Partial Hedging for Options Based on Extreme Values and Passage Times*

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Abstract

A hedger of a contingent claim may decide to partially replicate on some states of nature and not on the others: A partial hedge initially costs less than a perfect hedge. However, a partial hedge may lead to a default position. It is of interest in that context to estimate the gain and the default risk. Some partial hedging strategies based on the final primitive asset price, its maximum over the trading period, and the time to maximum, are analyzed. Closed-form solutions are derived in the Black and Scholes (1973) model and efficient Monte Carlo estimates are computed using a stochastic volatility model. The results show how the gain and the default risk inversely change depending on the hedging event.

1 Introduction

The basic idea used to evaluate a contingent claim in a frictionless, arbitrage-free and complete market is the possibility of hedging it with certainty by

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managing an associated replication portfolio. The initial cost of this portfolio is equal to the price of the contingent claim. The value of this portfolio matches with certainty its payoff at the exercise date. A hedger of a contingent claim may opt for partial replication since it costs less than a full replication, allowing thus an initial gain. However, this agent may fall in default, with probability that depends on the partial hedging event. We call this probability the *default risk*. Sellers of contingent claims who do not perfectly hedge their risk are commonplace in real life.

The aim of this paper is to compute the (initial) gain and the (final) default risk when the hedging event depends on the final primitive asset price, its maximum over the trading period, and the time at which this maximum occurs.

In the following, we consider models where no arbitrage and hedging arguments are used to evaluate contingent claims. Significant contributions dealing with hedging contingent claims are found in Harrison and Kreps (1979) and Harrison and Pliska (1981). In a frictionless, arbitrage-free, and complete market, a perfect hedge cannot be realized with an initial wealth which is less than the price of the contingent claim. A result derived by Föllmer (1995), and reported in Karatzas [(1996), page 57], shows how to minimize the default risk of a standard option in the Black and Scholes model when starting with an initial wealth which is less than required. Similar results for more complex contingent claims and for more general models were not available so far, to the best of our knowledge.

In the second section, a general model for options pricing is recalled. In the third section, some partial hedging strategies are analyzed. First, closed-form solutions are derived for the gain and the default risk in the Black and Scholes (1973) model. Then, efficient Monte Carlo estimates are computed in a stochastic volatility model using two correlation induction techniques: Antithetic Variates and Control Variates. In the simulation experiments, the closed-form solutions derived from the Black and Scholes model are used as control variables to improve the precision of the crude Monte Carlo estimators. In the conclusion, further applications are suggested.

2 The Model

The model described in this section is presented in detail in Karatzas [(1996), Chapters 0 and 1]. Let M be a market with $p+1$ traded assets in which trading takes place continuously over the period $[0, T]$. In our presentation, a stochastic process $X(t)$ for $t \in [0, T]$ is denoted by $X(\cdot)$. Let (Ω,F,P) be a probability space, $W(\cdot) = (W_1(\cdot),\ldots,W_d(\cdot))'$ a d-dimensional Brownian motion, and $F(\cdot)$ the P-augmented (natural) filtration of $W(\cdot)$. The σ-algebra F can be chosen as $F(T)$. The Brownian motions $W_j(\cdot)$, for $j = 1,\ldots,d$, can be interpreted as d sources of systematic risk and the filtration $F(\cdot)$ as a collection of the increasing sets of information available to investors over time. All the stochastic processes are assumed to be adapted to the filtration $F(\cdot)$ and verify some additional conditions which guarantee their existence and uniqueness. The adaptability requirement allows dependence on past realizations and precludes anticipation of future realizations. Assume that the market M is frictionless, that is, there are no taxes, transaction costs, information asymmetries, constraints on short selling, or any other friction.

The first asset, called the *bank account*, does not pay dividends. It starts at unity, and moves according to the differential equation

$$dB(t) = B(t)r(t)dt, \quad \text{for } B(0) = 1 \text{ and } 0 \le t \le T, \tag{1}$$

where $r(\cdot)$ is the interest rate process. The solution of (1) yields the following definition for the *discount factor*

$$\gamma(t) = 1/B(t) = e^{-\int_0^t r(s)ds}, \quad \text{for } 0 \le t \le T.$$

If the interest rate process is constant, $r(\cdot) = r$, the discount factor can be written as

$$\gamma(t) = e^{-rt}, \quad \text{for } 0 \le t \le T. \tag{2}$$

The p remaining assets, called the *primitive assets*, move according to the stochastic differential equations

$$\begin{aligned} dS_i(t) &= \mu_i(t)S_i(t)dt + \sum_{j=1}^{d} \sigma_{ij}(t)S_i(t)dW_j(t), \qquad (3) \\ &\quad \text{for } S_i(0) > 0,\ i = 1,\ldots,p, \text{ and } 0 \le t \le T, \end{aligned}$$

where $\mu(\cdot) = (\mu_1(\cdot),\ldots,\mu_p(\cdot))'$ is the vector-process of the *appreciation rates* and $\sigma(\cdot) = (\sigma_{ij}(\cdot))$ is the matrix-process of *volatility*. For simplicity, assume that the primitive assets do not pay dividends.

The Black and Scholes (1973) model assumes a constant interest rate, one source of systematic risk, and one primitive asset. Specifically:

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t), \quad \text{for } S(0) > 0 \text{ and } 0 \le t \le T, \tag{4}$$

where μ and σ are assumed to be (positive) constants. Another interesting case is a stochastic volatility model similar to those discussed by Hull and White (1987), Johnson and Shanno (1987), Scott (1987), and Wiggins (1987). Specifically,

$$dS(t) = \mu S(t)dt + \sigma(t)S(t)dW_1(t), \quad \text{for } S(0) > 0 \text{ and } 0 \leq t \leq T, \tag{5}$$

where the volatility process $\sigma(\cdot)$ is random and depends on a two-dimensional Brownian motion $W(\cdot) = (W_1(\cdot), W_2(\cdot))'$, as we shall see later.

