}-} F(u, \nu) * G(u, \nu)$$

### Convolution 2D -Version continue-

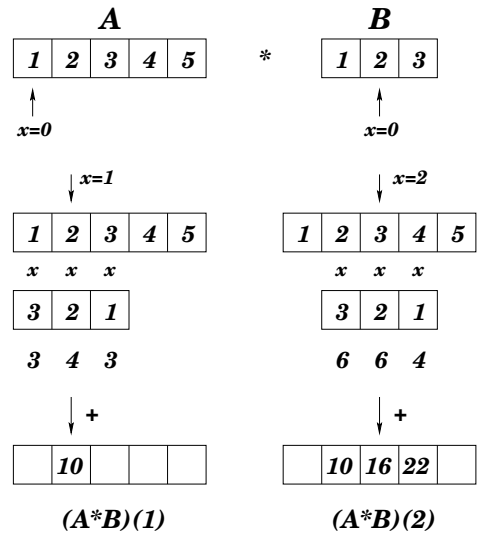
$$(f * g)(x, y) = \int_0^\infty \int_0^\infty f(\alpha, \beta) g(x - \alpha, y - \beta) d\alpha d\beta$$

### Convolution 1D -Version discrète-



$$(A * B)(x) = \sum_i A(i) B(x - i)$$

-Exemple-



**TF DISCRÈTE**  
**CONVOLUTION DISCRÈTE 1D & 2D (2)**

**Convolution des bords**

***A) Ajout de zéro***

0 0 

1	2	3	4	5
---	---	---	---	---

 0 0

4	10	16	22	22
---	----	----	----	----

***B) Enroulement***

4 5 

1	2	3	4	5
---	---	---	---	---

 1 2

19	10	16	22	23
----	----	----	----	----

***C) Réflexion***

3 2 

1	2	3	4	5
---	---	---	---	---

 4 3

10	10	16	22	26
----	----	----	----	----

***D) Défaut***

1	2	3	4	5
---	---	---	---	---

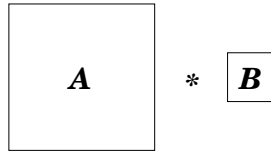
X	10	16	22	X
---	----	----	----	---



# TF DISCRÈTE

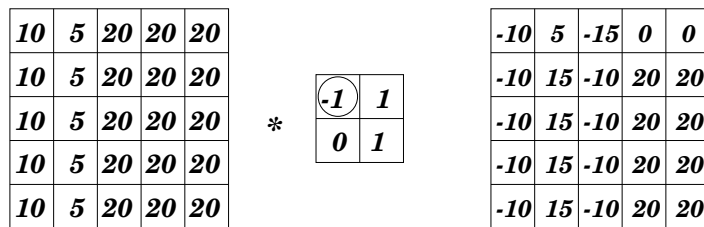
## CONVOLUTION DISCRÈTE 1D & 2D (3)

