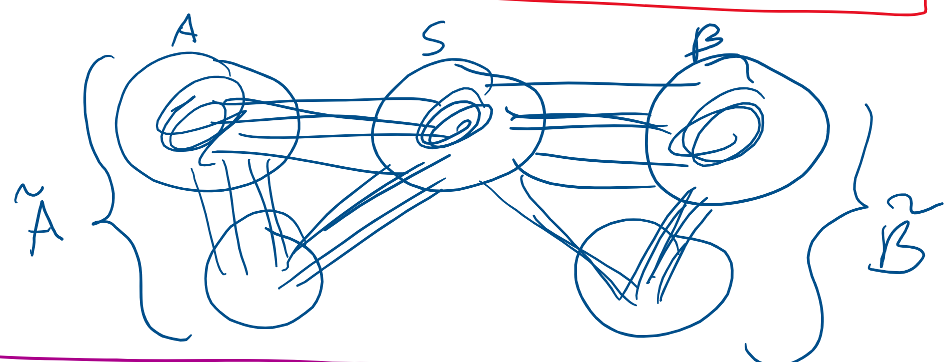
Lecture 13 - UGM

Monday, October 28, 2024 12:30 PM

<u>today</u>: UGM (cont.)
DGM vs. UGM

<u>cond. independence for UGM</u>

<u>def</u>: we say that p satisfies <u>global Markov property</u> (with respect to an undirected graph G)
iff $\forall A, B, S \subseteq V$ <u>disjoint</u> s.t. S <u>separates</u> A from B in G
then $X_A \perp\!\!\!\perp X_B \mid X_S$



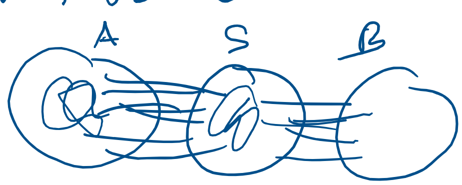
<u>prop</u>: $p \in \mathcal{L}(G) \Rightarrow p$ satisfies the global Markov property for G

<u>proof</u>: WLOG, we assume that $A \cup B \cup S = V$ & A separated from B by S

[why? if not, $\tilde{A} \triangleq A \cup \{a \in V : a \text{ and } A \text{ are not separated by } S\}$
$\tilde{B} \triangleq V \setminus (S \cup \tilde{A})$ $\Rightarrow \tilde{A} \cup \tilde{B} \cup S = V$ disjoint

then if have $X_{\tilde{A}} \perp\!\!\!\perp X_{\tilde{B}} \mid X_S$ [exercise: show that if $b \in \tilde{B}$ $\Rightarrow b$ is separated from $\tilde{A}$ by S]
by decomposition $\Rightarrow X_A \perp\!\!\!\perp X_B \mid X_S$
since $A \subseteq \tilde{A}$, $B \subseteq \tilde{B}$]

$V = A \cup S \cup B$



let $C \in \mathcal{C}$

cannot have $C \cap A \neq \emptyset$ <u>and</u> $C \cap B \neq \emptyset$

thus $p(x) = \frac{1}{Z} \prod_{C \in \mathcal{C}} \psi_C(x_C) \prod_{C' \in \mathcal{C}} \psi_{C'}(x_{C'})$

thus $p(x) = \frac{1}{Z} \underbrace{\prod_{\substack{C \in \mathcal{C} \\ C \subseteq A\cup S}} \psi_C(x_C)}_{f(x_{A\cup S})} \underbrace{\prod_{\substack{C' \in \mathcal{C} \\ C' \not\subseteq A\cup S}} \psi_{C'}(x_{C'})}_{g(x_{B\cup S})}$ $\quad \Rightarrow C' \subseteq B\cup S$

$$p(x_A | x_S) \propto p(\underbrace{x_A, x_S}_{x_{A\cup S}}) = \sum_{x_B} p(x_V) = \sum_{x_B} f(x_{A\cup S})\, g(x_{B\cup S})$$

$$= f(x_{A\cup S}) \underbrace{\sum_{x_B} g(x_{B\cup S})}_{\text{constant w.r.to } x_A}$$