An investor trades continuously on the $p + 1$ traded assets by managing a self-financing portfolio which generates the wealth process

$$X(t) = \sum_{i=1}^{p} \pi_i(t) + \left(X(t) - \sum_{i=1}^{p} \pi_i(t) \right), \quad \text{for } 0 \leq t \leq T,$$

where $\pi(\cdot) = (\pi_1(\cdot),\ldots,\pi_p(\cdot))'$, called the *portfolio strategy*, is the vector-process of the dollar amounts invested in the primitive assets. Notice that the components of this portfolio strategy can be positive or negative, depending on the position (long or short) of the investor. The wealth process is denoted by $X^{x,\pi}(\cdot)$ since it depends on the initial wealth $X(0) = x$ and the portfolio strategy $\pi(\cdot)$.

An *arbitrage opportunity* is a strategy $\pi(\cdot)$ such that

$$P(X^{0,\pi}(T) \geq 0) = 1 \text{ and } P(X^{0,\pi}(T) > 0) > 0.$$

A rational investor should take this opportunity when it appears because it costs nothing to hold and may lead to a positive final wealth. A market without arbitrage opportunities is called *arbitrage-free*. There are two sufficient conditions for the arbitrage-free property to be verified. Firstly, there exists a process $\theta : [0, T] \times \Omega \rightarrow \mathbb{R}^d$, called the *market-price of risk*, which verifies

$$\mu(\cdot) - r(\cdot)1_p = \sigma(\cdot)\theta(\cdot), \tag{6}$$

where $1_p = (1,\ldots,1)' \in \mathbb{R}^p$. Secondly, the exponential process

$$Z(t) = e^{-\frac{1}{2}\int_0^t ||\theta(t)||^2 dt - \int_0^t \theta(t)' dW(t)}, \quad \text{for } 0 \leq t \leq T, \tag{7}$$

is a P-martingale. In practice, it is somewhat hard to check for (7). However, a sufficient requirement for (7) to hold is the so-called Novikov condition

$$E^P \left[e^{-\frac{1}{2}\int_0^t ||\theta(t)||^2 dt} \right] < +\infty. \tag{8}$$

The Black and Scholes market defined in (4) is arbitrage-free since $\theta = (\mu - r)/\sigma$ does exist and verifies the Novikov condition (8). In the stochastic volatility model defined in (5), assume that the volatility process $\sigma(\cdot)$ is chosen such that the model is arbitrage-free.

Assume that the conditions (6) and (8) hold and that $p \leq d$. The market is said to be *standard*. The exponential process $Z(\cdot)$ defined in (7) allows the construction of a collection of P-equivalent probability measures Q_t for $t \in [0, T]$, each defined on the corresponding σ-algebra $F(t)$ by

$$Q_t(A) = E^P\left[Z(t)1_A\right], \quad \text{for } A \in F(t).$$

The probability measure Q_T, denoted by Q, is called the *risk-neutral* probability measure. An important feature of Q is that the process

$$B(t) = W(t) + \int_0^t \theta(s)ds, \quad \text{for } 0 \leq t \leq T, \tag{9}$$

is a Q-Brownian motion. In turn, the equations described in (3) can be transformed into

$$\begin{aligned} dS_i(t) &= r(t)S_i(t)dt + \sum_{j=1}^{d} \sigma_{ij}(t)S_i(t)dB_j(t), \\ &\quad \text{for } S_i(0) > 0,\ i = 1, \ldots, p, \text{ and } 0 \leq t \leq T, \end{aligned} \tag{10}$$

or, equivalently, into

$$\begin{aligned} d(\gamma(t)S_i(t)) &= \gamma(t)\sum_{j=1}^{d} \sigma_{ij}(t)S_i(t)dB_j(t), \\ &\quad \text{for } S_i(0) > 0,\ i = 1, \ldots, p, \text{ and } 0 \leq t \leq T, \end{aligned} \tag{11}$$

where $B(\cdot) = (B_1(\cdot), \ldots, B_d(\cdot))'$ is a Q-Brownian motion as expressed in (9). Equation (10) shows that one can ignore the appreciation rate $\mu(\cdot)$ when computing expectations in the form $E^Q[f(S(t),\ t \in I \subset [0,\ T])]$ where $f : \mathbb{R}^{|I|} \rightarrow \mathbb{R}$. Equation (11) shows that each discounted primitive asset price is a Q-martingale. The notion of efficiency implied by this result is that the "best prediction" of $\gamma(T)S(T)$ is $S(0)$.

A *contingent claim* is any non-negative and $F(T)$-measurable random variable Y such that $E^Q[\gamma(T)Y] < +\infty$. One can think of a contract that gives a payoff of Y at time T. This contract should be interpreted as a privilege since it gives its holder a non-negative amount. Contingent claims that take the form $f(S(t),\, t \in I \subset [0,\, T])$ are called *derivatives*. *Option*contracts are a subclass of derivatives. For example, a *call*option pays $C(T) = (S(T) - E)^+$ at time T where $x^+ = \max(x,\, 0)$ for $x \in \mathbb{R}$. This contract gives the holder the right to buy the primitive asset at time T at a specified *exercise price* E. A *put* option pays $P(T) = (E - S(T))^+$. This contract gives the holder the right to sell the primitive asset at time T at a specified exercise price E. Notice that

$$C(T) - P(T) = S(T) - E. \tag{12}$$

A *complete market* is a market on which any contingent claim Y is attainable by a well-selected self-financing portfolio strategy $\pi(\cdot)$:

$$\forall\ Y,\ \exists\ \pi(\cdot) \text{ such that } X^{v,\pi}(T) = Y, \tag{13}$$

where

$$v = E^Q\left[\gamma(T)Y\right]. \tag{14}$$

In a complete market, one can start at the initial wealth v defined in (14) and find a portfolio strategy $\pi(\cdot)$ as in (13), called the *replication portfolio*, such that the final wealth matches with certainty the contingent claim payoff. In a standard and complete market, the discounted wealth process $\gamma(\cdot)X^{v,\pi}(\cdot)$ satisfies the martingale property

$$\gamma(s)X^{v,\pi}(s) = E^Q\left[\gamma(t)X^{v,\pi}(t) \mid F(s)\right], \quad \text{for } 0 \le s \le t \le T. \tag{15}$$

Equation (14) can be obtained from (15) at $s = 0$ and $t = T$. In fact, the wealth process $X^{v,\pi}(\cdot)$ matches with certainty the contingent claim's value during all the trading period $[0, T]$. This is why the wealth process is called the *price-process* of the contingent claim. In a standard and complete market, the value v defined in (14) is the unique rational price of the contingent claim Y. If the market is only standard, there exists in general a whole range of prices

$$I_v = [v_{\min},\, v_{\max}],$$

including the value defined in (14), that are consistent with the arbitrage-free property.

