Lecture 2 - prob. review

Thursday, September 7, 2023 2:24 PM

today: • themes
• prob. review

key themes:

I) representation: how to represent structured prob. dist.?
(probability)
• graph ⟶ factorization
• parameterization ⟶ full table
vs.
"exponential family"

II) estimation (statistics): given data, how to learn/estimate the parameters of dist.
⟶ learning e.g. maximum likelihood
max. entropy
moment matching

III) "probabilistic inference" (CS): answer questions about data
e.g. compute $p(y \mid x)$ (conditional) or $p(x)$ (marginal)
y: query, x: observation
⟶ computation e.g.
• message passing
• approximate inference
- sampling
- variational methods

probability review

why? ⟶ principled framework to model uncertainty

sources of uncertainty

1) intrinsic uncertainty ⟶ quantum mechanics

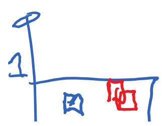
2) partial information/observation:
- card game
- rolling a die
⟶ don't know the initial conditions "exactly enough"

3) incomplete modeling of a complex phenomenon
(computational issues are also important)

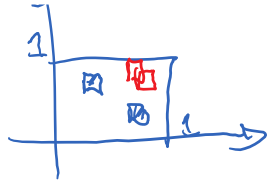
example: "most bird can fly"
⟶ simple rule can be advantageous but then yields uncertainty

• "AI"

probabilities ⟶ like areas



probabilities → like areas



15h07 "aleatoric"

Notation: $X_1 \; X_2 \; X_3 \quad X \; Y \; Z$

random variables (usually real-valued)

$x_1 \; x_2 \; x_3 \quad x \; y \; z$

their realizations

X is a random variable → represents an uncertain quantity

$X=x$ represents the "event" that X takes the value x

e.g. X "result of a roll of a die"

Ω → the sample space of "elementary events"

possible values of my R.V.

$\Omega = \{1,2,3,4,5,6\}$

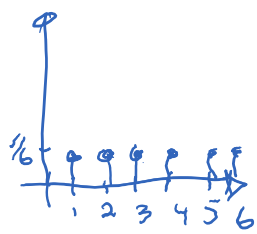
two types of R.V. — discrete where Ω is countable; continuous " Ω " uncountable

I) [assume Ω is countable → discrete case]

discrete R.V. is characterized by a probability mass function (pmf)

"lower case" → $p(x)$ for $x \in \Omega$

pmf p is s.t. $\begin{cases} p(x) \geq 0 \; \forall x \in \Omega \\ \sum_{x\in\Omega} p(x) = 1 \end{cases}$



probability distribution "big" P

is a mapping $P: \mathcal{E} \to [0,1]$

$\mathcal{E} = 2^{\Omega} \triangleq$ set of all subsets of Ω

set of "events"

("σ-field" in measure theory)

crucial when Ω is uncountable

that satisfies the Kolmogorov axioms

$P(E) \geq 0 \; \forall E \in \mathcal{E}$

$P(\Omega) = 1$

$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$ when E_i's are mutually disjoint

notation: $P(\{X=x\}) = p(x)$

"formal proba world"

$\{X=x\} \triangleq \{\omega : X(\omega)=x\}$

$P(\{X=x \text{ or } X=x'\}) = P(\{X=x\}) + P(\{X=x'\}) = p(x) + p(x')$ (pmf)

$E = \{x, x'\}$ (if $x \neq x'$)

$\{X=x\} \triangleq \{\omega : X(\omega)=x\}$

$E=\{x, x'\}$ (if $x \neq x'$)

for discrete R.V. $\boxed{P(E) = \sum_{x \in E} p(x)}$

(15h30)

continuous R.V.

continuous R.V. is characterized by a probability density function p (pdf)

$p : \Omega \to \mathbb{R}$

$p(x) \geq 0 \quad \forall x \in \Omega$

p is integrable and $\int_\Omega p(x)\,dx = 1$

prob. of events

$\Omega = \mathbb{R}$; $P([a,b]) = \int_a^b p(x)\,dx$

$\mathcal{E} \to$ Borel σ-fields

$p(x) = \lim_{n\to\infty} \frac{P(B_n)}{\mathrm{vol}(B_n)}$ where B_n are balls centered around x with decreasing radius

recap: discrete R.V. X; pmf $p(x)$ ~~st~~ $P\{X=x\} = p(x)$

cts. R.V. X; pdf $p(x)$ $\quad P\{X=x\}=0$

"$P\{X \in x \pm \frac{dx}{2}\}$" $\approx p(x)\,dx$

$P\{X \in x \pm \frac{\Delta}{2}\} = \int_{x-\frac{\Delta}{2}}^{x+\frac{\Delta}{2}} p(u)\,du$

