Information Causality is a Special Point in the Dual of the Gray-Wyner Region

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Information Causality puts restrictions on the amount of information learned by a party (Bob) in a one-way communication problem. Bob receives an index b, and after a one-way communication from the other party (Alice), tries to recover a part of Alice's input. Because of the possibility of cloning, this game in its completely classical form is equivalent to one in which there are several Bobs indexed by b, who are interested in recovering different parts of Alice's input string, after receiving a *public* message from her. Adding a *private* message from Alice to each Bob, and assuming that the game is played many times, we obtain the Gray-Wyner problem for which a complete characterization of the achievable region is known. In this paper, we first argue that in the classical case Information Causality is only a single point in the dual of the Gray-Wyner region. Next, we show that despite the fact that cloning is impossible in a general physical theory, the result from classical world carries over to any physical theory provided that it satisfies a new property. This new property of the physical theory is called the Accessibility of Mutual Information and holds in the quantum theory. It implies that the Gray-Wyner region completely characterizes all the inequalities corresponding to the game of Information Causality. In other words, we provide infinitely many inequalities that Information Causality is only one of them.

In the second part of the paper we prove that Information Causality leads to a non-trivial lower bound on the communication cost of simulating a given non-local box when the parties are allowed to share *entanglement*. We also consider the same problem when the parties are provided with preshared randomness.

Non-locality is arguably the most fundamental feature of quantum physics. Bell's theorem [1], as verified by experiments [2], states that there are correlations in nature that cannot be explained by local realistic (classical) theories. Bell's inequalities restrict the strength of classical correlations, while in the quantum theory correlations are characterized by Tsirelson's bounds [3]. The latter bounds, however, heavily rely on the seemingly ad hoc postulates of quantum mechanics. On the other hand, non-locality, the property that makes physical theories to depart from the classical ones, is a fundamental feature of nature rather than quantum mechanics by itself. Tsirelson's bounds then do not provide a satisfactory answer to the problem of quantifying non-locality in nature.

Recently, there has been a stream of works to understand non-locality from more fundamental principals. No-signaling as the first such principal does not describe correlations of quantum physics since non-signaling PR-boxes [4] maximally violate the Tsirelson bound (and then Bell's inequality) for the CHSH expression [5] and do not seem to be physical. Nevertheless, the recently proposed principal of Information Causality [6], a generalization of no-signaling, exactly gives Tsirelson's quantum bound for the CHSH. Thus this is a natural question whether Information Causality or other information theoretic principals can further our understanding of non-locality.

Information Causality and the Gray-Wyner problem: The underlying game of Information Causality is as follows. Alice receives the bit-string $\overrightarrow{a} = (a_1, \dots, a_N)$ consisting of i.i.d. random bits, and Bob gets an index $1 \leq b \leq N$. Bob's goal is to output a_b upon receiving a classical message x from Alice. Assuming that β_i is Bob's guess of a_i when $b = i$, Information Causality states that

$$H(x) \geq \sum_{i=1}^{N} I(a_i; \beta_i | b = i). \tag{1}$$

It turns out one can write this inequality in terms of entropies:

$$H(x) + \sum_{i=1}^{N} H(a_i | \beta_i, b = i) \geq H(\overrightarrow{a}). \tag{2}$$

Now consider this game in its completely classical form. Classicality enables us to assume that there are N Bobs instead of one. We denote these N Bobs by $\mathrm{Bob}_1, \dots, \mathrm{Bob}_N$, where the goal of Bob_i is to recover a_i. Moreover, we may assume that shared randomness is indeed shared amongst Alice and all Bobs, and all of them receive the

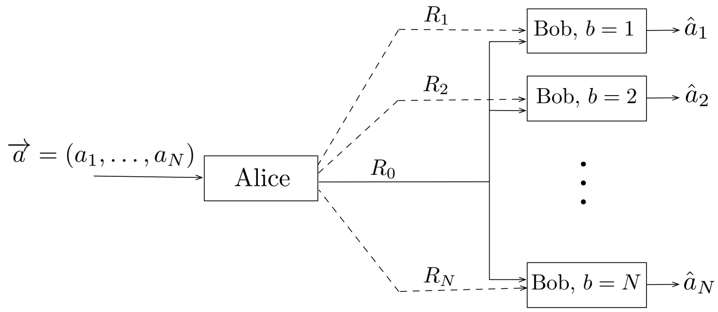

FIG. 1: The Gray-Wyner game consists of $N+1$ players, Alice and N Bobs who are indexed by $b = 1, \ldots, N$. Alice receives the i.i.d. copies of $(a_1, \ldots, a_N)$, sends public information at rate R_0 to all Bobs and private information at rate R_i to Bob_i. The goal of Bob_i is to recover a_i.

