

Equivalence of Topological Codes and Fast Decoding Algorithms[1]

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Topological stabilizer codes (TSC), introduced by Kitaev [14], are among the most promising quantum error correcting codes that could lead to realistic quantum computer architectures. In particular, because they have geometrically local check operators, they are especially well suited to architectures in which qubits are arranged in a lattice and only local operations are available. In addition, logical operators are closely related to the topology of the lattice, providing a macroscopic minimum distance. TSCs can be shown to operate reliably in the presence of a considerable amount of noise, with a fault-tolerant threshold of 0.75% [18]. They are to be contrasted with, e.g., concatenated codes where the number of qubits involved in each syndrome measurement grows linearly with the size of the code, thus increasing the complexity of the error correction procedure and lowering the threshold.

In addition, TSCs have been a very fertile playground for condensed matter physics because they provide exactly solvable models with complex phenomena such as topological order and anyonic excitations [7, 10, 15]. The connection is as follows. One can think of a stabilizer error correcting code as the ground subspace of a local Hamiltonian

$$H = - \sum_a S_a, \quad \text{with } [S_a, S_b] = 0 \quad (1)$$

where the S_a are stabilizer generators defining the code space $\mathcal{C} = \{|\psi\rangle : S_a|\psi\rangle = |\psi\rangle, \forall a\}$. In this language, errors will cause excitations in the system—a local energy increase above the ground energy—and the error syndrome will correspond to an excitation configuration.

Together with Kitaev's Toric Code (KTC), there exist other interesting examples of topological codes in the literature. One of us introduced Topological Color Codes (TCC)[7], similar in many respects to KTCs, but with the advantage of allowing transversal logical Clifford operations, a crucial asset for fault tolerance. From a condensed matter perspective, both give rise to anyonic excitations, but in such a way that the topological charges in a TCC correspond to two copies of a KTC.

The fundamental motivation of our work is to rigorously classify the distinctions and equivalences between such codes. The natural notion of equivalence here is that coming from the condensed matter perspective, where one can introduce a phase equivalence for topologically ordered systems [9]. Simply put, two TSCs are equivalent if there exists a finite-depth local quantum circuit

that takes one code and maps it to the other, and vice versa. Our main result is the following [1, 4]:

Theorem 1 *Every 2D TSC is locally equivalent to a finite number of copies of KTC.*

To define precisely what we mean by local, consider an operator P acting on some qubits contained within a region of finite radius r . Then, a unitary transformation U is a local mapping if UPU^\dagger is contained within a region of radius less or equal to $r + c$, where c is a constant.

This result is important for many reasons. Firstly, in the context of error correction, the local equivalence to KTC enables us to directly extend a number of properties of this code to all 2D TSCs. For instance, thermal instability [3, 17], code tradeoffs [8], logical operator geometry [13], and scale invariance [2] all become trivial corollaries of our mapping. In addition, our mapping provides a method to decode any 2D TSC code, while only a handful of special cases previously had solutions [10, 11, 19]. Secondly, the local mapping can be used to change encoding during a quantum computation. Because the mapping is local, this change will not propagate errors and is therefore fault-tolerant. This allows to put together the features of different codes—such as having transversal Clifford gates [7], lower weight stabilizer generators [6, 10, 15], etc.—and suggests a natural generalization of the notion of transversality for topological codes to include all local gates. Thirdly, from the condensed matter perspective it provides a large class of models where the definition of topological order based on local equivalence [9] can be directly applied in a rigorous manner.