Assume that the interest rate is constant as in (2). Using equations (12) and (14) and the martingale property of $\gamma(\cdot)S(\cdot)$ described in (15), one obtains the put-call parity relationship

$$c - p = S(0) - e^{-rT}E, \tag{16}$$

where c is the price of the call option and p is the price of the corresponding put option. Equation (16) shows that the value of a put option can often be deduced from the value of a call option.

A simple criterion exists to check for the completeness property in a standard market. A standard market is complete if and only if

$$p = d, \tag{17}$$

and

$$\sigma(t, \omega) \text{ is regular for } t \in [0,\ T] \text{ and } \omega \in \Omega. \tag{18}$$

Roughly speaking, the completeness property is a question of dimension: There must be as many sources of systematic risk as primitive assets. The Black and Scholes model described in (4) is complete since it is standard and verifies $p = d = 1$ and $\sigma(t,\omega) = \sigma > 0$ for $t \in [0,\ T]$ and $\omega \in \Omega$. On the other hand, based on equation (17), the stochastic volatility model described in (5) is *incomplete*.

3 Partial Hedging in Complete Markets

Assume that the market is standard and complete. The seller of a contingent claim Y can hedge perfectly all risk by starting at the initial wealth v defined in (14) and managing the replication portfolio $\pi(\cdot)$ mentioned in (13). Let 1_A be the indicator of an event $A \in F(T)$. Instead of hedging Y, this investor may want to hedge $Y 1_A$. This is less expensive to replicate, since if we define $u = E^Q\left[\gamma(T) Y 1_A\right]$ and $v = E^Q\left[\gamma(T) Y\right]$, then

$$g = v - u > 0. \tag{19}$$

The replication takes place only on the hedging event $H = \{Y = 0\} \cup A$. The *gain*is defined as $g = v - u$ and the *default event* is defined as $H^c = \{Y > 0\} \cap A^c$. The *default risk*is

$$P(H^c) = E^P\left[1_{H^c}\right]. \tag{20}$$

Notice that $P(H^c)$ depends on the appreciation rate $\mu(\cdot)$ which is assumed to be constant in the following.

In the next subsections, some partial hedging strategies are analyzed in terms of the gain and default risk. These strategies account for the final primitive asset price, its maximum during the trading period, and the time at which this maximum occurs. In the first subsection, closed-form solutions are derived for the Black and Scholes model. In the second subsection, efficient Monte Carlo estimators are developed for a stochastic volatility model.

3.1 Partial Hedging in the Black and Scholes Model

3.1.1 Partial Hedging when $A = \{E \leq S(T) \leq a\}$

Consider a partial replication of the call option $Y = (S(T) - E)^+$ when $A = \{E \leq S(T) \leq a\}$ for some real a greater than E. By equation (19), the gain is

$$\begin{aligned} g &= v - u, \\ &= E^Q\left[(S(T)-E)^+\right] - E^Q\left[(S(T)-E)1_{\{E \leq S(T) \leq a\}}\right]. \end{aligned}$$

The cost of the perfect hedge, denoted by v, is the Black and Scholes price which is known in closed-form. The cost of the partial hedge, denoted by u, can also be computed in closed-form as follows. In the Black and Scholes model defined in (4), the final primitive asset price can be written as

$$S(T) = S(0)e^{(r-\sigma^2/2)T+\sigma\sqrt{T}Z},$$

where Z is a standard normal random variable. The primitive asset price $S(T)$ is then lognormal, and from this we can derive (after some algebraic manipulations) the following expression for u:

$$\begin{aligned} u &= S(0)\left[N(d_1) - N(d_1')\right] - Ee^{-rT}\left[N(d_2) - N(d_2')\right], \qquad (21) \\ d_1 &= \left[\log(S(0)/E) + (r+\sigma^2/2)T\right]/\sigma\sqrt{T}, \\ d_2 &= d_1 - \sigma\sqrt{T}, \\ d_1' &= \left[\log(S(0)/a) + (r+\sigma^2/2)T\right]/\sigma\sqrt{T}, \\ d_2' &= d_1' - \sigma\sqrt{T}, \end{aligned}$$

where $N(\cdot)$ is the cumulative normal distribution.

Notice the similarity with the Black and Scholes formula since $u \to v$ when $a \to +\infty$. An agent who applies this partial hedging strategy $\pi(\cdot)$ must hold initially $N(d_1) - N(d_1')$ shares of the primitive asset, while a perfect hedge requires $N(d_1)$. The default risk measured under Q can be derived in a similar way. One obtains

$$Q(H^c) = E^Q\left[1_{\{S(T)>a\}}\right] = N(d_2').$$

The default risk $P(H^c)$ is deduced from $Q(H^c)$ by substituting μ for r where μ is the appreciation rate of $S(\cdot)$.

The parameters of the option to be evaluated are: $S(0) = 100$, $E = 100$, $T = 0.5$, $\sigma = 0.15$, and $r = 0.05$. The partial hedging parameter is a. A numerical illustration is given in Table 1 whose last column, denoted by ∞, reports the cost v of a full replication. Each cell of this table contains the exact solution computed by numerical integration.

