

Fault-tolerant quantum computing with color codes

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We have just completed a thorough, multi-year study of fault-tolerant quantum computation with color codes. These codes are amazing—one way to think about them is as a family of codes which generalize the Steane code not by concatenation but by lattice embedding [1]. Our study of fault-tolerant computing with these codes is complete with analytic bounds, numerical simulations, multiple noise models, and comparisons to other topological codes. This is the first in-depth study of how to make quantum computers fault-tolerant since concatenated coding and coding with Kitaev’s toric codes.

In the course of our study, we developed a variety of new analysis techniques, decoding algorithms, and protocols for encoded computation. We expect that these ideas will be of broad interest to anyone working on fault-tolerant quantum computing, especially in scenarios where technology limits quantum processing to be spatially local in 2D. We hope that our work will draw in a broader community to study and explore color codes, much as the work by Dennis, Kitaev, Landahl, and Preskill [2] did for Kitaev’s toric codes.

The specific results of our paper draw upon ideas ranging from theoretical computer science to theoretical physics, including integer programming, self-avoiding walk theory, code deformation, and phase transitions in the three-body random-bond Ising model. In retrospect, our paper could have been spun off into many smaller papers, but we believe that as a single corpus it is stronger and will be a better resource to the quantum computing community.

Space limitations in this abstract necessarily restrict our ability to survey all of our results and new ideas, not to mention the hefty list of future directions we catalogued that others may be interested in pursuing. We also appreciate that you have many abstracts to read for QIP 2012, and our goal is just to whet your appetite here and encourage you to read the full paper for details. For these reasons, we will give only the briefest of highlights in mini-sections. **If you want a really quick read, just read the bolded text.** The full paper is available on the arXiv at [arXiv:1108.5738](https://arxiv.org/abs/1108.5738). Happy reading!

Integer-program-based decoding algorithm. We show how to formulate the decoder that finds the most likely error given the syndrome of a color code as an integer pro-

gram. It is quite different from the analogous minimum-weight perfect matching algorithm used for Kitaev’s toric codes. **Our decoder generalizes quite far—it works for all CSS codes.** While the decoder isn’t efficient, it opens connections to order-disorder transitions in the three-body random-bond Ising model. We also show how the structure and symmetries of the color codes allow this IP to be simplified so that it is tractable for moderate-sized instances using modern desktop computers.

Numerical estimate of accuracy threshold. We calculate the probability of failure of fault-tolerantly correcting errors in 4.8.8 color codes as a function of the probability of error per elementary gate. For the smallest codes and simplest noise models, we compute this curve exactly. For larger codes and more complex noise models, we estimate the curve using Monte Carlo techniques. From these curves, we estimate the accuracy threshold for fault-tolerant quantum computation for several noise models and two different proposed architectures, accompanied by a thorough error analysis of our data. In the most realistic noise model, taken down to the circuit level, **we estimate the threshold to be 0.082(3)%**. This is roughly 10 times larger than the largest estimate for a concatenated coding threshold in 2D but roughly 10 times smaller than the largest estimate for a surface code threshold.

(Nearly) Transversal universal computation. We show how to make an end-run around the Eastin-Knill theorem [3], which says that encoded universal quantum computations require some non-transversal operation, by using two non-transversal operations used for the error-correction process itself: syndrome extraction and coherent encoding. Some unique features of the color codes make this possible. For 3D color codes, one can get away with just using **syndrome extraction as the sole non-transversal operation**.

Proofs of lower bounds on the accuracy threshold. In addition to numerical estimates, we **prove lower bounds on the accuracy threshold** for 4.8.8 color codes in some noise models. Our proof technique **generalizes the self-avoiding polygon technique of Dennis et al.** [2].

The Nishimori conjecture is false for the three-body random-bond Ising model. Using our estimate of the accuracy threshold for our IP decoder, the results of Katzgraber et al. [4] and Ohzeki [5], and a mapping between quantum error correction of color codes and magnetic order in the three-body random-bond Ising model,

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we **demonstrate that the Nishimori conjecture is false for the 4.8.8 lattice**. Namely, the three-body random bond Ising model on this lattice actually can become *more* ordered, not less ordered, by increasing the ambient temperature. This is a rare example of a result in quantum information science solving an open problem in classical physics.

Circuit-level schedules for syndrome extraction. We **develop two parallelized schedules for extracting the syndrome** for 4.8.8 color codes. Obtaining these schedules is a nontrivial exercise because any schedule must satisfy a number of constraints. We estimate the accuracy threshold for both of these, and the seemingly simpler schedule leads to a worse (lower) threshold. This runs counter to the intuition suggested by concatenated-coding studies. Our results suggest that quantum circuit optimization for syndrome extraction is a nontrivial task that could have a substantial impact on the resulting accuracy threshold.

Reduction in magic-state overhead. We design a scheme for using one “defect” per qubit in a 4.8.8 color-code substrate which **eliminates the need for magic states to implement the encoded phase (S) gate**. This is an advantage over Kitaev’s surface codes, which require the $|0\rangle + i|1\rangle$ magic state to perform the S gate.

Individual gate thresholds for two architectures. We

sketch two possible layouts for a color-code quantum computer. The first is a “pancake” architecture with one planar surface per logical qubit and each plane stacked atop the other. The other is a single 2D “substrate” architecture in which logical qubits exist as “defects” in the substrate which can be driven to move around one another to perform two-qubit gates. **We estimate the accuracy threshold for each gate in a universal set for both architectures, not just the accuracy threshold for quantum memory.**

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