### Convolution 2D -Version discrète-

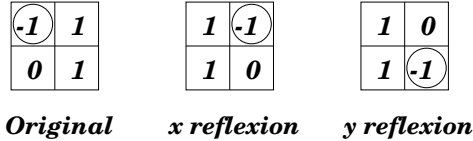


$$(A * B)(x, y) = \sum_i \sum_j A(i, j) B(x - i, y - j)$$

#### -Exemple-



*(Ajout de zéro)*

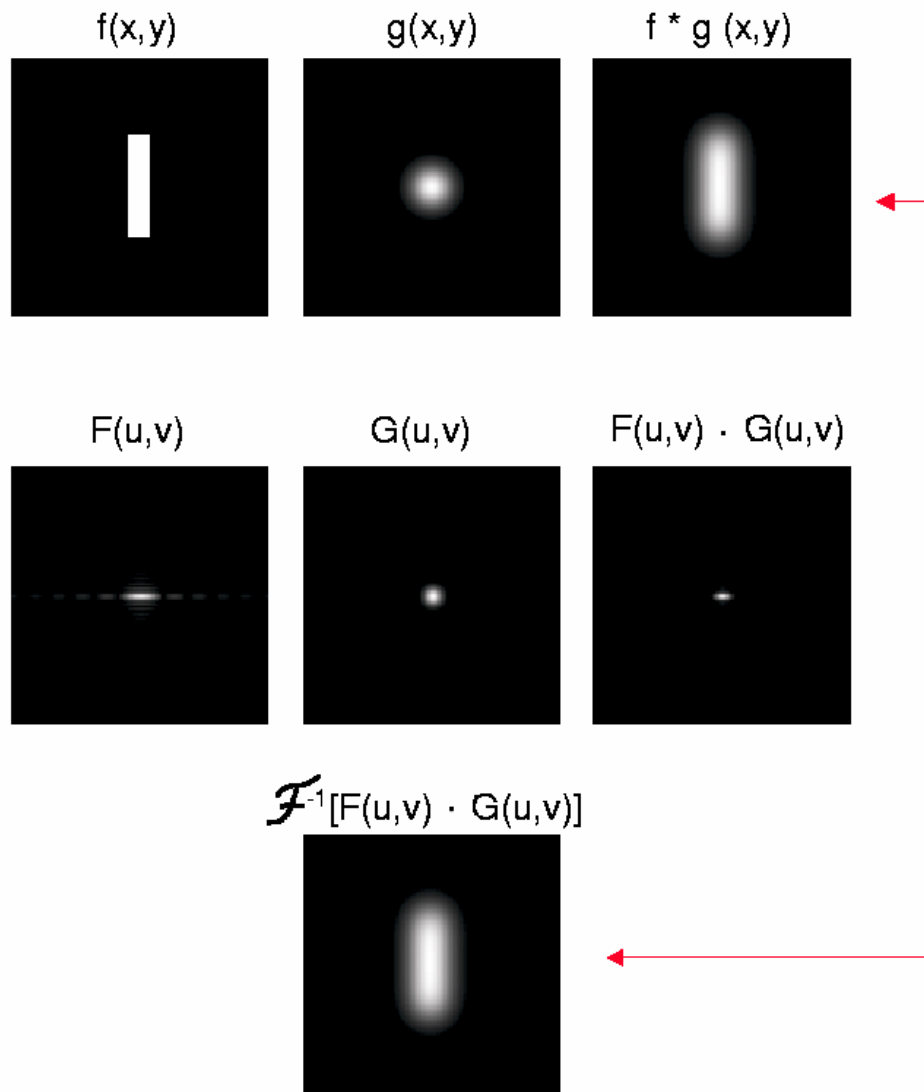


*Original      x reflexion      y reflexion*