message x. Then the first term of equation (2) is the amount of information that is sent to all Bobs; the second term $H(a_i|\beta_i, b = i)$ expresses the remaining uncertainty of Bob_i about a_i. We can interpret this as the average number of extra bits that Alice needs to privately send to Bob_i to enable the recovery of a_i by this party if they were to play multiple copies of this game in parallel (the Slepian-Wolf theorem). This is similar to the setup of the Gray-Wyner problem [7], see Fig. 1. In this problem the goal is to find the region of rates of public (R_0) and private ($R_1, \ldots, R_N$) messages such that Bobs' demands is achievable. The Gray-Wyner region explicitly characterizes the set of such tuples $(R_0, R_1, \ldots, R_N)$. It turns out that the rates $R_0 = H(x)$ and $R_i = H(a_i|\beta_i, b = i)$ have to lie in the Gray-Wyner region when the Information Causality game is played in the classical world, and (2) follows as a corollary of the characterization of the Gray-Wyner region.

A main contribution of our work is that despite the fact that the cloning of Bob is impossible in a general physical theory [8, 9], the tuple

$$(H(x), H(a_1|\beta_1, b = 1), \ldots, H(a_N|\beta_N, b = N)) \tag{3}$$

would still fall in the Gray-Wyner region if the underlying physical theory satisfies the following properties. First, a consistent definition of mutual information with the three properties of [6] should be available. Second, since we are writing Information Causality in terms of entropy rather than mutual information (inequality (2)), we need an entropy function consistent with the mutual information. Third and more importantly we need the following:

Accessibility of Mutual Information (AMI): Consider arbitrary subsystems A and B where A is classical. Let $(A_1, B_1), (A_2, B_2), \ldots, (A_n, B_n)$ denote n independent copies of (A, B). Then for any $\epsilon > 0$, there exists some n and a local state transformation $B_1, \ldots, B_n \to e_n$, such that e_n is classical and

$$\frac{1}{n} I(A_1 \ldots A_n; e_n) \geq I(A; B) - \epsilon.$$

AMI ensures the existence of certain maps in the underlying physical theory. From the other properties that we consider it is only the data processing inequality that talks about the set of valid state transformations. But this property becomes trivial for example in a physical theory whose only valid state transformation is the identity map. Thus to avoid such obscure examples, an information theoretic approach to study physical theories has to provide a postulate about the richness of the space of valid state transformations. From this point of view AMI is a natural postulate. Moreover, this is the property that formulates our intuition of mutual information, and otherwise the function of mutual information on non-classical systems has no tangible meaning.

Here we should point out that putting AMI aside, the other four properties mentioned above can be replaced by only two constraints on the entropy [10, 11]. Moreover, AMI holds in the quantum theory because the Holevo outer bound on the accessible information is asymptotically achievable by the pretty good measurement.

As a result of our work, Information Causality (2) is a special point in the dual of the Gray-Wyner region. Moreover, the Gray-Wyner region completely characterizes all the inequalities corresponding to the game of Information Causality in the following sense. On one hand, (3) has to be in the Gray-Wyner region. On the other hand, any point in the Gray-Wyner region is achievable, meaning that it can be obtained through a communication scheme in the classical world.

These new inequalities are indeed strictly stronger than (2). For instance, consider the following scenario where Bob receives either an index $1 \leq b \leq N$ or two indices $1 \leq b_1, b_2 \leq N$. In the former case he wants to recover a_b

