



An Introduction to **Generative Flow Networks**

Juan Ramirez juan.ramirez@mila.quebec

Outline

Motivation

• Applications

• GFlowNets

- O Flow Networks
- O Generative Flow Networks
- O Training GFlowNets

• Comparison to meethods seen in class

Important References

- GFlowNet Foundations. Yoshua Bengio, et al. JMLR 2023.
- [NeurIPS 2021 GFlowNet paper] Flow Network based Generative Models for Non-Iterative Diverse Candidate Generation. Emmanuel Bengio, et al. NeurIPS 2021.
- [Structure Learning with GFlowNets] *Bayesian Structure Learning with Generative Flow Networks*. Tristan Deleu, et al. UAI 2022.

Setting

We want to do **inference** over an *intractable* distribution:

- Sampling: $x \sim p(x)$.
- Computing expectations: $E_{x \sim p(x)} [f(x)]$.

Sampling is intractable, but:

- Samples are discrete* and can be built **compositionally**.
- Can not evaluate p(x), but can evaluate a reward function "**R**" such that $p(x) \propto R(x)$.

*For continuous GFlowNets, see A Theory of Continuous Generative Flow Networks (S. Lahlou et al., 2023)

Simplified Taxonomy for Sampling

- **Easy**: Know p(x), can efficiently sample from p(x)
 - O Uniform
 - O Mixture of Gaussians
 - O HMM (with Gaussian emission probabilities)

• Approximate: sampling is expensive or impossible, but can evaluate p(x)

- O Rejection sampling: for using an easy distribution q(x)
- O Importance sampling
- O MCMC: exploit structure and low dimensionality of $p(x_i | x_{j})$
- O Variational Inference: sample from $easy q \in argmin KL(q || p)$
- **GFlowNets** (can evaluate $\mathbf{R}(x) \propto \mathbf{p}(x)$; next slide)

GFlowNets Main Idea

Given a reward function R(x),

- 1. Construct a DAG **G** over all possible samples
 - **a. States:** possible samples *x* (e.g. all possible chemical molecules)
 - **b. Transitions**: composition of parent to produce child (e.g. adding a basic molecule to produce a more complex one)
- 2. Learn a probability distribution $p \in L(G)$ such that $p(x) \propto R(x)$
 - a. In practice, GFlowNets model unnormalized "flows" such that F(x) = R(x)
 - b. Also, approximate F(x) variationaly (i.e. use a neural network to output F(x))
- 3. Sample over G.
 - a. Ancestral sampling, beam search, ...

Applications

Scientific Discovery

For instance, constructing **molecules**.

• **States**: sets of (partially constructed) molecules.

• **Transitions**: addition of a basic molecule.

Reward: property of the molecule.

Building blocks



GFlowNets for AI-Driven Scientific Discovery. Moksh Jain, et al. Digital Discovery 2023.

Reward

Reward

Reward

Naproxer

Combinatorial Optimization Problems

For instance, finding the **largest clique** in a UGM.

- **States**: sets of fully connected nodes.
- **Transitions**: addition of a basic molecule.
- **Reward**: size of the set.



Let the Flows Tell: Solving Graph Combinatorial Optimization Problems with GFlowNets. Dinghuai Zhang, et al. NeurIPS 2023.

Structure Learning

Let X_1, X_2, \ldots, X_d be rvs. We want to construct a "minimal" graph \mathcal{G} such that:

$$p(x_1, x_2, \ldots, x_d) \in \mathcal{L}(\mathcal{G})$$



Т

2

n



 X_1

Structure Learning

- States: DAGs.
- **Transitions**: addition of a directed edge.
- **Reward**: likelihood of data given the graph.

The transitions exploit the <u>compositional</u> aspect. Instead of looking at all possible DAGs without structure, the GFlowNet depth equals the **number of edges** in the DAG.