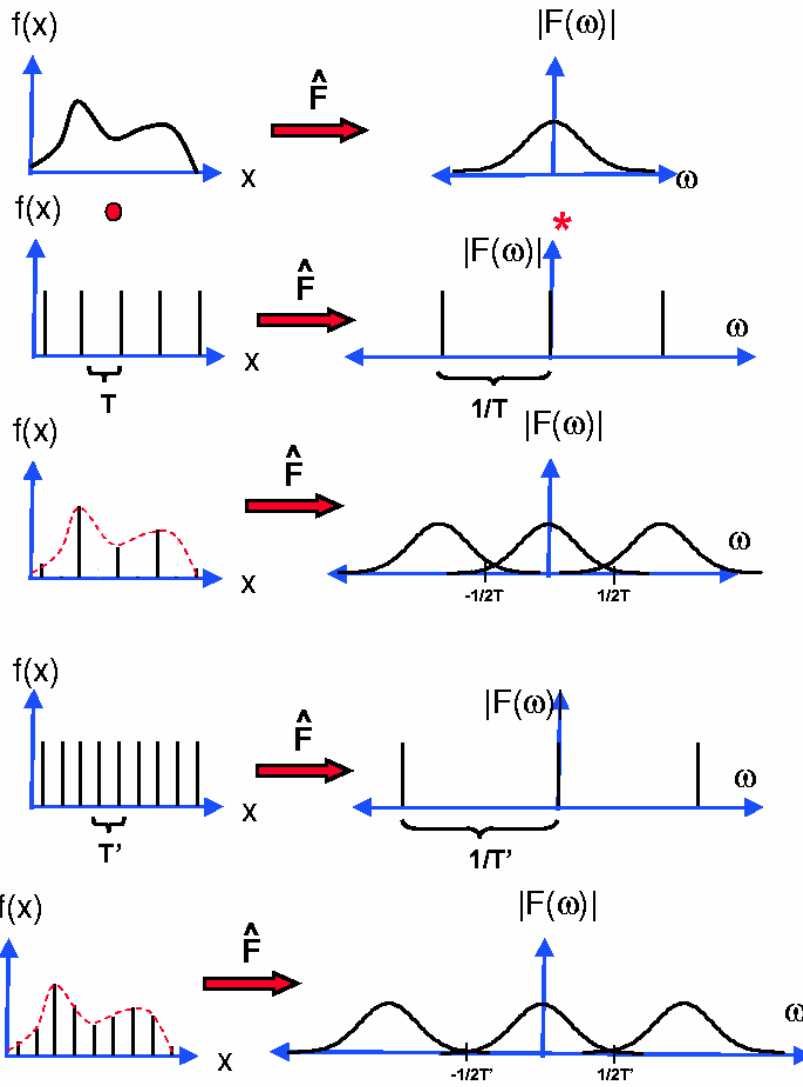
### PROPRIÉTÉS

- Commutatif ▶  $f_1 * f_2 * f = f_2 * f_1 * f$
- Associatif ▶  $(f_1 * f_2) * f = f_1 * (f_2 * f)$
- Distributif ▶  $(f_1 + f_2) * f = f_1 * f + f_2 * f$   
 ▶  $f * (f_1 + f_2) = f * f_1 + f * f_2$

TF DISCRÈTE  
CONVOLUTION DISCRÈTE 1D & 2D (4)