In 2D TSCs excitations are anyonic. They carry a topological charge defined by a notion of local equivalence. Consider a finite region of the system with a definite error syndrome or excitation configuration. Two such configurations carry the same topological charge if it is possible to transform one into the other by a unitary transformation with support only on that region. With this definition, KTCs have four topological charges, the vacuum corresponding to no excitations, an electric charge corresponding to a plaquette excitation, a magnetic charge corresponding to a star excitation, and a composite excitation containing both. Any excitation configuration on a finite region can be mapped to one of these four possibilities. Excitations with different charges are also characterized by different braiding statistics that describe the effect of exchanging two excitations. For

KTC, both the electric and magnetic particles are bosons because they have trivial self-statistics, while the composite particle is a fermion because it acquires a -1 sign upon particle exchange. In addition, mutual statistics are semionic: exchanging two non-trivial and different charges can be done in two topologically distinct ways that differ by a -1 sign.

The key intuition behind our result is that systems with equivalent anyonic excitations are equivalent. Indeed, a local mapping cannot change the topological charges of a system, because they are defined by a local equivalence relation.

Let us exemplify the theorem by demonstrating a local equivalence between various codes. More precisely, we consider the family of TCCs [7] defined on a 4-8 regular tiling of Fig. 1 and show that they can be locally mapped to two of copies of KTC. In the 4-8 TCC, qubits are located at the vertices of the 4-8 tiling, and there are two stabilizer operators associated to every plaquette p .

$$S_p^\sigma = \bigotimes_{j \in \partial p} \sigma_j, \quad \text{with } \sigma \in \{\sigma^x, \sigma^z\} \quad (2)$$

where ∂p is the set of vertices of the plaquette p and σ^x, σ^z are the usual Pauli matrices. The excitations in this model can carry 16 different topological charges, 10 bosons and 6 fermions. These correspond exactly to the charges obtained from two copies of KTC. Figure 2 shows the mapping between this TCC and two KTCs. These mappings were obtained by identifying hopping terms of elementary excitations with the same topological charges between the two codes. It can easily be verified that the mapping preserves stabilizer operators (cf. Fig. 1).

Decoding— To be of any practical use, any error-correcting code must have an efficient decoder—an algorithm that finds the most likely recovery given the measured error syndrome. Two of us have conceived a decoding algorithm for Kitaev’s toric code (KTC) that is exponentially faster than previously known decoding algorithms (run time $\log \ell$ rather than ℓ^6 where ℓ is the linear size of the torus) [11]. Note that the decoding runtime is a crucial factor for fault tolerance; proofs of the threshold theorem usually assume instantaneous classical side-computation to assist the error-correction procedure. Our algorithm is also very flexible, it enables various tradeoffs between complexity and error-correction performances. In particular, we were able to demonstrate that it can achieve a higher error-correction error threshold than what was achievable by previously known decoding algorithms [10, 12].

As a consequence of the local equivalence, we can, for the purpose of decoding, treat a topological code as a certain number of copies of KTC, and use any decoding algorithm suitable for KTC to complete the decoding on each of these copies. Crucially, the noise model induced on KTC remains essentially local: an error model that

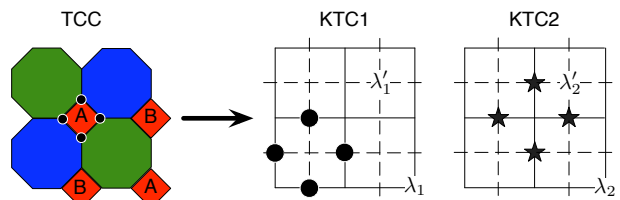


Figure 1: Regular square-octagon lattice for TCC. The diamonds can be labeled A or B according to a chessboard pattern. There are two stabilizers Eq. (2) associated to each plaquette. Here is an example of the mapping from one TCC to two copies of KTC. The black dots (stars) represent σ_z (σ_x) operators. A Z -plaquette on a A -diamond of the TCC gets mapped to a plaquette operator on the first KTC and to a site operator on the second KTC. The complete mapping for 1-qubit Pauli operators is shown in Fig.2.

is independent on each qubit will be mapped to an error model with some short-range correlations on a length scale c , but no long-range correlation. Because the existence of an error threshold is essentially a large scale property of a system, this decoding strategy should also produce a finite error threshold, albeit with a different critical error probability that depends on the value of c and other microscopic details of the mapping. Also, remember that, through the mapping, the syndrome information is readily available for decoding the KTCs after the stabilizers of the original code have been measured. Figure 3 shows the performances of the resulting decoding algorithm on a bit-flip channel.