Table 1: Partial Hedging of a Call Option when $A = \{E \leq S(T) \leq a\}$

a	120	125	130	135	150	∞
u	3.9642	4.7197	5.1534	5.3703	5.5199	5.5271
g	1.5630	0.8075	0.3737	0.1569	0.0072	0
$P(H^c)$ for $\mu = 0.05$	0.0622	0.0274	0.0110	0.0041	0.0001	0
$P(H^c)$ for $\mu = 0.10$	0.0967	0.0460	0.0199	0.0080	0.0003	0

When the hedging parameter a increases, the cost u of the partial replication increases and converges to the Black and Scholes price $v = 5.5271$. At the same time, the default risk converges to zero. For example, if the seller decides to hedge the call option only when $S(T) \leq 130$ and not on the others states of nature, he can do so with an initial wealth $u = 5.1534$. This results in a gain of $g = 0.3737$ over the perfect hedge. Nevertheless, the hedger will fall into default with probability $P(H^c) = 0.011$ for $\mu = 0.05$, and $P(H^c) = 0.0199$ for $\mu = 0.1$. Notice that $P(H^c)$ is an increasing function of μ.

3.1.2 Partial Hedging when $A = \left\{E \leq S(T) \leq a,\ M^S(T) \leq b\right\}$

We now consider a hedging event of the form

$$A = \left\{E \leq S(T) \leq a,\ M^S(T) \leq b\right\}, \quad \text{for } E < a < b,$$

where the random variable $M^S(T) = \max\{S(t),\ t \in [0,\ T]\}$ is the maximum attained by the primitive asset price over the trading period. By the Girsanov

Theorem [Karatzas and Shreve (1991), Section 3.5], there exists a probability measure $\widetilde{Q}$ under which the process

$$X(\cdot) = \log(S(\cdot)/S(0))/\sigma,$$

is a $\widetilde{Q}$-Brownian motion. The probability measure $\widetilde{Q}$ is defined by its Radon-Nikodym likelihood ratio

$$dQ/d\widetilde{Q} = \widetilde{Z}(T) = e^{(r-\sigma^2/2)X(T)/\sigma-(r-\sigma^2/2)^2T/2\sigma^2}.$$

This result has been used judiciously by Conze and Viswanathan (1991) to derive explicit formulas for several lookback options using the risk-neutral evaluation approach. The original results, solutions of a partial differential equation, are derived by Goldman, Sosin, and Gatto (1979).

This change of measure allows the use of the known density function of $(X(T), M^X(T))$ [Karatzas and Shreve (1991), Section 2.8]

$$\varphi(x,y) = 2(2\pi T^3)^{-1/2}(2y-x)e^{-(2y-x)^2/2T}, \quad \text{for } y \geq \max(x,0),$$

where $M^X(T) = \max\{X(t),\ t \in [0,\ T]\}$ is the maximum attained by the $\widetilde{Q}$-Brownian motion $X(\cdot)$ during $[0,\ T]$. This result allows for the derivation of the closed-form solutions

$$\begin{aligned} u &= E^Q\left[e^{-rT}(S(T)-E)1_{\{E\leq S(T)\leq a,\ M^S(T)\leq b\}}\right], \\ &= E^{\widetilde{Q}}\left[e^{-rT}\widetilde{Z}(T)(S(0)e^{\sigma X(T)}-E)1_{\left\{\widetilde{E}\leq X(T)\leq\widetilde{a},\ M^X(T)\leq\widetilde{b}\right\}}\right], \end{aligned}$$

and

$$\begin{aligned} Q(H^c) &= E^Q\left[1_{H^c}\right], \\ &= E^Q\left[1_{\{S(T)>E\}}\right] - E^Q\left[1_{\{E\leq S(T)\leq a,\ M^S(T)\leq b\}}\right], \\ &= N(d_2) - E^{\widetilde{Q}}\left[\widetilde{Z}(T)1_{\left\{\widetilde{E}\leq X(T)\leq\widetilde{a},\ M^X(T)\leq\widetilde{b}\right\}}\right], \end{aligned}$$

where $\widetilde{z} = \log(z/S(0))/\sigma$ for $z \in \{E,\ a,\ b\}$. The default risk $P(H^c)$ is deduced from $Q(H^c)$ by substituting μ for r. These expectations are basically 2-dimensional integrals.

The parameters of the option to be evaluated are: $S(0) = 100$, $E = 100$, $T = 0.5$, $\sigma = 0.15$, and $r = 0.05$. The partial hedging parameters are a and $b = a + 3$. Results are shown in Table 2 whose last column, denoted by ∞, reports the cost v of a full replication. Each cell of this table contains the exact solution computed by numerical integration.

Table 2: Partial Hedging of a Call when $A = \left\{E \leq S(T) \leq a,\ M^S(T) \leq b\right\}$

a	120	125	130	135	150	∞
u	3.6810	4.5688	5.0841	5.3418	5.5187	5.5271
g	1.8461	0.9583	0.4430	0.1853	0.0084	0
$P(H^c)$ for $\mu = 0.05$	0.0805	0.0347	0.0137	0.0050	0.0002	0
$P(H^c)$ for $\mu = 0.10$	0.1203	0.0564	0.0242	0.0095	0.0004	0

For example, if the seller decides to hedge the call option only when $S(T) \leq 130$ and $M^S(T) \leq 133$, and not on the others states of nature, he can do so with an initial wealth $u = 5.0841$. This results in a gain of $g = 0.4430$ over the perfect hedge. Nevertheless, the hedger will fall into default with probability $P(H^c) = 0.0137$ for $\mu = 0.05$, and $P(H^c) = 0.0242$ for $\mu = 0.1$. In comparison with the results of Table 1, here the cost of any partial hedging strategy is slightly smaller and the default risk is slightly larger.

