STABILITY OF FRUSTRATION-FREE HAMILTONIANS

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Abstract. We prove stability of the spectral gap for gapped, frustration-free Hamiltonians under general, quasi-local perturbations. We present a necessary and sufficient condition for stability, which we call *Local Topological Quantum Order*. This result extends previous work by Bravyi *et al.* on the stability of topological quantum order for Hamiltonians composed of commuting projections with a common zero-energy subspace. Moreover, we show that *Local Topological Quantum Order* implies a bound on the entanglement entropy of the groundstates.

Recent interest in topological quantum computation has focused the attention of the condensed matter and mathematical physics community on Hamiltonians whose low-energy sectors exhibit some form of topological order. In a seminal paper by Kitaev [15], a Hamiltonian known as the *toric code*, was constructed out of commuting spin-interaction terms, such that the groundstate subspace exhibits four-fold degeneracy which may only be detected under macroscopic operations. In other words, the different groundstates are indistinguishable on microscopic and mesoscopic scales, which implies that local errors cannot create logical errors in any encoding that utilizes such topologically ordered groundstates as the encoded state of two qubits.

Nevertheless, if the groundstate subspace were to become mixed with the non-topological higher-energy sectors, then any guarantee of protection from external errors would no longer be valid. At this point, another type of stability against errors is required, this time at the level of the spacing between the low-energy and high-energy sectors. Progress in this direction was made by Klich [16] using the method of cluster expansions, and soon after by Bravyi *et al.*[5, 4]. In particular, the latter result showed that all Hamiltonians with commuting spin-interaction terms have a spectral gap that is stable under local perturbations, as long as the groundstates satisfy some type of *local indistinguishability* and *frustration-freeness*.

Since the *toric code* Hamiltonian satisfies both conditions, it follows that encodings of qubits on the toric code, the groundstate subspace of the Hamiltonian, are robust against local errors, even if the Hamiltonian interactions are not precisely engineered. Even so, the toric code has a, seemingly, fatal flaw; at non-zero temperature, local excitations, once created, can travel around the torus to create logical errors, at no extra energy cost. Motivated by this issue, soon after the results of Klich and Bravyi *et al.*, a series of papers appeared that focused on the beneficial effects of Anderson-type localization of the undesired excitations in the presence of impurities, or external magnetic fields [14, 22, 23].

Still, until recently, no Hamiltonian model was known that combined the desirable properties of the *toric code*, with a rigorous and sufficiently large lower bound on the energy barrier restricting the mobility of unwanted excitations at non-zero

Date: November 3, 2011.

temperature. In fact, it was shown by Yoshida [24], that such Hamiltonians would need to either forgo commutativity or break certain "natural" conditions, such as translational invariance or scale invariance, in order to satisfy stability at non-zero temperature. Indeed, the only known family of Hamiltonians with all the desired properties, recently presented by Haah in [8], has logical operators with fractal geometry in three dimensions and no scale invariance, thus sidestepping Yoshida's no-go theorem. Yet, despite the rigorous bound [3] on the energy barrier of Haah's Hamiltonian, there remains a question of whether the barrier is large enough to allow for operations on the logical qubits, like read-out and error-correction, which may require times comparable to the time it takes for logical errors to appear. Moreover, it was recently shown by Hastings [11], that all two-dimensional Hamiltonians which are a sum of commuting terms, have no topological order at non-zero temperature.

Motivated by this line of research and the larger question of the classification of quantum phases [6, 1, 21], we present here a generalization of the result by Bravyi *et al.*, which removes the commutativity of the Hamiltonian terms as an assumption for stability. Some of the new candidate Hamiltonians now include, parent Hamiltonians of Matrix Product States (MPS) and Projected Entangled Pair States (PEPS) [20], as well as all other *frustration-free* Hamiltonians - that is, Hamiltonians whose groundstates minimize the energy of each local interaction term. Moreover, we generalize the conditions needed for the stability of the spectral gap, in hopes that in the future, one may be able to prove an equivalent result for general, gapped Hamiltonians, whose groundstates satisfy some type of topological order.

The stability of quantum phases was already studied by Borgs *et al.*[2] and Datta *et al.*[7], were "classical" systems were shown to be robust against small quantum perturbations up to some low-temperature, using the methods of contour and cluster expansions. Nevertheless, here, we draw heavily from the methods developed in [5, 4], following the more succinct format of [4], which uses Hastings' powerful *quasi-adiabatic continuation* [13]. We make extensive use of Lieb-Robinson bounds, both for the evolution of operators according to the Hamiltonians under consideration [9, 12, 18], as well as Lieb-Robinson bounds on the quasi-adiabatic evolution of the low-energy eigenspaces [10, 1].

We begin by defining the class of frustration-free Hamiltonians whose stability we proceed to study. We, then, introduce the *Local-TQO* and *Local-Gap* conditions sufficient for proving stability, clarifying with examples the extend to which the conditions are also necessary. Next, we define the class of perturbations we allow and proceed to transform the perturbed Hamiltonian, through a unitary transformation and a global energy shift, in a form amenable to studying its low energy sectors, using the concept of *relatively bounded perturbations*. We follow the bootstrapping argument of [4] to conclude that for weak enough perturbations, the spectral gap remains open.

To conclude, we show that topological phases corresponding to the groundstate sector of gapped, frustration-free Hamiltonians are stable under quasi-local perturbations. Moreover, our result implies the stability of symmetry-protected low-energy sectors under perturbations that obey the symmetry of that sector, as long as these perturbations satisfy the *Local-TQO* condition. Finally, it can be shown

[19] that parent Hamiltonians of Matrix Product States (MPS) satisfy the *Local-TQO* condition. Combined with the result of Nachtergaele [17] on the spectral gap of one-dimensional, frustration-free Hamiltonians, our result shows that parent Hamiltonians of MPS have stable low-energy spectrum against arbitrary, weak, local perturbations. We expect the same result to hold for two-dimensional, gapped parent Hamiltonians of Projected Entangled Pair States (PEPS).

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