Lecture 14 - Inference; GraphEliminate; sum-product

Friday, October 21, 2022 3:02 PM

today: inference: graphEliminate
sum-product alg.

graph elimination alg (for inference in generic UGM)

- consider $p \in \mathcal{G}(G)$ undirected $\qquad p(x) = \frac{1}{Z} \prod_{c \in \mathcal{C}} \psi_c(x_c)$

say want to compute $p(x_F)$ for some $F \subseteq V$ "query nodes"

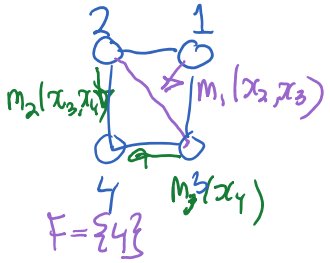
main trick: use distributivity of $\oplus$ over $\odot$ $\rightsquigarrow$ $c \odot (a \oplus b) = c \odot a + c \odot b$

$$\sum_{x_1, x_2} f(x_1) g(x_2) = \left(\sum_{x_1} f(x_1)\right)\left(\sum_{x_2} g(x_2)\right)$$ [convince yourself]

more generally

$$\sum_{x_{1:n}} \prod_{i=1}^{n} f_i(x_i) = \prod_{i=1}^{n} \left(\sum_{x_i} f_i(x_i)\right)$$

$O(K^n \cdot n) \rightsquigarrow O(K \cdot n)$



$$p(x_4) = \frac{1}{Z} \sum_{x_1,x_2,x_3} \psi(x_1,x_2)\,\psi(x_2,x_4)\,\psi(x_1,x_3)\,\psi(x_3,x_4)$$

$$= \frac{1}{Z} \left(\sum_{x_3} \psi(x_3,x_4) \left(\sum_{x_2} \psi(x_2,x_4) \left(\sum_{x_1} \psi(x_1,x_2)\,\psi(x_1,x_3)\right)\right)\right)$$

$\sum_{x_1} \psi(x_1,x_2)\,\psi(x_1,x_3) = m_1(x_2,x_3)$ "message"

$\sum_{x_2} \psi(x_2,x_4)\, m_1(x_2,x_3)$ — stored as a table $= m_2(x_3,x_4)$

$$= \frac{1}{Z} m_3(x_4)$$

↳ last message is proportional to marginal $p(x_4)$

$$\sum_{x_4} m_3(x_4) = Z$$

$$\boxed{p(x_4) = \frac{m_3(x_4)}{Z}}$$

general alg.: graph Eliminate:

init.: [a) choose an elimination ordering s.t. F are the last nodes

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b) put all $\psi_C(x_C)$ on "active list"

"update" c) repeat, in order of variables to eliminate

(say x_i is variable to eliminate)

1) remove all factors from active list with x_i in it & take their product

i.e. $\prod_{\alpha \text{ s.t. } i \in \alpha} \psi_\alpha(x_\alpha)$

2) sum over x_i to get a new factor $m_i(x_{S_i})$ (think as $\psi_{S_i}(x_{S_i})$)

i.e. S_i are all variables in these factors except i

get $m_i(x_{S_i}) \triangleq \sum_{x_i} \prod_{\alpha \text{ s.t. } i \in \alpha} \psi_\alpha(x_\alpha)$

new clique to sum over $S_i \cup \{i\}$

$S_i \triangleq \left(\bigcup_{\alpha \text{ s.t. } i \in \alpha} \alpha\right) \setminus \{i\}$

3) put back $m_i(x_{S_i})$ in active list $(\psi_{S_i}(x_{S_i}))$

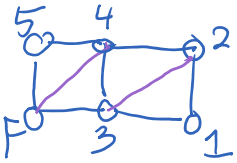
"normalize": d) last product of factors left has only $x_F \Rightarrow$ proportional $p(x_F)$

suppose $x_i \in \{0,1\}$ memory needed $\approx 2^{\max_i |S_i|} \cdot$ (# factor of factors)

computations $= \left(2^{\max_i |S_i|+1}\right) \cdot n$

later, related "treewidth" of a graph

(*) "augmented graph" → graph obtained by running graph Eliminate + keeping track of all edges added (for fixed ordering)



(**) note: augmented graph obtained after graph Eliminate is always a triangulated graph

def.: graph with no cycle of size 4 or more that cannot be broken by a "chord"

an edge between two non-neighboring nodes in cycle

not triangulated

Δ-graph

treewidth of a graph $\triangleq$ min over all possible elimination orderings { size of biggest clique in augmented graph -1 }

↳ convention treewidth(tree) = 1

both memory & running time of graph Eliminate is dominated by $2^{\text{size of biggest clique}}$

best ordering give $\sim 2^{\text{treewidth}+1}$

not all orderings are good

bad news:

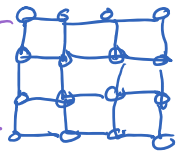
a) NP hard to compute treewidth of a general graph (or find best ordering)

b) NP hard to do (exact) inference in general UGM

$\Rightarrow$ approximate methods

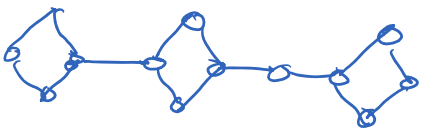
example:

treewidth of a grid $\approx \sqrt{|V|}$ = side = treewidth



good news

* inference is linear time for tree (treewidth = 1) ("sum-product alg.")
  ↳ $|V| + |E|$ (HMM, Markov chain)
* efficient for "small treewidth graph"
  ⇝ use junction tree alg.



