Quasi-Monte Carlo Node Sets from Linear Congruential Generators

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Abstract. In this paper we present a new approach to finding good lattice points. We employ the spectral test, a well-known figure of merit for uniform random number generators. This concept leads to an assessment of lattice points $\mathbf{g}$ that is closely related to the classical Babenko-Zaremba quantity $\rho(\mathbf{g}, N)$. The associated lattice rules are good uniformly over a whole range of dimensions. Our numerical examples suggest that this simple approach leads to quasi-Monte Carlo node sets that perform very well in comparison to the best available (t, m, s)-nets.

1 Introduction

There is no contradiction in the title of this paper. We show how to employ concepts that belong to the field of random number generation to obtain excellent node sets for quasi-Monte Carlo integration in high dimensions.

Our approach uses linear congruential generators ("LCGs") and the spectral test to find good lattice points ("GLPs") of the Korobov type. LCGs are a classical method for generating uniform random numbers, see Knuth [16] and L'Ecuyer [19] for further information. The spectral test, due to Coveyou and MacPherson [3], is a well-known figure of merit to assess uniform random number generators with a lattice structure, see Knuth [16] and L'Ecuyer and Couture [21]. GLPs, which are due to Korobov [17], are a fundamental concept in quasi-Monte Carlo integration for producing low-discrepancy node sets, in so-called *rank-1 lattice rules.* We refer the reader to the excellent monographs of Niederreiter [26] and Sloan and Joe [29] for details on all these notions and for an overview of known results.

Niederreiter [24] has been the first to propose this particular application of random number generators. He has proved general discrepancy estimates for

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the resulting node sets which indicate very good equidistribution properties. As a complement to Niederreiter's theoretical results, this paper proposes to use the spectral test as a selection criterion for the parameters of lattice rules. One important advantage of this criterion is its computational efficiency in practical searches for parameters. It can also be conveniently normalized to a number between 0 and 1, which permits one to compare the quality of lattices in different number of dimensions and with different number of points.

This paper is organized as follows. In Section 2 we discuss criteria for finding good lattice points and, hence, low-discrepancy point sets. We prove that the spectral test quantity σ gives a lower bound for a well-known figure of merit for lattice points, the Babenko-Zaremba index ρ (for the latter, see [26, p. 108], and [29, p. 75]). This inequality allows us to apply the comprehensive tables of LCGs in L'Ecuyer [20], which were found by a search based upon the spectral test as a selection criterion. Section 3 is devoted to a comparison of the performance of our rank-1 lattice rules based on L'Ecuyer's tables and the best available (t, m, s)-nets when integrating functions within a test setup that follows the guidelines of the well-known Genz [11,12] test battery of high-dimensional integrands. Our node sets performed very well in comparison to the sophisticated nets. This fact came as a notable surprise because the test functions we have used are optimally suited for (t, m, s)-nets due to their rapidly converging Haar series expansion (see Entacher [5,6]).

We refer the reader to Hellekalek [13] for the general concepts behind the figures of merit applied in this paper and to Hickernell [15] for the relationship between various generalizations of ρ and quadrature errors.

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2 Figures of Merit

In quasi-Monte Carlo integration we approximate high-dimensional integrals

$$I(f) := \int_{[0,1]^s} f(\mathbf{x})\, d\mathbf{x}$$

by means of the form

$$S_N(f, \omega) := \frac{1}{N} \sum_{n=0}^{N-1} f(\mathbf{x}_n),$$

where $\omega = (\mathbf{x}_n)_{n=0}^{N-1}$ is a finite sequence of nodes in the s-dimensional unit cube $[0,1]^s$ and $f : [0,1]^s \to \mathbb{R}$ is a Riemann-integrable function. In this context, it is essential that ω has low discrepancy, see [26]. The problem of the practitioner is to have appropriate node sets at his disposal for a broad range of sample sizes N and dimensions s. Such a readily available arsenal is highly desirable for numerical practice.