- DAG $G = (\mathcal{S}, \mathbb{A})$ with source and sink states, $s_0, s_f \in \mathcal{S}$
- Trajectories $au = (s_1, \dots, s_n)$, where $s_t {
 ightarrow} s_{t+1} \in \mathbb{A}$
- Let \mathcal{T} be the set of all trajectories.
- Forward transition probabilities $\hat{P}_F(s' \mid s)$

 $\hat{P}_F(s' \mid s) \text{ is a probability} \qquad \text{Chain rule}$ $\sum_{s' \in Child(s)} \hat{P}_F(s' \mid s) = 1 \qquad \hat{P}_F(\tau) := \prod_{t=1}^{n-1} \hat{P}_F(s_{t+1} \mid s_t)$

Instead of modeling probabilities, consider (unnormalized) flows:

$F: \mathcal{T} \mapsto \mathbb{R}^+$

$$F(s \to s') := F(\{\tau \in \mathcal{T} : s \to s' \in \tau\}) = \sum_{\tau \in \mathcal{T} : s \to s' \in \tau} F(\tau)$$

$$F(s) := F(\{\tau \in \mathcal{T} : s \in \tau\}) = \sum_{\tau \in \mathcal{T} : s \in \tau} F(\tau)$$



*Fig 2 in GFlowNet Foundations

The flow induces a distribution over the **terminating states.**

$$F(s_0) = \sum_{\tau \in \mathcal{T}} F(\tau) = Z, \qquad P(s) \coloneqq \frac{F(s)}{Z}$$
$$F(s_f) = \sum_{\tau \in \mathcal{T}} F(\tau) = Z. \qquad P(s \to s' \mid s) = \frac{F(s \to s')}{F(s)}$$

 $\mathbf{\Gamma}(\mathbf{A})$

The flow induces a distribution over the **terminating states.**

$$P_T(s) := P(s \to s_f) = \frac{F(s \to s_f)}{Z} \qquad \sum_{s \in \mathcal{S}^f} P_T(s) = 1.$$

If we learn a flow **F** that matches a reward function on terminating states:

- A probability distribution over terminating states follows.
- The probabilities are proportional to the reward.
- And the graph **G** can be used to efficiently **sample**.

But parameterizing a flow is expensive! -> One value per trajectory in G.

Markovian Flow Networks

Instead, consider Markovian flow networks:

$$P(s \rightarrow s' \mid \tau) = P(s \rightarrow s' \mid s)$$

- Cheaper to model.
- Induces a unique forward transition probability P(s'|s).
- **Proposition 23** in GFlowNet Foundations: the set of markovian flows is expressive enough to represent **all flow functions** over trajectories.

Markovian Flow Networks

- Even Markovian flows are expensive (need flow over every node).
- Approximate **F** with a neural network respecting flow conservation:

$$F(s) = \sum_{\substack{s' \in Child(s)}} F(s \to s')$$
$$F(s') = \sum_{s \in Par(s')} F(s \to s')$$

Generative Flow Networks - GFlowNets

A **GFlowNet** is a (Markovian) flow network where:

$$\forall s \in \mathcal{S}^f \quad F(s \to s_f) = R(s)$$

• **R** is a given reward function.

- O If **s** is not a valid terminating state, set a reward of 0.
- **F** is parameterized with (say) a NN.

The NeurIPS 2021 GFlowNet paper enforces flow matching:

$$\sum_{s \in \operatorname{Pa}(s')} F_{\theta}(s \to s') - \sum_{s'' \in \operatorname{Ch}(s')} F_{\theta}(s' \to s'') = R(s') \overset{0 \text{ for invalid states}}{ F(s \to s_f)}$$

Which leads to the following objective:

$$\mathcal{L}(\phi) = \mathbb{E}_{\pi} \left[\left[\log \frac{\sum_{s} F_{\phi}(s \to s')}{R(s') + \sum_{s''} F_{\phi}(s' \to s'')} \right]^2 \right]$$

The expectation is over all trajectories -> **intractable**.

$$\mathcal{L}(\phi) = \mathbb{E}_{\pi} \left[\left[\log \frac{\sum_{s} F_{\phi}(s \to s')}{R(s') + \sum_{s''} F_{\phi}(s' \to s'')} \right]^{2} \right]_{trajectories}$$

Sample trajectories instead

- If there is structure, the GFlowNet could generalize across trajectories.
- Trade-off between sampling likely trajectories and exploration.

The expectation is over all trajectories -> **intractable**.

$$\mathcal{L}(\phi) = \mathbb{E}_{\pi} \left[\left[\log \frac{\sum_{s} F_{\phi}(s \to s')}{R(s') + \sum_{s''} F_{\phi}(s' \to s'')} \right]^{2} \right]_{trajectories}$$

Note: this is different from supervised learning and reinforcement learning

- **SL**: the distribution over trajectories is non-stationary
- **RL**: *F* is not trained to *maximize* a reward, just to match it

Alternative Loss Functions

Detailed balance (Bengio et al., 2021):

 $F_{\theta}(s)P_F(s' \mid s) = F_{\theta}(s')P_B(s \mid s')$

Trajectory balance (Malkin et al., 2022):

$$Z\prod_{t=1}^{n} P_F(s_t \mid s_{t-1}) = R(s_n)\prod_{t=1}^{n} P_B(s_{t-1} \mid s_t)$$

Trajectory Balance: Improved Credit Assignment in GFlowNets. Nikolay Malkin et al., NeurIPS 2022.