3.1.3 Partial Hedging on $A = \{E \leq S(T) \leq a,\ M^S(T) \leq b,\ \theta^S(T) \leq s\}$

Consider now a partial hedging strategy on the event

$$\begin{aligned} A &= \left\{E \leq S(T) \leq a,\ M^S(T) \leq b,\ \theta^S(T) \leq s\right\}, \\ &\quad \text{for } E \leq a \leq b \text{ and } 0 < s < T, \end{aligned}$$

where $\theta^S(T) = \inf\left\{t \in [0,\ T],\ S(t) = M^S(T)\right\}$ is the first time when the primitive asset attains its maximum over the trading period. The random variable $\theta^S(T)$ is an example of a random time which is not a stopping time. The same change of measure as in Section 3.1.2 allows one to use the known density function of $(X(T),\ M^X(T),\ \theta^X(T))$ [Karatzas and Shreve (1991), Section 2.8]

$$\begin{aligned} \phi(x,\ y,\ \theta^X(T) \ &< \ s) = 2(2\pi T^3)^{-.5}\left[N(-\alpha_+/\beta)(2y-x)e^{-(2y-x)^2/2T} - \right. \\ &\left. N(-\alpha_-/\beta)xe^{-x^2/2T}\right], \\ &\text{for } y \geq \max(x, 0) \text{ and } 0 < s < T, \end{aligned}$$

where $\alpha_\pm = (y(T-s) \pm (x-y)s)/T$ and $\beta^2 = s(T-s)/T$. From this, we can derive closed-form solutions for the cost of a partial hedge and its associated default risk. The results are

$$\begin{aligned} u &= E^Q\left[e^{-rT}(S(T)-E)1_{\left\{E \le S(T) \le a,\ M^S(T) \le b,\ \theta^S(T) \le s\right\}}\right], \\ &= E^{\widetilde{Q}}\left[e^{-rT}\widetilde{Z}(T)(S(0)e^{\sigma X(T)}-E)1_{\left\{\widetilde{E} \le X(T) \le \widetilde{a},\ M^X(T) \le \widetilde{b},\ \theta^X(T) \le s\right\}}\right], \end{aligned}$$

and

$$\begin{aligned} Q(H^c) &= E^Q[1_{H^c}], \\ &= E^Q\left[1_{\{S(T)>E\}}\right] - E^Q\left[1_{\left\{E \le S(T) \le a,\ M^S(T) \le b,\ \theta^S(T) \le s\right\}}\right], \\ &= N(d_2) - E^{\widetilde{Q}}\left[\widetilde{Z}(T)1_{\left\{\widetilde{E} \le X(T) \le \widetilde{a},\ M^X(T) \le \widetilde{b},\ \theta^X(T) \le s\right\}}\right]. \end{aligned}$$

These expectations are basically 3-dimensional integrals transformed into 2-dimensional integrals. The default risk $P(H^c)$ can be deduced from $Q(H^c)$ by substituting μ for r.

The parameters of the option to be evaluated are: $S(0) = 100$, $E = 100$, $T = 0.5$, $\sigma = 0.15$, and $r = 0.05$. The partial hedging parameters are a, $b = a + 3$, and $s = 0.48$. Results are shown in Table 3 whose last column, denoted by ∞, reports the cost v of a full replication. In that way, for $a = \infty$, one has $s = 0.5$. Each cell of this table contains the exact solution computed by numerical integration.

Table 3: Partial Hedging a call when
$A = \left\{E \le S(T) \le a, M^S(T) \le b, \theta^S(T) \le s\right\}$

a	120	125	130	135	150	∞
u	2.4497	2.9545	3.2201	3.3401	3.4112	5.5271
g	3.0774	2.5727	2.3070	2.1870	2.1159	0
$P(H^c)$ for $\mu = .05$	0.2001	0.1721	0.1605	0.1563	0.1542	0
$P(H^c)$ for $\mu = 0.10$	0.2600	0.2219	0.2047	0.1977	0.1939	0

If the seller decides to hedge the call option only when $S(T) \le 130$, $M^S(T) \le 133$, and $\theta^S(T) \le 0.48$, and not on the others states of nature, he can do so with an initial wealth $u = 3.2201$. This results in a gain of $g = 2.3070$ over the perfect hedge. Nevertheless, the hedger will fall into default with probability $P(H^c) = 0.1605$ for $\mu = 0.05$, and $P(H^c) = 0.2047$ for $\mu = 0.1$. In comparison with the results of Table 1 and Table 2, here the cost of any partial hedging strategy is significantly smaller and the default risk is significantly larger. The reason is that the primitive asset is likely to attain its maximum at the end of the period: The density function of the time to maximum $\theta^S(T)$ obeys the arcsin law [Karatzas and Shreve (1991), Section 2.8]:

$$h(s) = 2\pi^{-1} \arcsin(\sqrt{s/T}), \quad \text{for } 0 \leq s \leq T.$$

Closed-form solutions can also be derived if the hedging event depends on the final primitive asset price, its first passage time at a certain level, its maximum during the trading period, and the time to maximum.

3.2 Partial Hedging in a Stochastic Volatility Model

3.2.1 A Monte Carlo Experiment

All the random variables introduced in the following sections are assumed to have finite variance. The stochastic volatility model introduced in (5) is arbitrage-free but incomplete (see Section 2 for a justification). The dynamic of the primitive asset under Q is

$$dS(t) = rS(t)dt + \sigma_1(t)S(t)dB_1(t), \quad \text{for } S(0) > 0 \text{ and } 0 \leq t \leq T,$$

where the volatility process $\sigma_1(\cdot)$ is a function of a Brownian motion $B(\cdot) = (B_1(\cdot),\ B_2(\cdot))'$. Several dynamics for volatility have been proposed in the literature (see Detemple and Osakwe (1997) for a general specification). One of these is the following mean-reverting process

$$\begin{aligned} d\sigma_1(t) &= \alpha(\overline{\sigma} - \sigma_1(t))dt + \theta\sigma_1(t)(\rho dB_1(t) + \sqrt{1-\rho^2}dB_2(t)), \\ &\quad \text{for } 0 \leq t \leq T, \end{aligned}$$

where the coefficients α (the reverting rate), $\overline{\sigma}$ (the long-term volatility), θ (the volatility of the volatility), and ρ (the correlation between the innovations) are assumed to be constants. Statistical methods are needed to estimate these coefficients. For simplicity, we assume here that $\rho = 0$ so that

$$d\sigma_1(t) = \alpha(\overline{\sigma} - \sigma_1(t))dt + \theta\sigma_1(t)dB_2(t), \quad \text{for } 0 \leq t \leq T.$$

To make the hedging of contingent claims possible, a second primitive asset $S_2(\cdot)$ is introduced in the market. It is assumed to move under Q according to the stochastic differential equation

$$dS_2(t) = rS_2(t)dt + \sigma_2 S_2(t)dB_2(t), \quad \text{for } S_2(0) > 0 \text{ and } 0 \leq t \leq T,$$

where σ_2 is a positive constant (see equation (10) for a justification). The asset $S_2(\cdot)$ could be interpreted as an index of the rest of the economy.