One of the most important techniques to construct good node sets is the method of good lattice points, see Korobov [17,18]. For a given integer $N \geq 2$ and a given dimension s, an integer point $\mathbf{g} = (g_1, \ldots, g_s) \in \mathbb{Z}^s$, where $\gcd(g_i, N) = 1$ for all i, is called a *good lattice point modulo* N if the finite sequence

$$\omega_{\mathbf{g}} = \left(\left\{ \frac{n}{N} \, \mathbf{g} \right\} \right)_{n=0}^{N-1}$$

is "very uniformly" distributed in the s-dimensional unit cube $[0,1[^s$. Here, $\{y\}$ denotes the fractional part of the number (or vector) y. The condition $\gcd(g_i, N) = 1$ assures that all points in $\omega_{\mathbf{g}}$ and in its projections to lower dimensions are distinct.

There exist several figures of merit to mold this heuristic concept into a solid mathematical form and to search for good lattice points $\mathbf{g}$. These quality measures are widely different in their properties. For practical reasons, searches are often restricted to the *Korobov type*

$$\mathbf{g} = (1, a, a^2, \ldots, a^{s-1}),$$

where a is an integer, $1 < a < N$, $\gcd(a, N) = 1$, and N is a prime number. We refer the reader to Niederreiter [26, p.125] and Sloan and Joe [29, Ch.4.4-5] for further information. In fact, the results and concepts of the present paper apply to general integration lattices, not only those generated by a single vector $\mathbf{g}$.

One important technique to obtain tables of GLPs $\mathbf{g}$ is to minimize the quantity

$$P_\alpha(\mathbf{g}, N) := \sum_{\substack{\mathbf{k} \neq \mathbf{0}: \\ \mathbf{k} \cdot \mathbf{g} \equiv 0 \pmod{N}}} \frac{1}{r(\mathbf{k})^\alpha}, \tag{1}$$

for particular parameters $\alpha > 1$, where

$$r(\mathbf{k}) := \prod_{i=1}^{s} \max\{1, |k_i|\}, \quad \mathbf{k} = (k_1, \ldots, k_s) \in \mathbb{Z}^s,$$

and $\mathbf{k} \cdot \mathbf{g}$ denotes the inner product of the vectors $\mathbf{k}$ and $\mathbf{g}$, see [26, p. 103] and [29, p. 72]. For even integers α, $P_\alpha(\mathbf{g}, N)$ can be computed efficiently. It is the quadrature error $S_N(f_\alpha, \omega_{\mathbf{g}}) - I(f_\alpha)$ for a certain function f_α constructed from the α-th Bernoulli polynomial. Available tables of GLPs were found by minimizing $P_2(\mathbf{g}, N)$, for different values of s ranging from 2 to 12, see [26, p. 145] and [29, p. 73] for references.

The so-called *Babenko-Zaremba index* $\rho(\mathbf{g}, N)$ plays a central role in the theory of good lattice points. It is defined as follows:

$$\rho(\mathbf{g}, N) := \min \left\{ r(\mathbf{k}) : \mathbf{k} \in \mathbb{Z}^s \setminus \{\mathbf{0}\}, \ \mathbf{k} \cdot \mathbf{g} \equiv 0 \pmod{N} \right\}. \tag{2}$$

In order to obtain a good lattice point $\mathbf{g}$, the quantity $\rho(\mathbf{g}, N)$ should be as large as possible. Other figures of merit for lattice points $\mathbf{g}$ like $P_\alpha(\mathbf{g}, N)$ or the discrepancy $D_N(\omega_{\mathbf{g}})$ of the point set $\omega_{\mathbf{g}}$ can be bounded from above and below in terms of $\rho(\mathbf{g}, N)$. Alas, the latter figure of merit has a major drawback with respect to applications: It is hard to compute.

For this reason, we introduce the *spectral test index* to assess lattice points $\mathbf{g}$ of the Korobov type.

Definition 1. Let $N \geq 2$ and $s \geq 1$ be integers, and let $\mathbf{g} = (g_1, \ldots, g_s) \in \mathbb{Z}^s$, $\gcd(g_i, N) = 1$ for all i, $1 \leq i \leq s$. The spectral test index of $\mathbf{g}$ modulo N is defined as

$$\sigma(\mathbf{g}, N) := \min \left\{ ||\mathbf{k}||_2 : \mathbf{k} \in \mathbb{Z}^s \setminus \{\mathbf{0}\},\ \mathbf{k} \cdot \mathbf{g} \equiv 0 \pmod{N} \right\}, \qquad (3)$$

where $||\mathbf{k}||_2$ denotes the usual euclidean norm of the vector $\mathbf{k}$.

