

Game Theory : An Overview

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Topics being covered ...

- Introduction
- Game Theory Terminology
- Two Person Games
- N Person Games

Traditional Game Theory

- Developed by John von Neumann and Oskar Morgenstern
- Studies bargaining between rational agents in a very general setting
- Agent "Second Guessing"
- notion of a "strategy"

Defining a "Game"

- At least Two 'Players' (Individual or Entity)
- Each player has a number of possible 'Strategies'
- The strategies chosen determine the 'outcome'
- Associated with each outcome is a collection of 'payoffs', one to each player

Representation of a Game

- Bimatrix Representation
 - Suppose Rose and Colin are playing a Game
 - A and B are the strategies of play

		Colin	
		A	B
Rose	A	(2,-2)	(-3,3)
	B	(-5,5)	(10, -10)

Définition

- Zero-sum game
- Quoi faire ?

$$\sum_i p_i = 0$$

		P2			
		-1	-3	-3	-2
P1	0	1	-2	-1	
	2	-2	0	1	

Choix de P1

- One thing that P1 might do is to ask "for each move I might make, what is the worst thing (for me) that P2 can do?"

- Best choice : max des minima = { 2, 3 }

P2

	-1	-3	-3	-2
P1	0	1	-2	-1
	2	-2	0	1

-1	-3	-3	-2
0	1	-2	-1
2	-2	0	1

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Choix de P2

- Similarly P2 could analyse looking for the move which will minimise his loss given that P1 will try to make this as big as possible.

- Best choice: minimum des maxima { 3 }

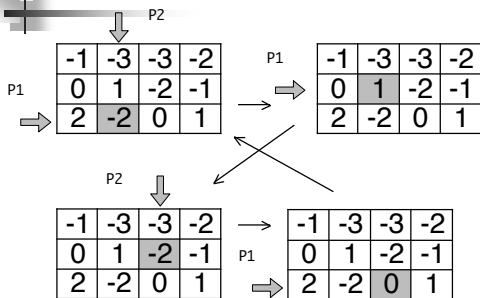
P2

	-1	-3	-3	-2
P1	0	1	-2	-1
	2	-2	0	1

-1	-3	-3	-2
0	1	-2	-1
2	-2	0	1

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Situation instable

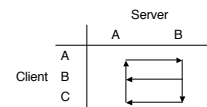


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Movement Diagram

- The game and the movement diagram analysis

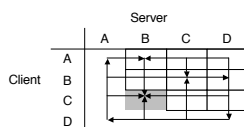
	Server	
	A	B
Client A	(2,-2)	(-3,3)
Client B	(0,0)	(2,-2)
Client C	(-5,5)	(10,-10)



Equilibrium Outcome

- An 'Equilibrium' exists when both players play a strategy such that the payoff to each player is higher for that combination of strategies than any other combination.

Client	Server			
	A	B	C	D
A	12	1	1	0
B	5	1	7	-20
C	3	2	4	3
D	-16	0	0	16



Equilibria .. contd ..

- Agents are economically rational
- Act to maximize expected utility w.r.t knowledge about other agents' actions and their own payoffs
- Weak Equilibrium**
 - No agent can 'gain' by changing his/her strategy; given the rest are fixed
- Strict Equilibrium**
 - Every Agent is strictly worst off; if he/she changes strategy

Dominance Principle

- Principle of Dominance**

A strategy S dominates T if every outcome in S is at least as good the corresponding outcome in T and at least one outcome in S is strictly better than the corresponding outcome in T

- Weakly dominating**

Removal of a row/column will also remove the use of an optimal strategy (although there is still 'at least' one left)

- Strongly or Strictly dominating**

If all elements of row (column) \geq row (column) then 1 dominates 2 but does not contain the optimal strategy

A rational player should never play a 'dominated' strategy

		Server			
		A	B	C	D
Client	A	12	-1	1	0
	B	5	1	7	-20
	C	3	2	4	3
	D	-16	0	0	16

Saddle Points

An outcome in a matrix game is called a 'Saddle Point' if the entry at that outcome is

- Both less than or equal to any entry in its row
- Both greater than or equal to any entry in its column

- Saddle Point Principle:**

If a matrix game contains a 'Saddle Point', both players should play strategies which contain it.