By equation (6), the components of the market-price of risk are $\theta_1(t) = (\mu_1 - r)/\sigma_1(t)$ and $\theta_2(t) = (\mu_2 - r)/\sigma_2$ for $0 \leq t \leq T$. By equation (9), the dynamics of $S(\cdot)$, $\sigma_1(\cdot)$, and $S_2(\cdot)$ under P are

$$\begin{aligned} dS(t) &= \mu_1 S(t)dt + \sigma_1(t)S(t)dW_1(t), \\ d\sigma_1(t) &= \alpha'(\overline{\sigma}' - \sigma_1(t))dt + \theta\sigma_1(t)dW_2(t), \\ dS_2(t) &= \mu_2 S_2(t)dt + \sigma_2 S_2(t)dW_2(t), \\ &\quad \text{for } S(0) > 0,\ \sigma_1(0) > 0, \text{ and } 0 \leq t \leq T, \end{aligned} \tag{22}$$

where $\alpha' = \alpha - \theta(\mu_2 - r)/\sigma_2$ and $\overline{\sigma}' = \alpha\overline{\sigma}/\alpha'$.

In the market defined in (22), starting at the initial wealth defined in (14), any contingent claim Y is attainable by a replication portfolio $\pi(\cdot) = (\pi_1(\cdot), \pi_2(\cdot))'$ as described in (13). The price of the call option written on the first primitive asset is $v = E^Q\left[e^{-rT}(S(T) - E)^+\right]$. The partial hedging strategy costs $u = E^Q\left[e^{-rT}(S(T) - E)^+ 1_A\right]$ and default risk is $P(H^c)$. Notice that the appreciation rate μ_1 of the first primitive asset and the parameters of the second primitive asset, μ_2 and σ_2, are needed for estimating $P(H^c)$, but not for v and u.

It is well known that option prices usually do not admit closed-form solutions in this model and that simulation is required. Since the final primitive asset price $S(T)$ cannot be simulated directly, a discrete-time approximation such that the Euler scheme with m periods of length $h = T/m$ can be performed:

$$\begin{aligned} \widehat{S}(kh) - \widehat{S}((k-1)h) &= \widehat{S}((k-1)h)(rh + \widehat{\sigma}_1((k-1)h)\sqrt{h}Z_1(k)), \\ \widehat{\sigma}_1(kh) - \widehat{\sigma}_1((k-1)h) &= \alpha(\overline{\sigma} - \widehat{\sigma}_1((k-1)h)h + \theta\widehat{\sigma}_1((k-1)h)\sqrt{h}Z_2(k), \\ &\quad \text{for } k = 1, \ldots, m, \end{aligned} \tag{23}$$

where the $\sqrt{h}Z_1(k) = B_1(kh) - B_1((k-1)h)$ and the $\sqrt{h}Z_2(k) = B_2(kh) - B_2((k-1)h)$ are the increments of the Brownian motions $B_1(\cdot)$ and $B_2(\cdot)$.

Here $Z_1(1),\ldots,Z_1(m),Z_2(1),\ldots,Z_2(m)$ are independent and identically distributed normal variables.

The error of the Euler approximation when computing an expectation in the form $E\left[f(S(t), t \in I \subset [0,T])\right]$, defined as

$$e(m) = |E[f(\widehat{S}(t),\, t \in I \subset [0,\, T])] - E[f(S(t),\, t \in I \subset [0,\, T])]|,$$

where $f : \mathbb{R}^{|I|} \to \mathbb{R}$, is known to be in $O(m^{-1})$. Given a computational budget, a trade-off between the number of time increments m of the Euler approximation and the sample size n of the simulation experiment must be found. Duffie and Glynn (1995) argue that n must increase as $O(m^2)$ so that doubling m necessitates quadrupling n. In the following, $f(\widehat{S}(t),\, t \in I \subset [0,\, T])$ is denoted $f(S(t),\, t \in I \subset [0,\, T])$.

The Euler approximation is used to simulate n copies (we take $n = 4000$) of $(S(T),\, M^S(T),\, \theta^S(T))$, which serve to simulate as many copies of $f(S(T),\, M^S(T),\, \theta^S(T))$ where $f : \mathbb{R}^3 \to \mathbb{R}$. Depending on the function f and the probability measure used, the parameter $w = E\left[f(S(T), M^S(T), \theta^S(T))\right]$ matches v, u, or $P(H^c)$. The crude Monte Carlo estimator of w based on n replications is

$$\widehat{w} = n^{-1} \sum_{i=1}^{n} f((S(T),\ M^S(T),\ \theta^S(T))_i),$$

where the $(S(T),\, M^S(T),\, \theta^S(T))_i$, for $i = 1,\ldots,n$, are the n copies of $(S(T),\, M^S(T),\, \theta^S(T))$. The estimated error of $\widehat{w}$ can be defined as the half-length of the asymptotic 95% confidence interval of w based on the normality assumption,

$$e = 1.96S/\sqrt{n},$$

where S is the sample standard error of $f(S(T),\, M^S(T),\, \theta^S(T))$.

Through each path, the global maximum $M^S(T)$ is simulated following Beaglehole, Dybvig, and Zhou (1996). The time to maximum is simulated as the midpoint of the subinterval $[(k^* - 1)h,\ k^*h]$ containing the global maximum.

The parameters of the option to be evaluated are $S(0) = 100$, $E = 100$, $T = 0.5$, $\sigma = 0.15$, and $r = 0.05$. The parameters of the volatility are $\alpha = 1.5$, $\overline{\sigma} = 0.15$, and $\theta = 0.08$. The appreciation rate of the first primitive asset is $\mu_1 = 0.1$, and the parameters of the second primitive asset are $\mu_2 = 0.08$ and $\sigma_2 = 0.12$. Results in Table 4 are obtained at $s = 0.5$, that is, the constraint

$\theta^S(T) \leq s$ can be ignored. Each cell of this table contains the Monte Carlo estimate and its estimated error. The partial hedging parameters are a and $b = a + 3$, and the parameters of the simulation are: $m = 60$ and $n = 4000$.