The figure of merit $\sigma(\mathbf{g}, N)$ is invariant under rotations and does not increase under projections of the vector $\mathbf{g}$ into lower dimensions. It has an intuitive interpretation (see Knuth [16]) and it is easy to compute as long as s is not excessively large, for s up to at least 40 or so, via the branch-and-bound algorithm proposed by Fincke and Pohst [7]. An implementation, together with a search procedure for good parameters, with several examples and timings, are described in [21].

The spectral test index yields a lower bound on $\rho(\mathbf{g}, N)$. Due to the following theorem, lattice points $\mathbf{g}$ with large $\sigma(\mathbf{g}, N)$ also have large $\rho(\mathbf{g}, N)$. This method allows one to use the comprehensive tables of parameters for LCGs with excellent spectral test indices that are available from L'Ecuyer [20]. The search strategy employed in [20] assures that all projections of the associated node sets onto the first d coordinates, $d \leq 32$, also have an excellent uniform distribution. This stability with respect to projections is an important aspect for numerical practice. For smaller dimensions and a sample size N equal to $2^{31} - 1$, 2^{32}, or 2^{48}, additional tables can be found in Fishman and Moore [10] and Fishman [8,9].

It would have been much more costly to search for the best LCGs in terms of the figure of merit P_2 than in terms of the spectral test index. Computing P_2 takes $\mathcal{O}(sm)$ time in general and $\mathcal{O}(m)$ time for a full period LCG, where m denotes the modulus and the number of points. Searching the best in a given dimension thus takes $\mathcal{O}(m^2)$ time. For the spectral test, the *worst-case* time complexity is exponential in $\sigma(\mathbf{g}, N)$ and polynomial in s for a fixed $\sigma(\mathbf{g}, N)$ in general (see [7]). Nevertheless, the branch-and-bound algorithm, coupled with a few heuristics to improve the speed, works very well in practice. For example, with the programs of [21], computing the spectral test index in dimensions 2 to 32 for an arbitrary Korobov rule with $N \approx 2^{30}$ takes approximately one second on a Pentium II computer. Computing P_2 for the same values of s and N and on the same machine takes approximately 31 minutes with our programs. As another example, to make an exhaustive search for the best Korobov rules for which a is a primitive element modulo N, in terms of the spectral test in dimensions up to 8, 16, and 32 for $N = 2^{22} - 3$ (the last entry of page 253 in [20]), it takes 25 minutes of computing time on a Pentium II. Performing the search in terms of P_2 (assuming that we find a proper way to normalize it) would take approximately 10000 hours (more than a year) on the same computer.

Theorem 2. *Let the integers N and s and the integer vector $\mathbf{g} \in \mathbb{Z}^s$ be as in Definition 1. Then*

$$\rho(\mathbf{g}, N)^2 \geq \sigma(\mathbf{g}, N)^2 - (s-1). \tag{4}$$

Proof. For the ease of notation, we write $\sigma := \sigma(\mathbf{g}, N)$ and $\rho := \rho(\mathbf{g}, N)$. For any $\mathbf{k} \in \mathbb{Z}^s \setminus \{\mathbf{0}\}$, we have the inequality

$$r(\mathbf{k})^2 \geq ||\mathbf{k}||_2^2 - (s-1).$$

This relation is easily verified by induction on the dimension s.

Any integer point $\mathbf{k} \neq \mathbf{0}$ with $\mathbf{k} \cdot \mathbf{g} \equiv \mathbf{0} \pmod{N}$ has an euclidean distance of at least σ from the origin $\mathbf{0}$, which implies

$$r(\mathbf{k})^2 \geq \sigma^2 - (s-1).$$

The relation

$$\rho = \min \left\{ r(\mathbf{k}) : ||\mathbf{k}||_2 \geq \sigma, \ \mathbf{k} \cdot \mathbf{g} \equiv \mathbf{0} \pmod{N} \right\}.$$

yields the result. □

Theorem 2 shows that good (i.e. large) values of $\sigma(\mathbf{g}, N)$ imply large values for $\rho(\mathbf{g}, N)$. The reverse inequality to (4) has the form

$$\rho(\mathbf{g}, N)^{1/s} \leq s^{-1/2} \, \sigma(\mathbf{g}, N).$$

It is derived from the inequality

$$r(\mathbf{k})^{1/s} \leq s^{-1/2} \, ||\mathbf{k}||_2,$$

which is elementary to prove, see, for example, Beckenbach and Bellman [1, Ch. 16, p. 16].