# TF DISCRÈTE THÉOREME DE SCHANNON

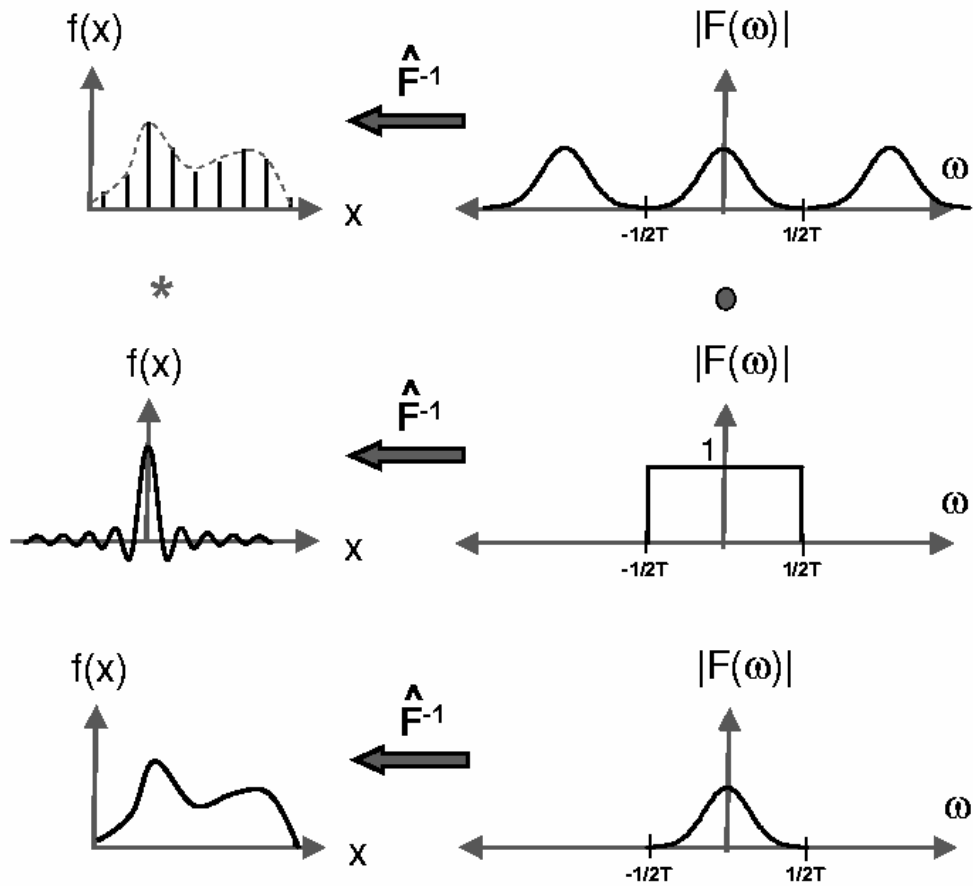


$$T_e \leq \frac{1}{2\nu_0}$$

► sinon phénomène d'aliasing

# TF DISCRÈTE

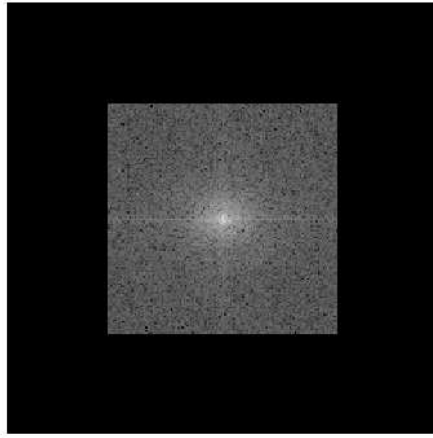
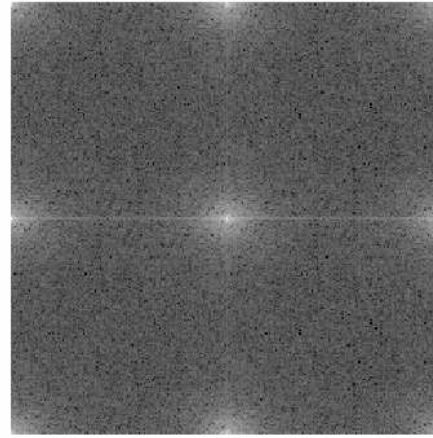
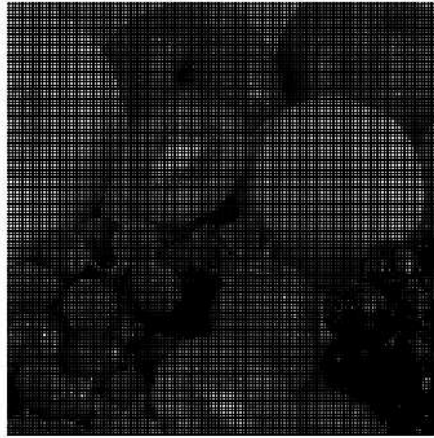
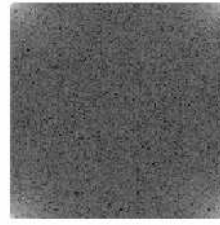
## INTERPOLATION OPTIMALE (1)



► Possibilité de reconstruire totalement le signal 2D à partir de ses échantillons

# TF DISCRÈTE

## INTERPOLATION OPTIMALE (2)



# TF DISCRÈTE

## FFT (1)

$$F(u) = \frac{1}{N} \sum_{x=0}^{N-1} f(x) \exp\left(\frac{-2\pi j u x}{N}\right) \quad u = 0, 1, 2, \dots, N-1$$

$O(N^2)$  opérations

$$F(u) = \underbrace{\frac{1}{N} \sum_{x=0}^{N/2-1} f(2x) \exp\left(\frac{-2\pi j u 2x}{N}\right)}_{x \text{ pair}} + \underbrace{\frac{1}{N} \sum_{x=0}^{N/2-1} f(2x+1) \exp\left(\frac{-2\pi j u (2x+1)}{N}\right)}_{x \text{ impair}}$$

$$= \frac{1}{2} \left[ \underbrace{\frac{1}{N/2} \sum_{x=0}^{N/2-1} f(2x) \exp\left(\frac{-2\pi j u x}{N/2}\right)}_{\text{TF de la partie paire}} + \exp\left(\frac{-2\pi j u}{N}\right) \underbrace{\frac{1}{N/2} \sum_{x=0}^{N/2-1} f(2x+1) \exp\left(\frac{-2\pi j u x}{N/2}\right)}_{\text{TF de la partie impaire}} \right]$$

