

today: probability review

why? → principled framework to model uncertainty

Sources of uncertainty

1) intrinsic uncertainty → quantum mechanics

2) partial information/observation:  
• card games  
• rolling a dice → don't know the initial conditions

3) incomplete modeling of a complex phenomenon  
(computation issues also important)

example: "most birds can fly"  
→ simple rule is advantage but then yields uncertainty

notation:  $X_1, X_2, X_3$        $X, Y, Z$

↳ random variables  
(usually real-valued)

$x_1, x_2, x_3$        $x, y, z$

↳ their "realizations"

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$X$  a random variable  $\rightarrow$  represents an uncertain quantity

$X=x$  represents the "event" that  $X$  takes the value  $x$

$\Omega \rightarrow$  the sample space of "elementary events"  
possible values for my R.V.

e.g.  $X$  is result of a die throw  
 $\Omega = \{1, 2, 3, 4, 5, 6\}$

two types of R.V.: discrete where  $\Omega$  is countable  
continuous ( $\Omega$  is uncountable)

[assume  $\Omega$  is countable]

R.V.  $X$  is characterized by a probability mass function (pmf)

$p(x)$  for  $x \in \Omega$

$$\text{s.t. } \begin{cases} p(x) \geq 0 \quad \forall x \\ \sum_{x \in \Omega} p(x) = 1 \end{cases}$$

a probability distribution  $\mathcal{P}$

is a measure  $\mathcal{P}: \Omega^{\mathcal{R}} \rightarrow [0, 1]$

is a mapping  $P: 2^\Omega \rightarrow [0,1]$

$2^\Omega = \mathcal{E} \triangleq$  set of all subsets of  $\Omega$

↑ set of "events"

('σ-field' in measure theory needed when  $\Omega$  uncountable)

notation:  $P\{X=x\}$

$P\{X=x \text{ or } X=y\} = p(x) + p(y)$  if  $x \neq y$

Kolmogorov axioms

$$\begin{cases} P(E) \geq 0 \quad \forall E \in \mathcal{E} \\ P(\Omega) = 1 \\ P(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} P(E_i) \text{ when } E_i \text{'s are disjoint} \end{cases}$$

for discrete R.V.  $P(E) = \sum_{x \in E} p(x)$

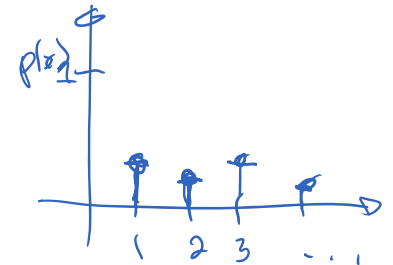
continuous R.V. is characterized by a probability density function  $p(\text{pdf})$

$$\begin{cases} p(x) \geq 0 \quad \forall x \in \Omega \\ p \text{ is integrable and } \int_{\Omega} p(x) dx = 1 \end{cases}$$

$\Omega = \mathbb{R} :$   $P([a,b]) = \int_a^b p(x) dx$

It has measure ..

discrete case



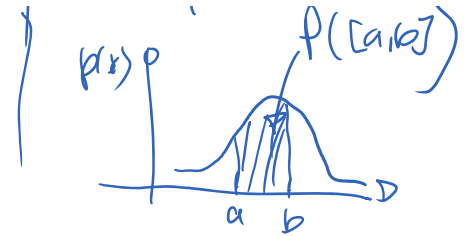
cts. case



Ja

[ sidenotes can change p on countable # of pts without changing "anything" ]

It has measure zero according to Lebesgue measure



4.1.4.0

notation:  $f_X(x)$   $f_Y(y)$   
 $f_Z(x)$  when  $x=y$

Recap:

discrete R.V.  $X$ ; pmf  $p(x)$   $\Leftrightarrow P\{X=x\} = p(x)$

cts R.V.  $X$ ; pdf  $p(x)$  " $P\{X \in x \pm \frac{dx}{2}\} = p(x)dx$ "

$$P\{X \in x \pm \frac{a}{2}\} = \int_{x-\frac{a}{2}}^{x+\frac{a}{2}} p(u) du$$

joint, marginal, etc. (multivariate R.V.)

$$Z = (X, Y) \quad \Omega_Z \triangleq \Omega_X \times \Omega_Y$$

pmf of  $Z$  is "joint pmf" on  $X$  &  $Y$

$$p(x, y) = P\{X=x, Y=y\}$$

... result ...

$$p(x,y) = P\{X=x, Y=y\}$$

↑ joint distribution

Say  $X$  is whether die<sup>result</sup> is even  
 $Y$  is whether die is odd

can represent elementary events of  $\Omega_{(x,y)}$  as table

	$X=0$	$X=1$
$Y=0$	0	$1/2$
$Y=1$	$1/2$	0

"and"  
↓  
if  $X$  &  $Y$  are dts.,

$$P\{\text{box}\} = \int_{\text{box}} p(x,y) dx dy$$

↑ joint pdf

marginal distribution (in the context of the joint)

↳ distribution of a component of a random vector

$$P\{X=x\} = \sum_{y \in \Omega_Y} P\{X=x, Y=y\}$$

"sum rule"

"marginalizing out  $Y$ "

independence :

$$X \text{ is independent of } Y \Leftrightarrow p(x,y) = p(x)p(y) \quad \forall (x,y) \in \Omega_X \times \Omega_Y$$

notation:  $X \perp\!\!\!\perp Y$

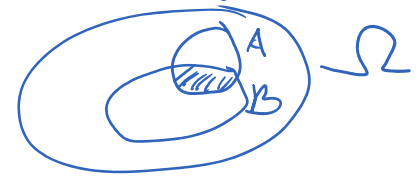
R.V.  $X_1, \dots, X_n$  are "mutually independent"  $\Leftrightarrow p(x_1, \dots, x_n) = \prod_{i=1}^n p(x_i) \quad \forall x_{1:n} \in \prod_{i=1}^n \Omega_{x_i}$

Conditioning:

- for events  $A$  &  $B$ , suppose  $P(B) \neq 0$

then define  $P(A|B) \triangleq \frac{P(A \cap B)}{P(B)}$

we know:  $P(B) = \sum_{A \in \text{partition of } \Omega} P(A \cap B)$



- for a discrete R.V.

define "conditional pmf"

$$p(x|y) \triangleq P(X=x | Y=y) \triangleq \frac{P(\{X=x, Y=y\})}{P(\{Y=y\})} \leftarrow \text{joint pmf}$$

$\leftarrow$  marginal pmf

normalization constant  $\rightarrow$

- for cts. random variable

"conditional pdf"  $p(x|y) \triangleq \frac{p(x,y)}{p(y)}$  } density

subtle point:  $p(x|y)$  is undefined when  $p(y) = 0$

• example: "P(having cancer | tumor measurement = a)"

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$$\text{cdf } F_X(x) \triangleq P\{X \leq x\}$$