In fact, σ and ρ are figures of merit for *lattices*, and the numbers $\sigma(\mathbf{g}, N)$ and $\rho(\mathbf{g}, N)$ are just the figures of merit $\sigma(L)$ and $\rho(L)$ of the special lattice L spanned by the vector $(1/N)\mathbf{g}$ and the vectors $\mathbf{e}_2, \ldots, \mathbf{e}_s$ of the canonical basis in $\mathbb{R}^s$, see [26] and [29]. Theorem 2 holds for general lattices in $\mathbb{R}^s$, not only for Korobov rules. The next example shows that for general lattices L, the ratio $\rho(L)/\sigma(L)$ can be made arbitrarily large by a proper choice of parameters. Therefore, while a large σ implies a large ρ, the opposite is not true in general.

Example 3. (Unboundedness of $\rho(L)/\sigma(L)$ for general lattices) We will exhibit lattices L where the ratio $\rho(L)/\sigma(L)$ can be made arbitrarily large by a proper choice of parameters. Let L be the s-dimensional lattice given by the basis

$$\begin{aligned}
\mathbf{v}_1 &= (c, 0, 0, \ldots, 0)/m \\
\mathbf{v}_2 &= (-1, 1, 0, \ldots, 0)/m \\
\mathbf{v}_3 &= (-1, 0, 1, \ldots, 0)/m \\
\vdots &\quad \vdots \\
\mathbf{v}_s &= (-1, 0, 0, \ldots, 1)/m,
\end{aligned}$$

where b and m are powers of 2, with $b < m$, and $c = m/b$. The dual lattice L^* of L is generated by the basis

$$\begin{aligned}
\mathbf{w}_1 &= (b, b, b, \ldots, b) \\
\mathbf{w}_2 &= (0, m, 0, \ldots, 0) \\
\mathbf{w}_3 &= (0, 0, m, \ldots, 0) \\
\vdots &\quad \vdots \\
\mathbf{w}_s &= (0, 0, 0, \ldots, m).
\end{aligned}$$

Now, assume that $s \geq 2$ and that $b^s \leq m$. Then a shortest vector in L^* with respect to the Euclidean norm is $\mathbf{w}_1$. As a consequence, its square length sb^2 is equal to the spectral test index $\sigma(L)^2$ of the lattice L. A vector $\mathbf{w} = (w_1, \ldots, w_s) \in L^*$ with minimal value of $r(\mathbf{w}) = \prod_{j=1}^{s} \max(1, |w_i|)$ is also $\mathbf{w}_1$. For large dimensions s, the figure of merit $\rho(L)^2 = b^{2s}$ is much larger than $\sigma(L)^2 = sb^2$. The ratio $\rho(L)/\sigma(L) = b^{s-1}/\sqrt{s}$ can be made arbitrarily large by proper choices of the parameters s and b. In fact, the lattice L has $m^{s-1}b$ points in the hypercube $[0,1[^s$, and all these points are contained in only sb equidistant hyperplanes which are orthogonal to the vector $\mathbf{w}_1$.

For $i = 1, \ldots, s$, let L_i be the lattice generated by the vector $\mathbf{v}_i$ and by $\mathbb{Z}^s$. Then L can be viewed as the direct sum of the lattices $L_1, \ldots, L_s$. Each L_i can be written as

$$L_i = \{a\mathbf{v}_i \pmod 1 : a \in \mathbb{Z}\} + \mathbb{Z}^s,$$

and one has

$$L_i \cap [0,1[^s = \{a\mathbf{v}_i \pmod 1 : a = 0, \ldots, m-1\}.$$

In other words, in the terminology of Sloan and Joe [29], each L_i determines a lattice rule of rank 1 and L gives a lattice rule of rank s.

