Disorder-assisted error correction in Majorana chains*

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It was recently realized that quenched disorder may enhance the reliability of topological qubits by reducing the mobility of anyons at zero temperature. Here we compute storage times with and without disorder for quantum chains with unpaired Majorana fermions - the simplest toy model of a quantum memory. Disorder takes the form of a random site-dependent chemical potential. The corresponding one-particle problem is a one-dimensional Anderson model with disorder in the hopping amplitudes. We focus on the zero-temperature storage of a qubit encoded in the ground state of the Majorana chain. Storage and retrieval are modeled by a unitary evolution under the memory Hamiltonian with an unknown weak perturbation followed by an error-correction step. Assuming dynamical localization of the one-particle problem, we show that the storage time grows exponentially with the system size. We give supporting evidence for the required localization property by estimating Lyapunov exponents of the one-particle eigenfunctions. We also simulate the storage process for chains with a few hundred sites. Our numerical results indicate that in the absence of disorder, the storage time grows only as a logarithm of the system size. We provide numerical evidence for the beneficial effect of disorder on storage times and show that suitably chosen pseudorandom potentials can outperform random ones.

Introduction

Topologically ordered quantum systems hold great promise for the robust storage and manipulation of quantum information. Prime examples are Kitaev's toric code [17] and his quantum wire with unpaired Majorana modes [18]. The latter describes a wide range of experimentally accessible systems [3, 13, 15, 19, 21]. The ground state of these models is an error-correcting code with macroscopic distance (if superselection rules are taken into account) and the Hamiltonian is a sum of commuting stabilizers. In particular, logical errors occur only if an error-chain affects a macroscopic number of sites – or – in the language of localized excitations or anyons, if an anyon propagates along a topologically non-trivial trajectory. Naïvely, this suggests that the robustness of the memory as measured in terms of e.g., the storage time, scales extensively with the system size N (i.e., the number of qubits or the number of fermions).

Unfortunately, this is not the case because anyons, once created, can propagate freely without any energy cost. As a result, such topological memories are not robust if left to evolve on their own, a fact recognized early on [11]. This no-go statement has since been elaborated in various ways, ranging from system-size independent bounds on relaxation times [4, 9], investigations of non-zero temperature topological entanglement entropy [8], to constant upper bounds on the energy barrier needed to cause a logical error for arbitrary $2D$ stabilizer and subsystem codes [7]. Even at zero temperature, constant-strength local perturbations V can effectively cause a logical error in a time scaling as $O(\log N)$ [20]. This scaling, albeit extensive, is far from what can be considered a robust memory.

In pioneering work, Wootton and Pachos [24], and, independently, Stark et al. [22], recently proposed using quenched disorder as a means for reducing the propagation of anyons. Since disorder may be naturally present in a physical implementation, or may be artificially engineered by tuning interactions, this appears to be a realistic and potentially experimentally feasible way of improving the reliability of a topological quantum memory. The intuition underlying this proposal is analogous to the reasoning used to explain the suppression of zero-temperature electron conductivity in wires: the anyons, like electrons, get trapped in the many local minima present in a random potential. A mathematical expression of this phenomenon, commonly referred to as Anderson localization, is the fact that the eigenfunctions of the (random) Hamiltonian are exponentially localized (in some appropriate sense) on average over disorder realizations.

To study the effect of disorder on storage in the toric code, the authors of [22, 24] restrict their analysis to the dynamics of subspaces with a fixed particle number, neglecting processes creating and destroying anyons. Within this approximation, localization properties of the corresponding multi-particle states have been established both numerically [24] and analytically [2, 10, 22]. Using these results, some estimates on the failure probability starting from a two-particle configuration as well as certain geometrically arranged multi-particle initial configurations were obtained in [22, 24]. However, these estimates neglect the full many-particle dynamics, and hence only provide a somewhat qualitative picture of the effect of localization.