Table 4: Partial Hedging of a Call when $A = \left\{E \leq S(T) \leq a,\ M^S(T) \leq b\right\}$

a	120	125	130	135	150	∞
u	3.46 ± 0.16	4.40 ± 0.19	4.84 ± 0.21	5.09 ± 0.22	5.26 ± 0.23	5.26 ± 0.23
$P(H^c)$	0.120 ± 0.010	0.062 ± 0.007	0.026 ± 0.005	0.011 ± 0.003	—	0

Results in Table 4 are similar to those in Table 2, except for the statistical error, which we shall now try to reduce. The relative error of $P(H^c)$, defined as the ratio of the statistical error over the statistical estimation, increases as the parameter a increases, i.e., as the event H^c becomes rarer. This is a typical situation when estimating probability of rare events. At the extreme case $a = 150$, the default event H^c is so rare that we have observed no realization of $f(S(T),\ M^S(T),\ \theta^S(T))$ in this region for our 4000 simulation runs. The variance reduction technique, called *Importance Sampling* (see Boyle, Broadie, and Glasserman (1997) and L'Écuyer (1994) for a discussion), provides a way to handle this type of situation and could be used for large values of a. The idea is to select a change of measure so that the integrand, here $f(S(T), M^S(T), \theta^S(T))$, goes more frequently into the most important regions of the sample space, here H^c. For $a = 150$, one can also see that the simulation could not distinguish between the cost of the partial replication and the cost of the full replication.

In the next subsections, correlation induction techniques are used to reduce the estimated error of the crude Monte Carlo estimators. These variance reduction techniques, namely *Antithetic Variates* and *Control Variates*, induce correlation between estimators in attempt to reduce the variance. The techniques used are discussed, e.g., in Bratley, Fox, and Schrage (1987) and L'Écuyer (1994).

3.2.2 Antithetic Variates

Let $\widehat{w}_1$ to be an unbiased estimator of w. For simplicity, take $\widehat{w}_1$ as the crude Monte Carlo estimator of w based on one replication. Assume that one can build a second unbiased estimator $\widehat{w}_2$ of w which is negatively correlated with

$\widehat{w}_1$. Thus, the unbiased estimator $\widehat{w} = (\widehat{w}_1 + \widehat{w}_2)/2$ of w is expected to have lower variance than each of its components:

$$\mathrm{Var}\,[\widehat{w}] = \mathrm{Var}\,[\widehat{w}_1]\,/4 + \mathrm{Var}\,[\widehat{w}_2]\,/4 + \mathrm{Cov}\,[\widehat{w}_1,\widehat{w}_2]\,/2,$$

if $\mathrm{Cov}[\widehat{w}_1,\widehat{w}_2] < 0$ and $\widehat{w}_2$ is well selected. Roughly speaking, if $\widehat{w}_1$ takes high values above its mean w, $\widehat{w}_2$ takes low values below its mean w. Thus, their deviations are mutually compensated in $\widehat{w}$ whence the terminology "Antithetic Variates".

The estimator $\widehat{w}_1$ is often written as a monotone function of some independent and identically basic uniforms $U_1,\ldots,U_q$

$$\widehat{w}_1 = f(U_1,\ldots,U_q),$$

where $f : \mathbb{R}^q \to \mathbb{R}$. Taking

$$\widehat{w}_2 = f(1 - U_1,\ldots,1 - U_q),$$

ensures the condition $\mathrm{Cov}[\widehat{w}_1,\widehat{w}_2] < 0$ [Bratley, Fox, and Schrage (1987), page 46] and variance reduction. In the case analyzed here, the output $\widehat{w}_1$ is a function of some inputs as shown in (23):

$$\widehat{w}_1 = f(Z_1(1),\ldots,Z_1(m),Z_2(1),\ldots, Z_2(m)),$$

where $Z_1(1),\ldots,Z_1(m),Z_2(1),\ldots,Z_2(m)$ are independently and identically distributed normal random variables. By the same argument, taking

$$\widehat{w}_2 = f(-Z_1(1),\ldots, - Z_1(m),Z_2(1),\ldots,Z_2(m)),$$

ensures the condition $\mathrm{Cov}[\widehat{w}_1,\widehat{w}_2] < 0$ and variance reduction. One can focus only on the components where the function is monotone and synchronize between the estimators to induce the attempted negative correlation. For the estimation of $P(H^c)$, we observed no variance reduction with the antithetic variates.

The parameters of the option to be evaluated are $S(0) = 100$, $E = 100$, $T = 0.5$, $\sigma = 0.15$, and $r = 0.05$. The parameters of the volatility are $\alpha = 1.5$, $\overline{\sigma} = 0.15$, and $\theta = 0.08$. Results are shown in Table 5. Each cell of this table contains the Antithetic Variates estimate and its estimated error. The partial hedging parameters are a and $b = a + 3$, and the simulation parameters are $m = 60$ and $n = 4000$. The estimated errors, given in Table 5, show a modest variance reduction in comparison with those of Table 4.

