Increasing Entanglement by Separable Operations and New Monotones for W-type Entanglement

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In this talk, we seek to better understand the structure of local operations and classical communication (LOCC) and its relationship to separable operations (SEP). To this end, we compare the abilities of LOCC and SEP for distilling EPR entanglement from one copy of an N-qubit W-class state (i.e. that of the form $\sqrt{x_0}|00...0\rangle + \sqrt{x_1}|10...0\rangle + ... + \sqrt{x_n}|00...1\rangle$). In terms of transformation success probability, we are able to quantify a gap as large as 37% between the two classes. Our work involves constructing new analytic entanglement monotones for W-class states which can increase on average by separable operations. Additionally, we are able to show that the set of LOCC operations, considered as a subset of the most general quantum measurements, is not closed.
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In the "distant lab" setting of quantum information processing, entanglement is shared among spatially separated parties, and its manipulation is implemented through local quantum operations coordinated by classical communication (LOCC). However, despite its fairly intuitive physical description, the mathematical structure of LOCC operations is quite complex with many important questions still unanswered. As a result, the simpler and more general class of separable operations (SEP) is used as an approximation for LOCC. This class consists of all completely positive (CP) maps allowing for a Kraus representation with operators of the form $\Pi_\lambda = M_\lambda^{(1)} \otimes M_\lambda^{(2)} \otimes ... \otimes M_\lambda^{(N)}$.

Nevertheless, it is well-known that LOCC $\subsetneq$ SEP where the inclusion is proper [1]. A dramatic example of this latter fact is the phenomenon of "non-locality without entanglement," a term originally used to describe sets of orthogonal product states indistinguishable by LOCC [1]. Beyond the relationship LOCC $\subsetneq$ SEP, very little is known about the precise difference between these two. For instance, how much more powerful is SEP than LOCC? Do there exist functions which behave monotonically under LOCC but can be increased on average by SEP?

In this talk, we provide the first quantitative analysis of the difference between SEP and LOCC. Our measure for comparison is success probability in certain multiqubit entanglement transformations. Specifically, we turn to the task of random distilling an EPR pair from one copy of a multipartite state, as first initiated by Fortescue and Lo [2, 3]. Letting $|\varphi\rangle_{1...N}$ denote a general N-qubit state and $|\Phi^{(ij)}\rangle = \sqrt{1/2}\,(|00\rangle + |11\rangle)$ denote an EPR pair shared between parties i and j, a random EPR distillation is the multi-outcome transformation

$$|\varphi\rangle_{1...N} \to \{|\Phi^{(ij)}\rangle \text{ with probability } p_{ij}\}_{i<j} \tag{1}$$

and $\sum_{i<j}^N p_{ij} \leq 1$. We consider how the abilities of LOCC and SEP differ for this transformation.

Our investigation focuses on N-qubit random distillations when the initial state belongs to the W Class of states, thus taking the form

$$|\vec{x}\rangle = \sqrt{x_0}|00...0\rangle + \sqrt{x_1}|10...0\rangle + \sqrt{x_2}|01...0\rangle + \sqrt{x_N}|00...1\rangle$$

up to some local unitary (LU) operation. The primary reason for focusing on W-class states is that their entanglement properties are easy to analyze. Specifically, each W-class state can be represented by an N-component vector $\vec{x} = (x_1, ..., x_N)$ where the x_i correspond to the components in $|\vec{x}\rangle$, and it is not difficult to monitor how these components change under a local measurement [4].

However, even for W-class states, it is difficult to decide LOCC feasibility of a general random distillation specified by Eq. (1). In our talk, we focus on two particular types of transformation: a combing-type and a complete-type (see Fig. 1). A combing-type transformation is the one-shot analog of "entanglement combing" studied by Yang and Eisert [5] in which one particular party is selected to be a shareholder of the bipartite entanglement for each of the possible outcomes. In a

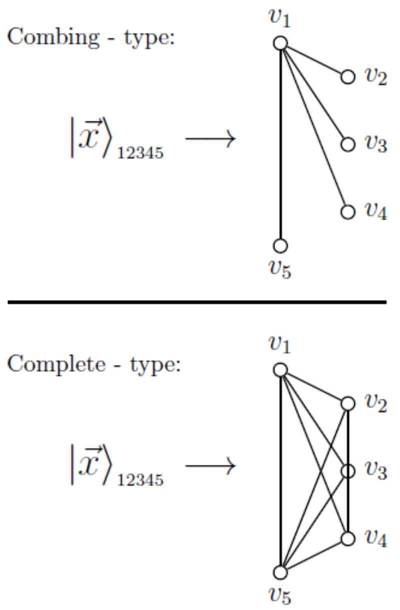

FIG. 1: A "combing-type" versus "complete-type" distillation for five parties. Each of the parties are assigned a node, and nodes v_i and v_j are connected if and only if it is desired for parties i and j to be EPR entangled with some nonzero probability.

complete-type transformation, the transformation is considered a success if any two parties ends up sharing an EPR state, regardless of who they are.