Storage fidelity and disorder for Majorana chains

Here we study the robustness of a qubit encoded in the unpaired edge modes of a Majorana chain at zero temperature. We consider the process of encoding into the ground space of the unperturbed Majorana chain

*A full technical version is available at [1]

Hamiltonian H_0 (i.e., preparing a ground state $|g\rangle$), time-evolving for some time t under $H_0 + V$, where V is an unknown perturbation, followed by readout, that is, error correction. The latter step, described by a CPTPM Φ_{ec}, consists of a syndrome measurement followed by an error correction operation. This is analogous to the minimal matching algorithm used to restore ground states of the toric code [11]. We measure the robustness of the memory in terms of its storage fidelity, the overlap between the initial encoded state and the final error-corrected state:

$$F_{|g\rangle}(t) = \langle g|\Phi_{ec}(e^{i(H_0+V)t}|g\rangle\langle g|e^{-i(H_0+V)t})|g\rangle \ .$$

Error correction is necessary here to give a fair assessment of the information recoverable from the memory. It is important to stress, however, that it plays no direct role in preserving coherence in the interval $[0,t]$. This is in contrast to e.g., [11], where it is shown that continuous error-correction in short time intervals allows to preserve quantum information indefinitely in the toric code.

We also consider the worst-case fidelity $F(t) = \min_{|g\rangle} F_{|g\rangle}(t)$ minimized over all encoded states, and the *storage time* $T_{\mathrm{storage}} = T_{\mathrm{storage}}(F_0)$, the minimal time it takes for the storage fidelity $F(t)$ to drop below a given threshold F_0, e.g., $F_0 = 0.99$. Clearly, these quantities directly characterize the performance of the system as a quantum memory.

For concreteness, we focus on perturbations corresponding to a non-zero chemical potential. The 'clean' case is modeled by a uniform chemical potential μ, while the disordered case will be modeled by a site-dependent chemical potential

$$\mu_j \equiv \mu + \eta x_j$$

where x_j are identically and uniformly distributed random variables on $[-1,1]$ and η controls the strength of the disorder. This choice of perturbation and disorder is physically motivated, but our results apply to more general (quadratic) perturbations.

Main results

We identify a dynamical localization condition for the corresponding one-particle problem, which, when satisfied, implies that the storage time scales as

$$\mathbb{E}\left[T_{\mathrm{storage}}\right] \sim e^{\Omega(N)} \qquad (1)$$

on average over disorder realizations, in the limit of large system size N. The localization condition concerns moments of the orthogonal matrix R describing time evolution of the Majorana operators in the Heisenberg picture. In particular, it is satisfied if the entries of R decay exponentially away from the main diagonal. We conjecture that it holds in the limit of weak perturbations and strong disorder, $\mu \ll \eta \ll 1$. By computing Lyapunov exponents, we also give supporting evidence that it is satisfied in the limit of weak perturbations, $\mu \to 0$, when the ratio η/μ is kept constant. This computation is based on a method developed by Eggarter et al [12] for analyzing tight-binding chains with off-diagonal disorder.

We also compute the storage time numerically using an adaptation of the Monte Carlo technique for simulating dynamics and measurements for non-interacting fermions developed by Terhal and DiVincenzo [23]. The running time of this algorithm scales as N^3/δ^2, where δ is the precision up to which the storage fidelity is estimated. It allows us to compute the storage time for chains with a few hundred sites (up to $N = 256$) in the regime of strong perturbations[1], that is, $\mu \sim 1$ and $\eta = 0$ (clean case), and $\eta \sim \mu \sim 1$ (disordered case). The simulation shows that in the absence of disorder the storage time grows as a logarithm of the system size:

$$T_{\mathrm{storage}} \sim O(\log N). \qquad (2)$$

This scaling recently has been predicted by Kay [16] based on mean-field arguments. In the presence of disorder we observe an approximately linear scaling $\mathbb{E}\left[T_{\mathrm{storage}}\right] \sim N$, see Fig. 1. This confirms the expected enhancement of the storage time, although the enhancement is much weaker than our theory predicts, see Eq. (1). This discrepancy could be accounted for by the fact that the system size N is comparable with the localization length ξ of single-particle wavefunctions in the simulated regime, whereas Eq. (1) is expected to hold only when $N \gg \xi$. It could also point to an interesting possibility that a crossover from a polynomial to an exponential scaling of the storage time occurs as one interpolates between strong ($\mu \sim \eta \sim 1$) and weak ($\mu \ll \eta \ll 1$) perturbations. Finally, we give examples where an artificially engineered deterministic disorder potential leads to improved storage times compared to random disorder.

Techniques

The exact solvability of the Majorana chain model means that the full many-body dynamics can be related to a single-particle problem. This gives rise to powerful tools which may be of independent interest. For example, we derive a strengthened form of the quasi-adiabatic continuation technique [14] for free fermion Hamiltonians: it shows that ground states of the perturbed and unperturbed Hamiltonian can be connected by constant-time evolution under a time-dependent Hamiltonian with exponentially decaying interactions. In contrast, applying the standard quasi-adiabatic continuation technique based on suitable filter functions only gives a stretched-exponential decay. This is insufficient for our purposes.

[1] We show that the storage fidelity is close to 1 whenever $\epsilon < 1/\sqrt{N}$ simply because the perturbed ground state has a large overlap with the unperturbed one. To explore the asymptotic scaling of the storage time for weak perturbations, say, $\epsilon \sim 10^{-2}$, one would need to simulate chains with at least $N \sim 10^4$ sites.

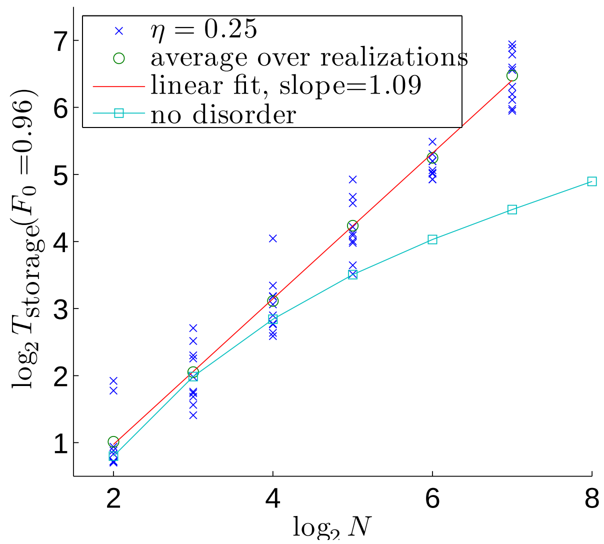

FIG. 1: Plot of $\log_2 T_{\text{storage}}(F_0)$ versus $\log_2 N$ for the fidelity threshold $F_0 = 0.96$ (for the clean case with $\mu = 0.5$ and with disorder of strength $\eta = 0.25$, see [1] for details). For each system size, 10 different disorder realizations are considered. A straight line is fitted to the average over disorder realizations, showing a linear relationship between $\mathbb{E}\left[T_{\text{storage}}(F_0)\right]$ and N. In contrast, $T_{\text{storage}} \sim \log N$ when no disorder is present.

One application of this quasi-adiabatic continuation technique is a strengthening of the gap stability result of [5, 6] in our setting: we can show that the energy splitting of the ground states is exponentially small in N and derive tight bounds on the gap stability radius.

Conclusions

In summary, our results establish a direct connection between Anderson localization and operational quantities characterizing the quality of a quantum memory. Our analytical and numerical results strongly support the idea that disorder can significantly enhance topological quantum memories. Future work may try to further quantify this relationship for other models, and – some day, try to observe this effect in the laboratory.

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