IGM II

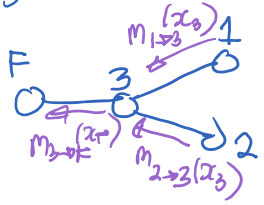
Inference on trees

graph Eliminate on a tree
→ good order is to eliminate leaves first

$m_{i \to j}(x_j) = 1$

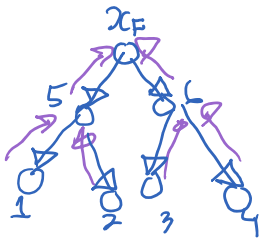
$p(x) = \frac{1}{Z} \prod_i \psi_i(x_i) \prod_{i,j} \psi_{ij}(x_i, x_j)$

→ good order is to eliminate leaves first



$$p(x) = \frac{1}{Z}\prod_i \psi_i(x_i)\prod_{\{i,j\}\in E}\psi_{ij}(x_i,x_j)$$

order: make a directed tree by using x_F as a root (singleton)



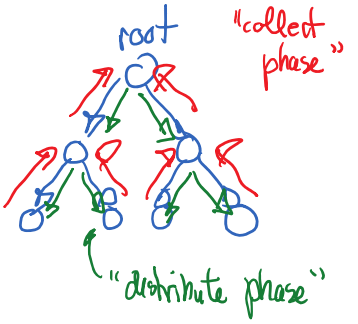
$$m_{i\to j}(x_j) = \sum_{x_i}\psi_i(x_i)\psi_{ij}(x_i,x_j)\prod_{k\in \text{children}(i)} m_{k\to i}(x_i)$$

i: child, j: parent

$k\in$ children(i): new factors containing i in the active list

Sum-product alg. (for trees)

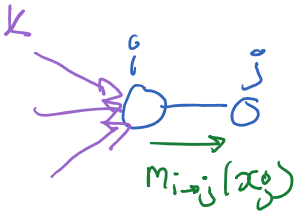
alg. to get all node/edge marginals cheaply by storing (caching) & re-using messages (dynamic programming)



goal: $\{i,j\}\in E$, compute $m_{i\to j}(x_j)$

$m_{j\to i}(x_i)$

rule: i can only send message to neighbor j when it has received all messages from other neighbor



$$m_{i\to j}(x_j) = \sum_{x_i}\psi_i(x_i)\psi_{ij}(x_i,x_j)\prod_{k\in N(i)\setminus\{j\}} m_{k\to i}(x_i)$$

at end:

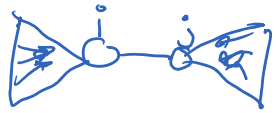
(node marginal) $p(x_i) \propto \prod_{j\in N(i)} m_{j\to i}(x_i)\psi_i(x_i)$

normalize! $Z = \sum_{x_i}(\ \ )$

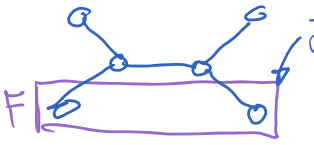
(edge marginal)

$$p(x_i,x_j) = \frac{1}{Z}\psi_i(x_i)\psi_j(x_j)\psi_{ij}(x_i,x_j)\cdot \prod_{\ldots} m_{\ldots}(x_\cdot)\cdot$$

(edge marginal)



$$p(x_i, x_j) = \frac{1}{Z} \psi_i(x_i)\psi_j(x_j)\psi_{ij}(x_i, x_j) \cdot \prod_{k \in N(i)\setminus\{j\}} m_{k\to i}(x_i) \cdot \prod_{k' \in N(j)\setminus\{i\}} m_{k'\to j}(x_j)$$



for this marginal, need to use graph Elimination

Sum-product schedules:

a) above, collect / distribute schedule

b) (flooding) parallel schedule:

1) initialize all $m_{i\to j}(x_j)$ messages to uniform dist $\forall (i,j)$ s.t. $\{i,j\} \in E$ (j,i)

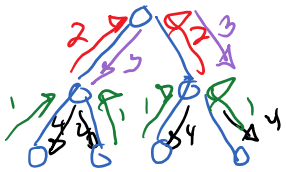
2) at every step (in parallel) compute $m_{i\to j}^{\text{new}}(x_j)$

as if the neighbor messages were correctly computed

→ one can prove that "diameter of tree" # of steps

all messages are correctly computed (for a tree)

(and are fixed à fixed pt.)



Loopy belief propagation (loopy BP): approximate inference for graphs with cycles

$$m_{i\to j}^{\text{new}}(x_j) = \left(m_{i\to j}^{\text{old}}(x_j)\right)^{1-\alpha} \left(\sum_{x_i} \psi_i(x_i)\psi_{ij}(x_i, x_j) \Big/ \left(m_{k\to i}^{\text{old}}(x_i)\right)_{k \in N(i)\setminus\{j\}}\right)^{\alpha}$$

$\alpha \in [0,1]$ ("damping")

↑ step-size

(*) this gives exact answer on trees (fixed pt. → yields correct marginals)

* on (not too loopy) graphs → approximate solution