- TF [N éléments] = 2 TF [N/2 éléments]  
+ N/2 multiplications et N/2 additions complexes (en fait N/4 pour les multiplications)
- $Op(N) = 2Op(N/2) + N$

$$F_N(u) = \frac{1}{2} \left[ F_{N/2}^{\text{paire}}(u) + \exp\left(-\frac{2\pi j u}{N}\right) F_{N/2}^{\text{impaire}}(u) \right]$$

$$\text{Pour } u' = u + \frac{N}{2}$$

$$\exp\left(-\frac{2\pi j u'}{N}\right) = \exp\left(-\frac{2\pi j (u + \frac{N}{2})}{N}\right) = \exp\left(-\frac{2\pi j u}{N}\right) \exp(-\pi j) = -\exp\left(-\frac{2\pi j u}{N}\right)$$

# TF DISCRÈTE

## FFT (2)

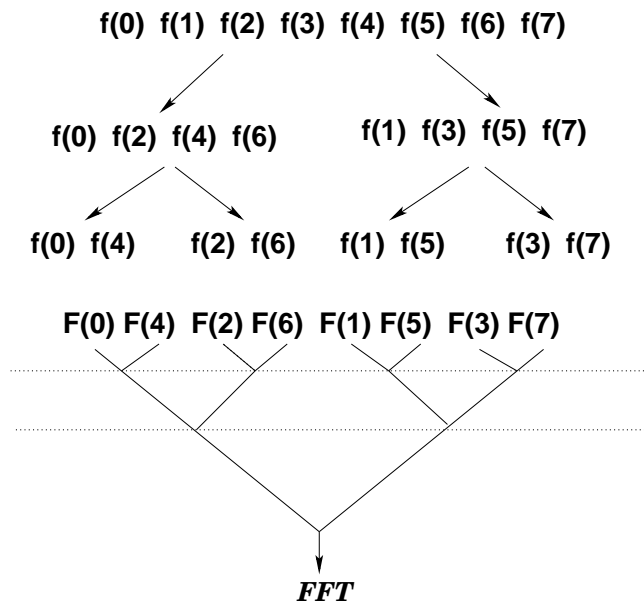
Ainsi

$$F_N(u) = \frac{1}{2} \left[ F_{N/2}^{\text{paire}}(u) + \exp\left(-\frac{2\pi ju}{N}\right) F_{N/2}^{\text{impaire}}(u) \right]$$

$$F_N\left(u + \frac{N}{2}\right) = \frac{1}{2} \left[ F_{N/2}^{\text{paire}}(u) - \exp\left(-\frac{2\pi ju}{N}\right) F_{N/2}^{\text{impaire}}(u) \right]$$

Pour  $u = 0, 1, 2, \dots, N/2 - 1$

1 multiplication est nécessaire pour deux termes



FFT 1D ►  $O(N \log N)$   
 FFT 2D ►  $O(N^2 \log N)$

# TF DISCRÈTE

## FFT (3)

### Vérifions que $O(N \log N)$

(Par induction, sachant que  $Op(N) = 2Op(N/2) + N$ )

- Vrai pour  $N = 1$ ?  $\blacktriangleright Op(1) = 1 \log 1 = 0 \blacktriangleright$  oui
- Vrai pour  $N = 2$ ?  $\blacktriangleright Op(2) = 2 \log 2 = 2 \blacktriangleright$  oui
- Vrai pour  $N \log N$ ?

- $Op(N) = N \log N$ ?
- $Op(N) = 2N/2 \log(2N/2)$
- $Op(N) = 2N/2(\log(N/2) + \log 2)$
- $Op(N) = 2N/2 \log N/2 + 2N/2 \log 2$
- $Op(N) = 2Op(N/2) + N$

C.Q.F.D.

### Rehaussement d'image