		Server			
		A	B	C	D
Client	A	4	2	5	2
	B	2	1	-1	-20
	C	3	2	4	2
	D	-16	0	16	1

Not a saddle point (pointing to cell A, B)

Saddle Points (pointing to cells B, C and C, D)

Strategies

- Pure Strategies**

Playing one strategy with certainty

- Mixed Strategies**

A plan that involves a mixture of strategies according to certain fixed probabilities

		Server	
		A _S	B _S
Client	A _C	2	-3
	B _C	0	3

2 → -3
↑ ↓
0 ← 3

Pure strategy doesn't work

Mixed strategies

- What we want is a "spy-proof" strategy.
- This is one which works even if the other player knows what the strategy is.
- We manage this by moving from a pure strategy to a mixed strategy in which a player makes a random choice across a set of pure strategies.
- Random BUT with specific probabilities
- The 'expected value' of getting payoffs a_1, a_2, \dots, a_k with respective probabilities p_1, p_2, \dots, p_k is

$$p_1 a_1 + p_2 a_2 + \dots + p_k a_k$$

Expected Value Principle

		Server	
		A _S	B _S
Client	A _C	2	-3
	B _C	0	3

Client assumes that Server will choose strategies A_S and B_S with probabilities x and $(1-x)$ respectively.

Payoff from A_C: $P(A_C) = x \cdot 2 + (1-x) \cdot (-3)$

Payoff from B_C: $P(B_C) = x \cdot 0 + (1-x) \cdot 3$

The Client wants a payoff independent of the strategy of the server

$$P(A_C) = P(B_C)$$

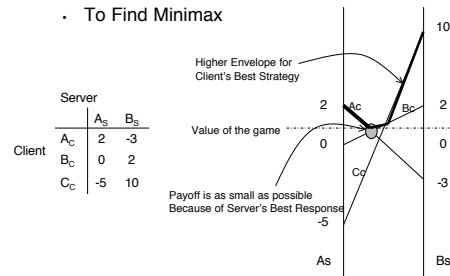
If Client plays with (3/8 A_C, 5/8 B_C)

$$P(A_C) = P(B_C) = _ = \text{Value of the game}$$

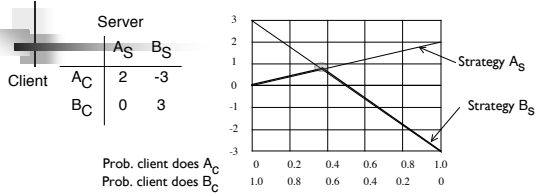
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Envelope Method

To Find Minimax



Client Strategy



- λ If Client plays ($3/8 A_C, 5/8 B_C$)
Pay(A_S) = Pay(B_S) = $3/4$ = Value of the game
- λ No matter what what strategy S adopts !!!

Minimax theorem for finite games

- There is a mixed strategy for Player I (Client) s.t.
- V is the value of the Game
- I's gain $\geq V$
- There is a mixed strategy for Player II (Server) s.t.
- II's gain $\leq V$

Solution of a Game

- Value of the Game
- Optimal Strategies of the Players

Server's Optimal Strategy = $3/4 A, 1/4 B$
Client's Optimal Strategy = $3/8 A, 5/8 B$

Minimax Theorem

- Finite Game: When both the strategy sets of game (X, Y, A) are finite sets.
- Minimax theorem for finite games sets
 - V is the value of the Game
 - There is a mixed strategy for Player I (Client) s.t.
I's gain $\geq V$
 - There is a mixed strategy for Player II (Server) s.t.
II's gain $\leq V$

Review Zero-Sum Games

- Dominant strategies
 - B_C is always better for **Client**
 - A_S is always better for **Server**

		Server	
		A_S	B_S
Client	A_C	3	1
	B_C	2	4

Review Zero-Sum Games

- Minimax - **Pure** strategy
 - Strategy which maximizes gains supposing that opponent will minimize loss
 - A_C is always better (dominant) for **Client**
 - Given that, **Server** should play B