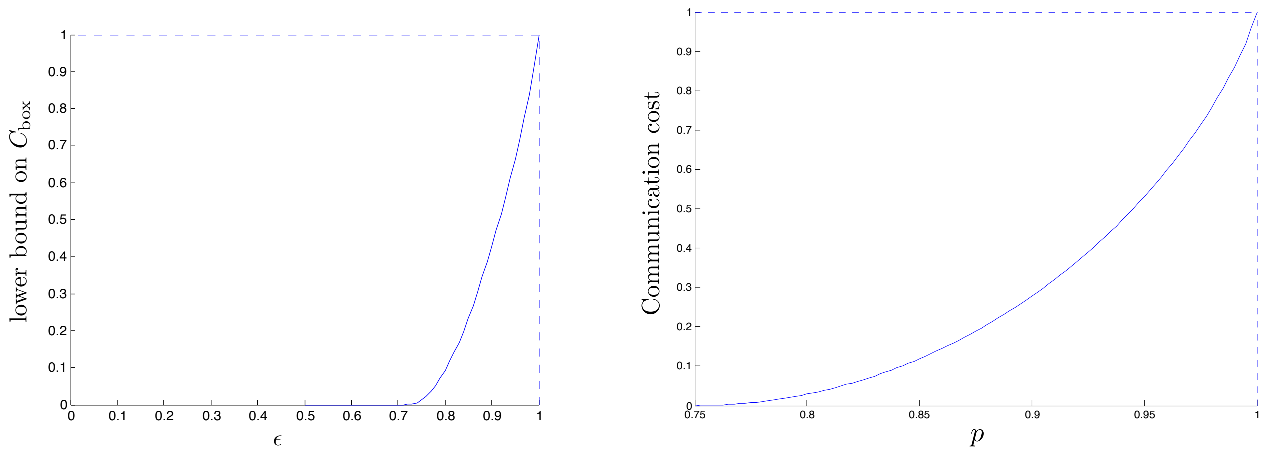

FIG. 2: (I) The left plot is a lower bound on C_{box}, the entanglement-assisted one-way communication cost of simulating imperfect PR-boxes with parameter ϵ, where $p(x \oplus y = ab) = \frac{1+\epsilon}{2}$. This lower bound is an implication of Information Causality. (II) The right plot gives the one-way communication cost of winning the CHSH game with probability $p = \frac{1+\epsilon}{2}$ assuming preshared randomness.

and in the latter both a_{b_1} and a_{b_2}. In this case it is easy to see that (2) is loose while we offer much stronger inequalities.

Simulation of Non-Local Correlations: Finding the amount of classical communication required to simulate non-local correlations is a well-known way to quantify non-locality (see e.g. [12–19]). Given a certain non-local box, we consider the problem of the minimum amount of one-way communication from Alice to Bob required to simulate the box. Here we assume that Alice and Bob have infinite shared randomness, and similarly the entanglement-assisted version of this problem can be considered. We prove that Information Causality leads to a lower bound on the communication cost of simulating a given non-local box when the two parties are provided with preshared *entanglement*.

This lower bound for imperfect PR-box with bias ϵ (the box with binary inputs a, b and outputs x, y such that $p(x \oplus y = ab) = \frac{1+\epsilon}{2}$) is given in Fig. 2. We see that the lower bound is equal to one at $\epsilon = 1$, thus it has to be tight at this point. By [6] this bound is also tight at the other end point $\epsilon \leq \frac{1}{\sqrt{2}} \simeq 0.70$. However, it may be loose in between because this lower bound holds more generally for any physical theory satisfying properties of mutual information given in [6] and not only for quantum physics. Nonetheless, observe that the lower bound at $\epsilon = 1$ is tight in any such physical theory.

Non-local box simulation from an Information theoretic perspective: We also do have a non-technical contribution if one is interested in the communication cost of simulating a given non-local box when the parties only share common randomness. Information theorists who have been interested in the area of control have independently studied the same problem in a different context [20–23]. The communication complexity formulation of the problem turns out to be a very difficult one. However, information theoretic formulation of the problem looks at the limits of the problem and takes the advantage of laws of large numbers. Connecting these two lines of research, we report a formula that gives an *exact* expression for the optimal amount of communication needed for non-local simulation given preshared randomness. It should be also noted that the information theoretic characterization of the communication cost serves as a lower bound on the communication complexity characterization of the bound, because the former setup considers asymptotic behaviors and is more relaxed.

Summarizing this result, take an arbitrary bipartite box $p(x, y|a, b)$ and fix a distribution $p(a, b)$ on a, b as well. The cost of simulating this box equals the maximum of $I(a; u|b)$ over all classical random variables u determined by $p(u|a, b, x, y)$ such that the joint distribution factorizes as $p(u, a, b, x, y) = p(a, b)p(u|a)p(x|u, a)p(y|u, b)$ [24]. See the right plot of Fig. 2 for the example of PR-boxes.

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