$$\Rightarrow p(x_A | x_S) = \frac{f(x_{A\cup S})}{\sum_{x'_A} f(x'_A, x_S)}$$

similarly, $p(x_B | x_S) = \dfrac{g(x_{B\cup S})}{\sum_{x'_B} g(x'_B, x_S)}$

$$p(x_A|x_S)\, p(x_B|x_S) = \frac{f(x_{A\cup S})\, g(x_{B\cup S})}{\sum_{x'_A} f(\text{—}) \sum_{x'_B} g(\text{—})} \frac{\}\, p(x_V)}{\}\, p(x_S)} = p(x_A, x_B | x_S)$$

i.e., $X_A \perp\!\!\!\perp X_B \mid X_S$ //

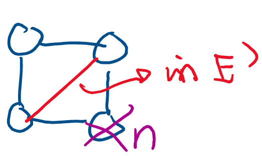
thm: (Hammersley-Clifford) if $p(x_V) > 0 \;\; \forall x_V$

then $p \in \mathcal{F}(G) \Leftrightarrow p$ satisfies global Markov property for G

proof: ch. 16 of Mike's book; use "Möbius inversion formula" (as exclusion-inclusion principle in prob.)

<u>more properties of UGM</u>

* closure with respect to marginalization



let $V' = V \setminus \{n\}$

E' = edges in $G \setminus \{n\}$ + connect all neighbors of n in G <u>together</u> (new clique)

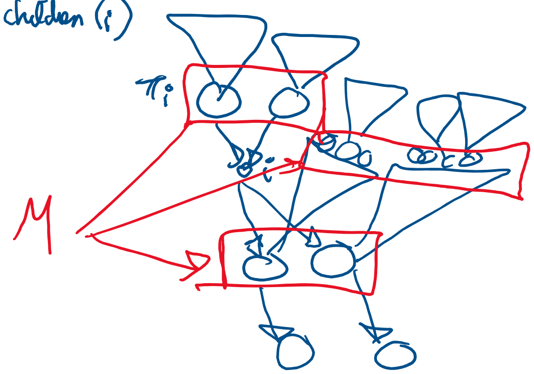
+ connect all neighbors of n in G together
(new clique)

$\{$ marginal on $x_{1:n-1}$ for some $p \in \mathcal{L}(G) \} = \mathcal{L}(G')$

DGM vs. UGM

def: Markov blanket for i (for graph G) is the smallest set of nodes M s.t. $X_i \perp\!\!\!\perp \underbrace{X_{V \setminus \{i\}}}_{\text{"rest"}} \mid X_M$

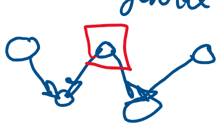
- for UGM: $M = \{ j : \{i,j\} \in E \}$ = set of neighbors of i
- for DGM: $M = \underbrace{\pi_i \cup \text{children}(i)}_{\text{like UGM}} \cup \bigcup_{j \in \text{children}(i)} \pi_j$



Recap

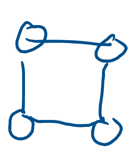
	DGM	UGM
factorization:	$p(x) = \prod_i p(x_i \mid x_{\pi_i})$	$\frac{1}{Z} \prod_C \psi_C(x_C)$
cond. indep:	d-separation	separation
marginalization:	not closed in general	closed (connect all neighbors of removed node)

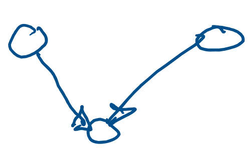
e.g.



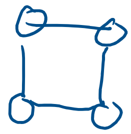
but is fine for leaves

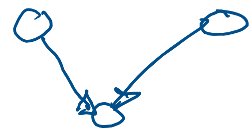
cannot exactly capture some families





cannot exactly capture some families





Moralization:

Let G a DAG; when can we get an equivalent UGM?

def: for G a DAG, we call $\bar{G}$ the moralized graph of G

where $\bar{G}$ is an undirected graph with same V

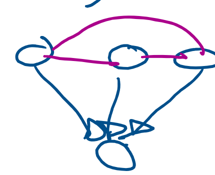
and $\bar{E} = \{ \{i,j\} : (i,j) \in E \}$ } undirected version of E