Subsystem TSCs— The most general method to protect quantum information is not of a subspace code, but a subsystem code [16]. A family of Topological Subsystem Color Codes (TSCC) [6] was recently constructed that pushes the features of topological codes to their extreme. Indeed, the syndrome extraction for these codes requires only two-qubit measurements, as simple as it can possibly get. Additionally, the entire Clifford group can be performed in a topological fashion on the qubits encoded in these codes. For these reasons, TSCC may well be the simplest and most efficient means of achieving fault tolerance.

To decode the TSCCs, we used a simplified version of the technique presented above. The TSCC has 2 elementary particles and 2 logical qubits, just like the KTC. We mapped the excitations of the TSCC to the excitations of the KTC, losing five qubits out of six, which corresponds to the number of gauge generators not in the stabilizer group of the TSCC. The nature of the 2 defects has changed, going from fermions to bosons. Nevertheless, the homology of string operators is preserved. This map, even though not a Clifford map, enabled the decoding of the TSCC. We obtained a 2% threshold under depolarizing noise (cf Fig. 3).

The importance of this decoding strategy for the TSCC

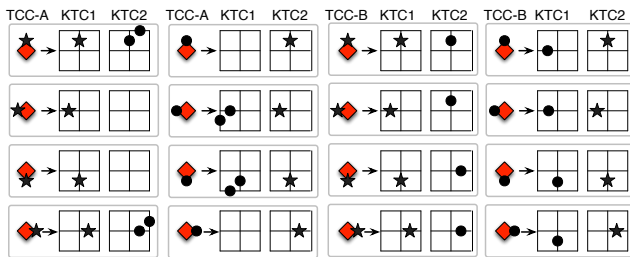


Figure 2: Mapping between the 1-qubit Pauli operators of the square-octagon TCC and two copies of Kitaev's code KTC1, KTC2. The first (last) two columns are for the A (B) sub-lattice. Circles (stars) represent σ_z (σ_x) operators. For instance, the upper left diagram indicates that a σ_x located at the top of a diamond of the A sub-lattice gets mapped to a σ_x on KTC1 and two σ_z on KTC2. All commutation relations are preserved by this mapping, so it is unitary and obviously local.

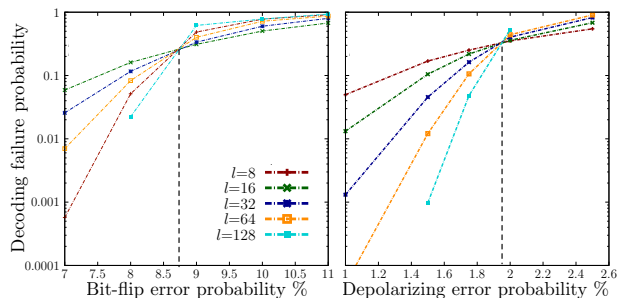


Figure 3: Decoding failure probability as a function of the error probability of each qubit for the octagon-square TCC (left) and TSCC (right), based on the algorithm of [11]. The different curves illustrate lattices of different linear size l : below a threshold probability (dotted lines), the decoding failure probability decreases with the lattice size, leading to a perfect recovery in the infinite-size limit.

can be stressed by the fact that it is not limited to this specific model but can be adapted to any topological subsystem stabilizer codes. See [19], for an example of decoder tailored for a specific topological subsystem code. Another motivation comes from Bravyi's recent proof that geometrically local subsystem codes are much more powerful than geometrically local subspaces codes in that they can achieve better tradeoffs between minimal distances and encoding rates [5].

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