3 Numerical Results

In this section, we compare the performance of GLPs selected with respect to the spectral test to some of the best available (t, m, s)-nets.

For our numerical comparisons we used (t, m, s)-nets to the base two (abbrev. NET) constructed over irreducible polynomials, see [2,25], and rank-1 lattice rules $\omega_{\mathbf{g}}$ defined by GLP-parameters from [20, Table 2]. From the latter paper we took every first multiplier a for each prime modulus $N \approx 2^m$, where $m = 16, \ldots, 23$. These parameters are optimized with respect to the spectral test value for dimensions up to 8. The exact values of t for our (t, m, s)-nets, which have been calculated by Schmid [28], and the GLP-parameters a that we have used are given in Tables 1 and 2, respectively.

The following class of test functions was chosen because (i) it fits well the testing approach of Genz [11] and Sloan and Joe [29, Ch. 11], and (ii) it belongs to classes of rapidly converging Haar series which are in a certain sense optimal for numerical integration with (t, m, s)-nets, see Entacher [5,6] for details.

For $\mathbf{c} := (c_1, \ldots, c_s) \in [0,1]^s$, define $f_{\mathbf{c}} : [0,1]^s \longrightarrow \mathbb{R}$ with

$$f_{\mathbf{c}}(x_1, \ldots, x_s) := (c_1 x_1 + \ldots + c_s x_s)^{\alpha}, \quad \alpha \in \mathbb{N}. \tag{5}$$

In accordance with Sloan and Joe [29, Ch. 11], for each dimension s considered and each sample size N, we have initialized twenty vectors $\mathbf{c}$, distributed uniformly over $[0,1[^s$, and determined the quasi-Monte Carlo approximations $S_N(f_{\mathbf{c}}, \omega)$. The initialization was carried out with an explicit-inversive congruential pseudorandom number generator (see Eichenauer-Herrmann [4] and Niederreiter [27]) with modulus $m = 2^{31} - 1$, multiplier $a = 1234567$, additive constant $b = 0$ and seed $n_0 = 1234567$. Further empirical research might focus on the influence of the distribution (normal, exponential, ...) of the vectors $\mathbf{c}$ on the integration errors and on a comprehensive study of integrands of the Genz suite.

Table 3 below shows the mean values $\frac{1}{20}\sum E_{\mathbf{c}}$ of the relative integration errors $E_{\mathbf{c}} := |I(f_{\mathbf{c}}) - S_N(f_{\mathbf{c}}, \omega)|/I(f_{\mathbf{c}})$ for dimensions $s = 5 \cdot j$, $1 \le j \le 6$ and $\alpha = 2, 3$. The smallest mean errors are indicated by boldface.

We would like to state explicitly that any empirical results like those presented below may only indicate but cannot prove the existence of phenomena, and the results may be quite different for other integrands and other test designs. This applies, of course, to any empirical testing in this field.

Other recent experiments with GLPs selected via the spectral test, for simulation problems related to finance and risk analysis, have confirmed their competitivity especially for problems in large dimensions, see Lemieux and L'Ecuyer [22,23].

Table 1. Values of t for the (t, m, s)-Nets

NET Parameters						
$m\backslash s$	5	10	15	20	25	30
16	3	9	10	13	13	13
17	3	9	10	13	13	13
18	3	9	11	13	14	14
19	3	10	12	14	14	15
20	3	11	12	14	15	15
21	3	12	13	14	15	16
22	3	12	14	14	16	16
23	3	12	14	15	17	17

Table 2. The GLP Parameters N and a

GLP Parameters	
N	a
65521	17364
131071	43165
262139	92717
524287	283741
1048573	380985
2097143	360889
4194301	914334
8388593	653276