$\cup \{ \{k,l\} : k \neq l \in \pi_i \text{ for some } i \}$ } "moralization"

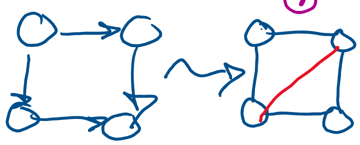
connect all the parents of i with i in big clique

"marrying the parents" (??)

only needed if $|\pi_i| > 1$ (i.e. v-structure)



e.g. G ⇝ $\bar{G}$



Prop: for a DAG with no v-structure [forest]
then $\mathcal{L}(G) = \mathcal{L}(\bar{G})$

(proof: see hwk 3?)

but in general, can only $\underset{\text{DAG}}{\mathcal{L}(G)} \subseteq \mathcal{L}(\bar{G})$

[note that $\bar{G}$ is the minimal undirected graph G' s.t. $\mathcal{L}(G) \subseteq \mathcal{L}(G')$]

$$p(x) = \prod_i \underbrace{p(x_i | x_{\pi_i})}_{\psi_C(x_C)} \text{ where } C = \{i\} \cup \pi_i$$

⇝ $\bar{G}$

13h31

<u>general themes in this class</u>

A) representation $\to$ DGM, UGM

parameterization $\rightsquigarrow$ exponential family (later)

} model high dim. distribution

B) inference $p(x_Q \mid x_E)$ — "query", "evidence" $\to$ next class: elimination algorithm; next: sum-product algorithm (for trees)

later: approximate inference e.g. MCMC, variational

C) statistical estimation / learning $\to$
- MLE
- maximum entropy (next week)
- method of moments

<u>Inference</u>:

want to compute:

a) marginal: $p(x_F)$ for some $F \subseteq V$

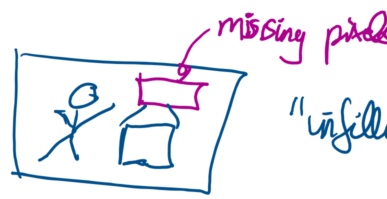
b) conditional: $p(x_F \mid x_E)$ — "query", "evidence"

c) for UGM: partition function $Z = \sum_{x_V} \left(\prod_C \psi_C(x_C) \right)$

$\uparrow$ exponential sum

<u>why?</u>:

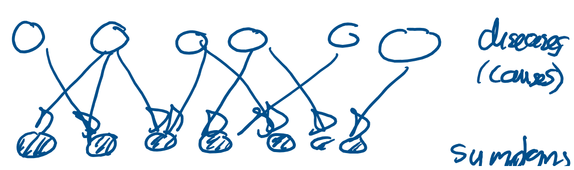
- missing data $p(x_{\text{unobv.}} \mid x_{\text{obs}})$

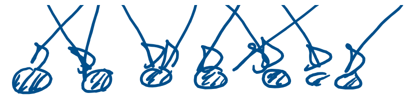


"infilling task"

- prediction $p(x_{\text{future}} \mid x_{\text{past}})$

- "latent cause" $p(x_{\text{cause}} \mid x_{\text{obs}})$





(causes)

symptoms

* also related inference

$$\underset{x_F}{\arg\max}\ p(x_F \mid x_E)$$ (Viterbi alg.) $\rightsquigarrow$ e.g. speech recognition

F could be big (max product)

* inference is also needed during estimation (parameter fitting MLE)

[e.g. during EM, E-step $p(z \mid x)$]

or computing Z for MLE in UGM

(*) we present inference alg. for UGM [for simplicity and more generality]

make DGM $\rightsquigarrow \subseteq$ UGM via moralization

i.e. $p \in \text{DGM}$; $p(x) = \prod_i p(x_i \mid x_{\pi_i})$ moralize: $C_i \triangleq \{i\} \cup \pi_i$

$$= \frac{1}{Z} \prod_i \psi_{C_i}(x_{C_i})$$ where $Z = 1$

$$\psi_{C_i}(x_{C_i}) \triangleq p(x_i \mid x_{\pi_i})$$

$$p(x_1) = \sum_{x_{2:n}} p(x) = \sum_{x_{2:n}} \prod_i \frac{1}{Z} \psi_{C_i}(x_{C_i})$$