Solution to LOCC Combing-Type and Complete-Type Transformations

For an N-party W-class state $\vec{x} = (x_1, x_2, ..., x_N)$, set $\{n_1, n_2, ..., n_N\} = \{1, 2, ..., N\}$ such that $x_{n_1} \geq x_{n_2} \geq ... \geq x_{n_N}$ and consider the functions:

$$\eta(\vec{x}) = x_{n_1} - \left(\frac{1}{x_{n_1}}\right)^{N-2} \prod_{i=2}^{N} (x_{n_1} - x_{n_i})$$

$$\kappa(\vec{x}) = \sum_{i=2}^{N} x_{n_i} + \eta(\vec{x}). \tag{2}$$

By decomposing each local measurement into a sequence of weak measurements, we prove the following theorem.

Theorem 1.

(I) *η is non-increasing on average for any single local measurement in which n_1 is the same value for the initial and all possible final states,*

(II) *κ is an entanglement monotone. It is strictly decreasing on average for any non-trivial measurement by party n_1.*

The functions η and κ provide tight upper bounds to combing-type and complete-type transformations.

Theorem 2. *For an N-party W-state $\vec{x} = (x_1, x_2, ..., x_N)$, let P_{tot} be the optimal total probability of obtaining an EPR pair by LOCC (i.e. a complete-type transformation), and P_k the optimal total probability of party k becoming EPR entangled (i.e. a combing-type transformation w.r.t. party k). Then*

(I) $P_{tot} < \kappa(\vec{x})$, *and*

(II) $P_k \leq \begin{cases} 2x_k & \text{if } x_k < x_l \text{ for some } l \\ 2\eta(\vec{x}) & \text{if } x_k \geq x_l \text{ for all } l. \end{cases}$

When $x_0 = 0$, the upper bound in (I) *can be approached arbitrarily close while in* (II) *it can be achieved exactly.*

General Random Distillation by Separable Operations

We are show that for W-class states, when given some collection of p_{ij} such that $\sum_{i<j}^{N} p_{ij} \leq 1$, deciding whether transformation (1) is possible by SEP can be solved using semi-definite programming. For the special case when the initial state is $|W_N\rangle = \sqrt{1/N}\,(|10...0\rangle + |01...0\rangle + ... + |00...1\rangle)$, the solution takes a relatively simple form.

Theorem 3. *For $|\varphi_{1...N}\rangle = |W_N\rangle$, let E be the set of (i,j) such that $p_{ij} > 0$. Then transformation* (1) *is possible by separable operations if and only if*

$$\frac{N^2}{4} \sum_{(i,j)\in E} p_{ij}^2 \leq 1, \quad \frac{N}{2} \sum_{(i,j)\in E_k} p_{ij} \leq 1, \quad 1 \leq k \leq N.$$

LOCC vs. SEP Comparison

Theorems 2 and 3 allow us to make a direct comparison between the powers of LOCC and SEP for implementing transformation (1). For example, with a general three-qubit W-class state $\sqrt{x_1}|100\rangle + \sqrt{x_2}|010\rangle + \sqrt{x_3}|001\rangle$ with $1/2 \geq x_1 \geq x_2 \geq x_3$ and $x_0 = 0$, separable operations can randomly distill with probability 1. What's particularly interesting is that such perfect distillation in the range $1/2 \geq x_1 \geq x_2 \geq x_3$ is precisely the same as if Bob and Charlie were allowed to work together in distilling entanglement across the bipartition A:BC. Thus, we see that SEP saturates the bipartite bound while LOCC is bounded by $\kappa(\vec{x})$. As an example, we compare κ-type distillation rates on the one parameter family of states