Table 3. Average Values of E_c for $\alpha = 2, 3$

$\alpha = 2$

	N	$s = 5$	$s = 10$	$s = 15$	$s = 20$	$s = 25$	$s = 30$
NET	65536	**1.89e-05**	2.69e-05	3.94e-05	1.27e-04	1.63e-04	1.53e-04
GLP	65521	5.84e-05	**1.52e-05**	**1.54e-05**	**2.92e-05**	**8.63e-06**	**8.71e-06**
NET	131072	**1.52e-05**	**2.61e-05**	**3.87e-05**	**9.19e-05**	**1.15e-04**	**1.15e-04**
GLP	131071	5.24e-04	9.31e-04	7.91e-04	6.38e-04	5.63e-04	4.84e-04
NET	262144	2.87e-06	1.59e-05	2.86e-05	1.96e-05	**2.10e-05**	3.55e-05
GLP	262139	**1.61e-06**	**6.98e-06**	**1.22e-05**	**1.59e-05**	2.18e-05	**2.75e-05**
NET	524288	**2.83e-06**	**1.10e-05**	**1.86e-05**	**1.41e-05**	1.61e-05	2.85e-05
GLP	524287	7.88e-06	1.59e-05	1.90e-05	1.66e-05	**1.46e-05**	**1.28e-05**
NET	1048576	**1.42e-06**	**1.78e-06**	**1.67e-06**	**2.83e-06**	5.81e-06	1.49e-05
GLP	1048573	6.96e-06	8.83e-06	6.99e-06	3.35e-06	**1.31e-06**	**4.25e-07**
NET	2097152	**5.02e-07**	**1.01e-06**	**1.22e-06**	2.75e-06	3.31e-06	2.65e-06
GLP	2097143	4.12e-06	3.31e-06	1.76e-06	**4.03e-07**	**3.51e-07**	**5.95e-07**
NET	4194304	**2.79e-07**	**9.39e-07**	**1.21e-06**	2.45e-06	2.91e-06	2.51e-06
GLP	4194301	2.50e-06	2.57e-06	2.68e-06	**2.02e-06**	**1.93e-06**	**1.84e-06**
NET	8388608	**2.33e-07**	**6.90e-07**	**9.85e-07**	1.75e-06	2.03e-06	1.82e-06
GLP	8388593	1.29e-06	1.59e-06	1.60e-06	**1.44e-06**	**1.36e-06**	**1.21e-06**

$\alpha = 3$

	N	$s = 5$	$s = 10$	$s = 15$	$s = 20$	$s = 25$	$s = 30$
NET	65536	**2.12e-05**	2.72e-05	5.09e-05	1.78e-04	2.33e-04	2.41e-04
GLP	65521	1.27e-04	**2.47e-05**	**1.95e-05**	**5.52e-05**	**8.57e-06**	**5.35e-05**
NET	131072	**1.65e-05**	**3.16e-05**	**5.45e-05**	**1.31e-04**	**1.65e-04**	**1.85e-04**
GLP	131071	1.34e-03	2.58e-03	2.25e-03	1.85e-03	1.64e-03	1.43e-03
NET	262144	**4.71e-06**	2.01e-05	4.05e-05	**2.36e-05**	**2.66e-05**	**5.24e-05**
GLP	262139	6.16e-06	**1.25e-05**	**2.75e-05**	3.86e-05	5.60e-05	7.28e-05
NET	524288	**3.11e-06**	**1.45e-05**	**2.64e-05**	**1.81e-05**	**2.13e-05**	**4.04e-05**
GLP	524287	2.28e-05	4.73e-05	5.74e-05	5.13e-05	4.59e-05	4.11e-05
NET	1048576	**1.71e-06**	**1.81e-06**	**1.67e-06**	**3.06e-06**	7.73e-06	2.20e-05
GLP	1048573	1.63e-05	2.27e-05	1.81e-05	7.83e-06	**2.51e-06**	**2.32e-06**
NET	2097152	**6.88e-07**	**1.11e-06**	**1.40e-06**	3.61e-06	4.45e-06	4.07e-06
GLP	2097143	1.13e-05	9.97e-06	5.85e-06	**1.43e-06**	**6.87e-07**	**8.63e-07**
NET	4194304	**3.66e-07**	**1.15e-06**	**1.58e-06**	**3.43e-06**	**4.10e-06**	**4.15e-06**
GLP	4194301	6.81e-06	7.51e-06	8.05e-06	6.26e-06	6.05e-06	5.83e-06
NET	8388608	**2.79e-07**	**8.94e-07**	**1.37e-06**	**2.49e-06**	**2.90e-06**	**3.18e-06**
GLP	8388593	3.15e-06	4.18e-06	4.32e-06	3.93e-06	3.73e-06	3.29e-06

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