GFlowNets in Context

Markov Chain Monte Carlo

MCMC

- No "setup" cost.
- Samples are not independent
- Sampling is costly
 - O Mixing time can scale poorly
 - O Mode mixing

GFlowNets

- Needs to be trained.
- Samples are independent.
- Sampling is efficient: do ancestral sampling

Generative Modeling

Generative Modeling

- Trained on *data*, to maximize its likelihood.
- Prone to overfitting.
- Improves with more data.

GFlowNets

- Trained to match a reward *function*.
- Prone to underfitting.
- Improves with more trajectories.

Reinforcement Learning

Reinforcement Learning

GFlowNets

See section 7.2 in *GFlowNet Foundations*



Thanks!

References

- *GFlowNets for AI-Driven Scientific Discovery.* Moksh Jain, et al. Digital Discovery 2023.
- Let the Flows Tell: Solving Graph Combinatorial Optimization Problems with GFlowNets. Dinghuai Zhang, et al. NeurIPS 2023.
- Trajectory Balance: Improved Credit Assignment in GFlowNets. Nikolay Malkin et al., NeurIPS 2022.
- *GFlowNet Foundations*. Yoshua Bengio, et al. JMLR 2023.
- Flow Network based Generative Models for Non-Iterative Diverse Candidate Generation. Emmanuel Bengio, et al. NeurIPS 2021.
- Bayesian Structure Learning with Generative Flow Networks. Tristan Deleu, et al. UAI 2022.



Generative Flow Networks and Bayesian Structure Learning

Structure Learning

Credit to Tristan Deleu





Structure Learning

- Graphical representation of the **conditional** independences in a distribution, represented as a Directed Acyclic Graph (DAG).
- The **joint distribution** is decomposed as:

$$P(X_1,\ldots,X_d) = \prod_{k=1}^d P(X_k \mid \operatorname{Pa}_G(X_k))$$

• Structure learning: Given a dataset of **observations** \mathcal{D} , find the graph structure G.




Markov Equivalence

- Multiple DAGs may encode the same Conditional Independence statements.



 Two DAGs encoding the same Conditional Independence statements are called Markov Equivalent.

Recall: A Directed Graphical Model encodes the Conditional Independence of a distribution.



Markov Equivalence

Theorem (Verma & Pearl, 1991)

Two DAGs G₁ and G₂ are **Markov Equivalent** if and only if they have the same skeleton and the same v-structures.



 Markov Equivalence Classes can be represe Graph (CPDAG).

• Markov Equivalence Classes can be represented as a Completed Partially Directed Acyclic

Faithfulness

A & B are d-separated by C in \mathcal{G}

Exercise: Violation of Faithfulness



 $X := N_X$ $Y := X + N_Y$ $Z := X - Y + N_Z$ with $N_X, N_Y, N_Z \stackrel{iid}{\sim} \mathcal{N}(0, \sigma^2)$

p(X, Y, Z) is a Multivariate Normal distribution, where the only conditional independence statements are: $X \perp \!\!\!\perp Z$ and $X \perp \!\!\!\perp Z \mid Y$.



Structure Identifiability

Theorem

graph. Two Markov Equivalent graphs may lead to different causal conclusions!



- Under different assumptions, an individual DAG may be identifiable

 - Using **interventional data** (i.e. data resulting from controlled experiments).



If p is faithful wrt. \mathcal{G}^0 , then the Markov Equivalence class of \mathcal{G}^0 is **identifiable** from p.

Only the Markov Equivalence class is identifiable from observations, **not an individual**

or

• Additive Noise Model (ANM): $X_j := f_j(X_{Pa_j}) + N_j$, $N_j \stackrel{iid}{\sim} \mathcal{N}(0, \sigma^2)$, where f are nonlinear.

Constraint-based methods

Step 1: Identify the skeleton

For each pair of nodes X & Y, and $\mathbf{A} \subseteq \mathbf{V} \setminus \{X, Y\}$, test if $X \perp_{\mathcal{D}} Y \mid \mathbf{A}$.