[sidenote: can change pdf p on a countable # of points without changing anything]

has measure zero according to Lebesgue measure

joint, marginal, etc. (context: multivariate R.V.)
"random vector"

$Z = (X, Y)$ $\quad \Omega_Z \triangleq \Omega_X \times \Omega_Y$

(discrete) pmf of Z

"joint pmf" on X & Y

$p(x,y) = P\{X=x, Y=y\}$

2 joint distribution

can represent elementary events of $\Omega_{(X,Y)}$ as table

e.g.:

	$X=0$	$X=1$
$Y=0$	0	1/2
$Y=1$	1/2	0

$p(1,1)$

say X is whether die result is even
" Y " " " " is odd

cts. case: if X & Y are cts. ("and")

$$P\{\underbrace{(X,Y)\in \text{box}}_{\text{region}}\} = \int_{\text{box}} \overbrace{p(x,y)}^{\text{joint pdf}}\,dx\,dy$$

marginal distribution (in context of a joint dist.)

↳ dist. of a component of a random vector

$$P\{X=x\} = \sum_{y\in\Omega_Y} \underbrace{P\{X=x, Y=y\}}_{p(x,y)}$$ "sum rule"

"marginalizing out Y"

cts. case joint pdf $p(x,y)$

$$p(x) = \int_y p(x,y)\,dy$$

mixed type: X cts., Y discrete

cts. → integrate

$p(x,y)$

discrete → sum

$$P\{(X,Y)\in \text{set}\} = \sum_y \int_{x\in \text{set}(y)} p(x,y)\,dx$$

independence

$$\boxed{X \text{ is independent of } Y \iff p(x,y) = p(x)\,p(y) \quad \forall (x,y)\in\Omega_X\times\Omega_Y}$$

notation: $X \perp\!\!\!\perp Y$

R.V. $X_1,\dots,X_n$ are "mutually independent" $\iff p(x_{1:n}) = \prod_{i=1}^{n} p(x_i)$

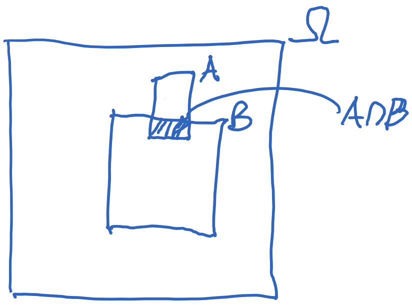
$$\forall x_{1:n} \in \bigtimes_{i=1}^{n} \Omega_{X_i}$$

conditioning:

· for events A & B, suppose $P(B)\neq 0$

then $P(A|B) \triangleq \dfrac{P(A\cap B)}{P(B)}$

normalization



$$P(B) = \sum P(A\cap B)$$

$A \in$ partition of Ω

for discrete R.V.

define "conditional pmf"

$$p(x|y) \triangleq P(\{X=x\} \mid \{Y=y\}) \triangleq \frac{P(\{X=x, Y=y\})}{P(\{Y=y\})} = \frac{p(x,y)}{p(y)} \begin{matrix} \} \text{ joint pmf} \\ \} \text{ marginal pmf} \end{matrix}$$

$p(x|y) \propto p(x,y)$ ← "proportional to"

normalization cst. $\sum_x p(x,y) = p(y)$

for cts. R.V.

"conditional pdf" $p(x|y) \triangleq \frac{p(x,y)}{p(y)}$

subtle point: $p(x|y)$ is undefined when $p(y)=0$

see Borel-Kolmogorov paradox for weird properties of cond. prob.

Borel-Kolmogorov paradox on wikipedia

(*) note: independent $X \perp\!\!\!\perp Y$; $p(x|y) = \frac{p(x,y)}{p(y)} \overset{\text{by } \perp\!\!\!\perp}{=} \frac{p(x)\cancel{p(y)}}{\cancel{p(y)}} = p(x)$