		Server	
		A_S	B_S
Client	A_C	1	-1
	B_C	-5	-2

Review Zero-Sum Games

- Minimax - **Mixed** strategy

- Play randomly but choose probabilities rationally

- Optimal for **Client**

50% A_C & 50% B_C

- Optimal for **Server**

75% A_S & 25% B_S

- Value of Game = 0.5 for Client

		Server	
		A_S	B_S
Client	A_C	1	-1
	B_C	0	2

Utility Theory

- One should evaluate a payoff by its 'utility to the player' rather than on its 'numerical monetary value'
- Existence of Saddle Points
 - Ordinal Scale
 - Higher Numbers represent higher preferred outcomes
 - Hence, only order of numbers matter
 - Ordinal Utilities
- Involvement of Mixed Strategies
 - Interval Scale
 - Order of numbers and ratio of differences matter
 - Cardinal Utilities

Two-Person Non-Zero Sum Games

- Interests of the players are

- 'not strictly' opposed
- 'not strictly' coincident

		Server	
		A_S	B_S
Client	A_C	(2,2)	(3,-1)
	B_C	(-1,3)	(4,4)

Equalizing Strategy

- Strategies in which the opponent earns a payoff regardless of the strategy the opponent plays.
- Client's Optimal Mixed Strategy (3/7 A_C , 4/7 B_C)
- Expected Payoff = 16/7
- Client's Equalizing Strategy

		Server	
		A_S	B_S
Client	A_C	(2,4)	(1,0)
	B_C	(3,1)	(0,4)

Nash Equilibrium

- Both Players play equalizing strategies
- Neither player can gain by deviating
- This is an 'equilibrium' called 'Nash Equilibrium' in the honor of John Nash.
- Proof states that 'Every Two-Person Game has **at least** one equilibrium in **either** Pure Strategies or Mixed Strategies'

Nash Equilibrium ... contd ...

- Example:

		Server	
		A_S	B_S
Client	A_C	(1,1)	(2,5)
	B_C	(5,2)	(-1,-1)

Equilibria

Nash Equilibrium : Complexity

- Christos Papadimitriou (UC Berkeley) *Algorithms, Games, and the Internet*
- Is there a polynomial algorithm for computing the Nash equilibrium in a 2-person game?
- Assuming that there is a guaranteed solution, it belongs to the class of problems 'between' P and NP

Equivalence & Interchangeability

- When the game has multiple saddle points
- The saddle points have the same 'value'
- Non-zero Sum Games can have multiple non-equivalent and non-interchangeable saddle points

		Server	
		A_S	B_S
Client	A_C	(1,1)	(2,5)
	B_C	(5,2)	(-1,-1)

Not equivalent Or interchangeable

Pareto Optimality Principle

- Vilfredo Pareto, Italian Economist 1900
- Existence of Unique Nash Equilibrium
- Pareto Optimal Outcome exists when in a game
 - There is an outcome that has higher payoffs to both players or
 - One of the players has a better outcome and the other player has the same outcome

		Server		Pareto Optimal
		A_S	B_S	
Client	A_C	(3,3)	(-1,5)	
	B_C	(5,-1)	(0,0) ← Equilibrium	

Prudential Strategies

- Prudential Strategy
In a Non-Zero Sum game, a player's optimal strategy in the player's own game
- Security Level
The Value of the Game with Prudential Strategy
- Counter-Prudential Strategy
Optimal Response of a player to his/her opponent's prudential strategy

The Prisoner's Dilemma

- Consider the strategies as follows
 - C : Payoff for Sharing
 - D : Payoff for 'Not' Sharing
 - R : Reward for Sharing
 - S : Sucker Payoff
 - T : Temptation Payoff
 - U : Uncooperative Payoff
- $T > R > U > S$ and $R > (S + T)/2$

		Agent 2	
		C	D
Agent 1	C	(R,R)	(S,T)
	D	(T,S)	(U,U)

Repeated Plays

- Games are not necessarily played once; but repeated
- Cooperating in early plays for beneficial outcomes
- Strict logic prevents cooperation from being started
- Next play occurrence with probability 'p'
 - If a player never chooses an equilibrium strategy (D) then $payoff_a = R / (1 - p)$
 - If a player chooses D in the m^{th} game $payoff_b = (R + p^m R + (1 - p)p^m T + p^{m+1} U) / (1 - p)$
 - The player never chooses D if $payoff_a > payoff_b$ for all values of 'm'
 - Hence, $p > (T - R) / (T - U)$
 - If the probability of continuing play is higher than a threshold, it makes sense for both the players to cooperate!

Qui perd et qui gagne ?

- Roulette au casino
- Loto
- ...