If there is no set **A** s.t. $X \perp _{\mathcal{D}} Y \mid \mathbf{A}$, then add an edge X - Y.







Constraint-based methods

Step 1: Identify the skeleton

For each pair of nodes *X* & *Y*, and $\mathbf{A} \subseteq \mathbf{V} \setminus \{X, Y\}$, test if $X \perp _{\mathcal{D}} Y \mid \mathbf{A}$.

If there is no set \mathbf{A} s.t. $X \perp _{\mathcal{D}} Y \mid \mathbf{A}$, then add an edge X - Y.

Step 2: Identify the v-structures

For each structure X - Z - Y with no edge between X & Y, orient $X \rightarrow Z \leftarrow Y$ iff $Z \notin \mathbf{A}$, where **A** is such that $X \perp\!\!\!\!\perp_{\mathcal{D}} Y \mid \mathbf{A}$.







Constraint-based methods

Step 2': Additional orientations

Use Meek's orientation rules to orient

some of the remaining edges.





IC Algorithm



Score-based methods

Treat the problem of learning the structure of the DAG as a **model selection problem**

 $\max_{\mathcal{G}\in \mathrm{DAG}}\mathrm{score}(\mathcal{G}\mid\mathcal{D})$

Choice of scores

- Likelihood score:
- Bayesian score:
- Bayesian Information Criterion (BIC):

score_{*B/C*}($\mathcal{G} \mid \mathcal{D}$) = log $p(\mathcal{D} \mid \hat{\theta}_{\mathcal{G}}^{\text{MLE}}, \mathcal{G}) - \frac{\log N}{2} \text{Dim}[\mathcal{G}]$



score_{*L*}($\mathcal{G} \mid \mathcal{D}$) = log $p(\mathcal{D} \mid \hat{\theta}_{\mathcal{G}}^{\text{MLE}}, \mathcal{G})$ score_B($\mathcal{G} \mid \mathcal{D}$) = log $p(\mathcal{D} \mid \mathcal{G})$ + log $p(\mathcal{G})$

Score-based methods

- How to search over the space of DAGs?
- The number of DAGs over *n* nodes is **super-exponential** in *n*: $2^{\Theta(n^2)}$

Theorem

maximizes a score is **NP-hard** for $d \geq 2$.

- Heuristic solutions:
 - Greedy algorithms: Hill climbing, GES
 - Genetic algorithms
 - Constrained continuous optimization: NOTEARS, Gran-DAG, DCDI, etc...



$\max_{\mathcal{G}\in \mathrm{DAG}}\mathrm{score}(\mathcal{G}\mid\mathcal{D})$

Let $G_{\leq d} = \{\mathcal{G} \ a \ DAG \mid every \ node \ has \ at \ most \ d \ parents\}$. Finding a DAG in $G_{\leq d}$ that

Bayesian Structure Learning

- When the **dataset is small**, we want to take into account the **epistemic uncertainty** over the graph structures of the Bayesian Network.
- Markov Equivalence: There may be multiple graphs encoding the same conditional independences.



 $X \perp Z$ and $X \perp Z \mid Y$ $X \perp\!\!\!\perp Z$ and $X \not\!\!\perp Z \mid Y$

• From the point of view of observations, Markov equivalent graphs fit the data equally well.

Bayesian Structure Learning: Instead of finding a single graph from observations, characterize the whole posterior distribution over graphs:

$$P(G \mid \mathcal{D}) = \frac{P(\mathcal{D} \mid G)P(G)}{P(\mathcal{D})}$$





Bayesian Structure Learning

Bayesian Structure Learning: Instead of finding a single graph from observations, characterize the whole **posterior distribution** over graphs:

Graphs are **discrete** and **composite** objects

The number of DAGs is **super-exponential** in the number of nodes (eg. there are 10⁷² DAGs over 15 nodes)

$$P(G \mid \mathcal{D}) = \frac{P(\mathcal{D} \mid G)P(G)}{\uparrow P(\mathcal{D})}$$

The marginal likelihood is in general intractable $P(\mathcal{D} \mid G) = \int_{\Theta} P(\mathcal{D} \mid \theta, G) P(\theta \mid G) d\theta$

We will choose models so that this can be computed efficiently in **closed form.**





Markov Chain Monte Carlo

- Approximate the posterior distribution using Markov Chain Monte Carlo (MCMC).
- Build a Markov chain by adding, removing, or reversing edges uniformly at random.
- Issue: Highly multimodal distribution (Markov equivalence), leading to **poor mixing** of the Markov chain.