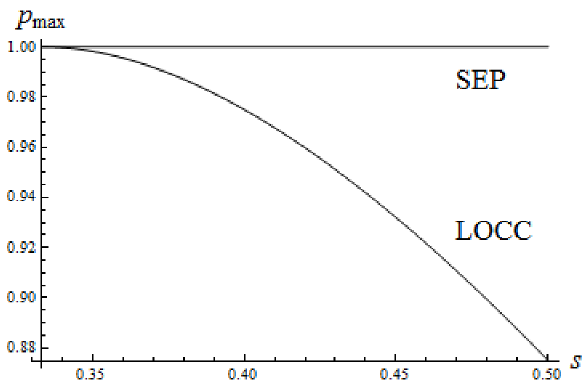

FIG. 2: LOCC vs. SEP for the maximum probability of obtaining an EPR pair between any two parties as a function of s when the initial state is $\sqrt{s}|100\rangle + \sqrt{\frac{1-s}{2}}(|010\rangle + |001\rangle)$. The LOCC probability is $2(1-s) - \frac{(1-s)^2}{4s}$. A gap of 12.5% exists between SEP and LOCC.

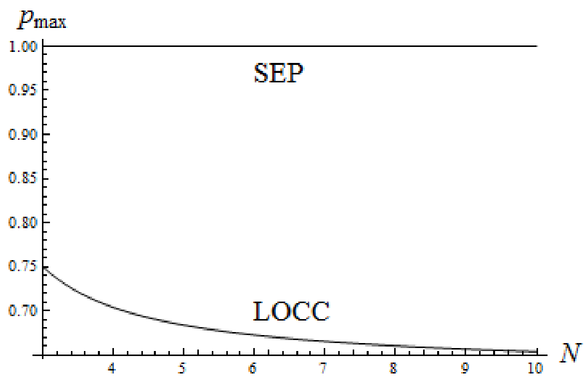

FIG. 3: LOCC vs. SEP for the maximum probability of party 1 become EPR entangled as a function of N when initial state is $\sqrt{\frac{1}{2}}|10....0\rangle + \sqrt{\frac{1}{2(1-N)}}(|01...0\rangle + ... + |00...1\rangle)$. The LOCC probability is $1 - (1 - \frac{1}{N-1})^{N-1}$. A gap of 37% exists between SEP and LOCC.

$|\psi_s\rangle = \sqrt{s}|100\rangle + \sqrt{\frac{1-s}{2}}(|010\rangle + |001\rangle)$ for $\frac{1}{3} \leq s \leq \frac{1}{2}$. The LOCC optimal probability is given by

$$\kappa(\psi_s) = 2(1-s) - \frac{(1-s)^2}{4s}. \tag{3}$$

The comparison of SEP versus LOCC is depicted in Fig. 2.

Next, we compare the probabilities for a combing-type transformation on states of the form $|\psi_{1/2}\rangle_{1...N} = \sqrt{\frac{1}{2}}|10....0\rangle + \sqrt{\frac{1}{2(1-N)}}(|01...0\rangle + ... + |00...1\rangle)$. By LOCC, the optimal probability is

$$2\eta(\psi_{1/2}) = 1 - (1 - \frac{1}{N-1})^{N-1} \longrightarrow 1 - e^{-1} \tag{4}$$

where we have taken the limit for large N. However, separable operations can achieve the transformation with unit probability. We plot this separation between LOCC and SEP as a function of N in Fig. 3.

LOCC is Not a Closed Set of Operations

An immediate consequence of our monotones pertains to the question of whether the set of LOCC operations is topologically closed. Intuitively, LOCC closure, denoted by $\overline{\text{LOCC}}$, consists of all LOCC maps and their sequential limits. Thus, to show that LOCC is not closed, it is sufficient to construct a sequence (indexed by i) of LOCC transformations

$$|\varphi\rangle \to \{p^i_j, |\varphi^i_j\rangle\}_{j=1...m} \qquad \text{for } i = 1, 2, ...$$

such that the transformation

$$|\varphi\rangle \to \{\overline{p_j}, |\overline{\varphi_j}\rangle\}_{j=1...m}$$

is not implementable by LOCC, where $p^i_j \to \overline{p_j}$ and $|\varphi^i_j\rangle \to |\overline{\varphi_j}\rangle$.

The Fortescue-Lo Protocol given in Ref. [2] describes a sequence of complete-type distillations $\mathcal{T}_n$ on $|W_3\rangle$ that succeeds with total probability $p_{AB} + p_{AC} + p_{BC} > 1 - 1/n$. Thus the limit transformation $\overline{\mathcal{T}}$ succeeds with probability 1. However, from our entanglement monotone κ, it is easy to see that a deterministic random distillation of $|W_3\rangle$ is impossible by LOCC. This is because at some point in the protocol party n_1 must perform a non-trivial measurement, and by Theorem 1 (II) this will cause κ to be strictly less than 1. Hence, a deterministic transformation is not possible and therefore LOCC is not a closed set of operations.

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