DAG-GFlowNet





GFlowNet over DAGs

- DAGs are constructed sequentially **one edge at** a time, starting from the empty graph.
- All the states of the GFlowNet are valid DAGs, meaning that all the states are terminating.
- A new edge to be added to a DAG:
 - must not **already be present;**
 - must not **introduce a cycle.**
- We can filter out invalid actions using a **mask**, that can also be **updated online**.







Detailed balance condition

Flow matching condition (Bengio et al., 2021)

$$\sum_{s \in \operatorname{Pa}(s')} F_{\theta}(s \to s') - \sum_{s'' \in \operatorname{Ch}(s')} F_{\theta}(s' \to s'') = R(s')$$

Detailed balance condition (Ours)

$$R(s')P_B(s \mid s')P_{\theta}(s_f \mid s) = R(s)P_{\theta}(s' \mid s)P_{\theta}(s_f)$$
Fixed backward
transition probability
e.g. Uniform distribution
$$Learned \text{ forward} \\ transition probability}$$

Yoshua Bengio, Tristan Deleu, Edward J. Hu, Salem Lahlou, Mo Tiwari, Emmanuel Bengio, GFlowNet Foundations (ArXiv 2021)

s')

$P(s_{t+1} \mid s_t) \propto F_{\theta}(s_t \to s_{t+1})$

- ✓ Valid when **all the states** of the GFlowNet **are terminating**
- \checkmark Induces a distribution $P(s) \propto R(s)$
- ✓ It does not depend on flows anymore (flow-matching or detailed balance conditions).
- \checkmark It does not depend on the total flow Z (trajectory balance condition).







Forward Transition Probabilities

Hierarchical model for the forward transition probabilities: $P_{\theta}(G' \mid G) = (1 - P_{\theta}(s_f \mid G))P_{\theta}(G' \mid G, \neg s_f)$



Independent of the order of edges. ✓ Set-to-set architecture.

Angelos Katharopoulos, Apoorv Vyas, Nikolaos Pappas, François Fleuret, Transformers are RNNs: Fast Autoregressive Transformers with Linear Attention (ICML 2020)

- The **number of parameters** does not scale \checkmark too much with the size of the graph.
- ✓ No quadratic scale with the input size.





Application to Bayesian Structure Learning

Bayesian Structure Learning

Characterize the **posterior distribution** over DAGs





 $R(G) = P(\mathcal{D} \mid G)P(G)$

GFlowNet

A GFlowNet induces a distribution

 $P(s) \propto R(s)$



DAG-GFlowNet





Tools from Reinforcement Learning

The GFlowNet is trained **off-policy**

We use a **replay buffer** to store transitions over the course of training, and **sample transitions** from the replay buffer

 $\mathcal{L}(\theta) = \mathbb{E}_{\pi} \left[\left[\log \frac{R(G')P_B(G \mid G')P_{\theta}(s_f \mid G)}{R(G)P_{\theta}(G' \mid G)P_{\theta}(s_f \mid G')} \right]^2 \right]$





Experimental results





Comparison with the **exact posterior** distribution $P(G \mid \mathcal{D})$ on graphs with d = 5 nodes, computed by enumerating the **29,281 possible DAGs**.



Experimental results – Accurate approximation





Experimental results – Simulated data







Experimental results – Flow cytometry data

- Real-world **flow cytometry data**, to learn protein signaling pathways.
- Data: continuous measurements of **11** phosphoproteins. There are 853 observations.
- The ground truth graph contains **11 nodes** and 17 edges.
 - The consensus graph may not represent a realistic description of the system.



Higher score is better Concentrated around the **mode** of the distribution







Experimental results – Flow cytometry data



BGe score



Experimental results – Interventional data

- The real world flow cytometry data also contain interventional data, based on experimentation where some phosphoproteins are inhibited.
 - We model these as perfect interventions. even though it may not be the case in prac
 - We know the **intervention targets**.
- We can adapt the reward function (computati of the marginal likelihood) to handle a mixture observational & interventional data.
- This is a first step toward **causal discovery.**

ains	
ons	

s, actice.		\mathbb{E} -# Edges	E-SHD	AUROC
	Exact posterior [*] MC^3	25.97 ± 0.01	25.08 ± 0.02	0.816 0.665
_	DAG-GFlowNet	30.66 ± 0.04	27.77 ± 0.03	0.700
tion re of				





OC

Thank you



github.com/tristandeleu/jax-dag-gflownet