Table 5: Partial Hedging a Call when $A = \left\{E \leq S(T) \leq a,\ M^S(T) \leq b\right\}$

a	120	125	130	135	∞
u	3.68 ± 0.08	4.58 ± 0.09	5.03 ± 0.09	5.29 ± 0.10	5.49 ± 0.11

3.2.3 Control Variates

Let X to be an unbiased estimator of w and let $C = (C_1,\ldots,C_q)'$ be a random vector with a known expected value $\nu = (\nu_1,\ldots,\nu_q)'$ presumably correlated with X. Assume that C is known to the simulator. Think of X as the crude Monte Carlo estimator of w based on one replication when the volatility moves randomly and $C = C_1$ as the synchronous crude Monte Carlo estimator of w when the volatility is constant. The idea behind this technique is to find a vector $\beta = (\beta_1,\ldots,\beta_q)'$ such that the unbiased estimator of w, namely the *controlled estimator*,

$$X_c = X - \beta'(C - \nu),$$

has a lower variance than X. The optimal choice for β, to yield the maximum variance reduction, is

$$\beta^* = \Sigma_C^{-1}\Sigma_{X,C},$$

where Σ_C is the variance matrix of C and $\Sigma_{X,C}$ is the covariance vector between X and the components of C. At β^*, a variance reduction takes place:

$$\mathrm{Var}\left[X_C\right] = (1 - R^2_{X,C})\mathrm{Var}\left[X\right],$$

where

$$R^2_{X,C} = \Sigma'_{X,C}\Sigma_C^{-1}\Sigma_{X,C}/\mathrm{Var}\left[X\right],$$

is the multiple coefficient of correlation between X and the components of C. In the particular case $q = 1$, these results can be written as

$$\beta^* = \mathrm{Cov}\left[X,C\right]/\mathrm{Var}\left[C\right],$$

and

$$\mathrm{Var}\left[X_C\right] = (1 - \mathrm{Corr}\left[X,C\right]^2)\mathrm{Var}\left[X\right].$$

Roughly speaking, if X increases and takes high values, $\mathrm{Cov}[X, C]\,(C - \nu)/\mathrm{Var}[C]$ necessarily increases. Thus, it controls the excess of X above its mean w via X_c, whence the terminology "Control Variates". In options pricing, the random variable $\gamma(T)S(T)$ is usually taken as a control variable since $S(0) = E^Q[\gamma(T)S(T)]$ is known: The process $\gamma(\cdot)S(\cdot)$ is a Q-martingale as mentioned in (15). Several authors, e.g., Clewlow and Carverhill (1994), select a priori $\beta^* = 1$ and report a significant variance reduction. In fact, this choice is not necessarily acceptable but it should work when C is simulated to be approximately equal to X. In that case, the optimal value for β is expected to be near unity since Cov$[X, C] \simeq$Var$[C]$.

Unfortunately, neither Σ_C nor $\Sigma_{X,C}$ are known in practice and β^* cannot be computed as shown above. An alternative idea is to simulate n copies of (X,C), estimate Σ_C and $\Sigma_{X,C}$ as usual, and define the observations of the controlled estimator as

$$X_{c,i} = X_i - \widehat{\beta}'(C_i - \nu), \quad \text{for } i = 1,...,n,$$

where

$$\widehat{\beta} = \widehat{\Sigma}_C^{-1}\widehat{\Sigma}_{X,C}.$$

The controlled estimator of w is defined as the sample mean of the $X_{c,i}$, for $i = 1,...,n$,

$$\overline{X}_c = \overline{X} - \widehat{\beta}'(\overline{C} - \nu).$$

The sample variance S_c^2 of X_c is defined as usual. Notice that the controlled estimator $\overline{X}_c$ is generally a biased estimator of w since $\widehat{\beta}$ and $\overline{C}$ are a priori correlated. However, Lavenberg and Welch (1981) showed that this bias vanishes when (X,C) is multinormal.

As pointed out by Nelson (1990), the controlled estimator often is convergent as $n \to +\infty$:

$$\begin{aligned}
\overline{X}_c &\to w \quad \text{in probability,} \\
S_c^2 &\to (1 - R_{X,C}^2)\text{Var}[X] \quad \text{in probability,} \\
\sqrt{n}(\overline{X}_c - w)/S_c &\to N(0,1) \quad \text{in distribution.}
\end{aligned}$$

Thus, $\overline{X}_c$ is an asymptotically unbiased estimator of w and has asymptotic smaller variance than the crude Monte Carlo estimator $\overline{X}$. Techniques for reducing the bias of $\overline{X}_c$ for small samples, such as Batching, Jackknifing, and Splitting, are described in Nelson (1990). By splitting optimally into three groups, Avramidis and Wilson (1993) build a controlled estimator which is somewhat more consistent than $\overline{X}_c$, as it converges to w always surely.

The parameters of the option to be evaluated are $S(0) = 100$, $E = 100$, $T = 0.5$, $\sigma(0) = 0.15$, and $r = 0.05$. The parameters of the volatility are $\alpha = 1.5$, $\overline{\sigma} = 0.15$, and $\theta = 0.08$. The appreciation rate of the first primitive asset is $\mu_1 = 0.1$, and the parameters of the second primitive asset are $\mu_2 = 0.08$ and $\sigma_2 = 0.12$. Results are shown in Table 6. Each cell of this table contains the Control Variates estimate and its estimated error. The partial hedging parameters are a and $b = a + 3$, and the simulation parameters are $m = 60$ and $n = 4000$. We observe a significant variance reduction resulting from the high correlation between the crude estimators and their associated control variables.

Table 6: Partial Hedging a Call when $A = \left\{E \le S(T) \le a,\ M^S(T) \le b\right\}$

a	120	125	130	135	∞
u	3.63 ± 0.04	4.46 $\pm .04$	4.95 $\pm .04$	5.19 $\pm .02$	5.39 $\pm .01$
$P(H^c)$	0.119 ± 0.002	0.056 ± 0.002	0.023 ± 0.002	0.010 ± 0.001	0

3.2.4 Integrating the Correlation Induction Techniques

Now, denote X as the antithetic variates estimator of w (based on one replication) when the volatility moves randomly and $C = C_1$ the synchronous antithetic variates estimator of w when the volatility is constant. As pointed out by Avramidis and Wilson (1996), the estimator X can be viewed as an aggregate response and the random variable C as an aggregate control variable.

The parameters of the option to be evaluated are $S(0) = 100$, $E = 100$, $T = 0.5$, $\sigma(0) = 0.15$, and $r = 0.05$. The parameters of the volatility are $\alpha = 1.5$, $\overline{\sigma} = 0.15$, and $\theta = 0.08$. Results are shown in Table 7. Each cell of this table contains the estimate based on the aggregate response and its estimated error. The partial hedging parameters are a and $b = a + 3$, and the simulation parameters are $m = 60$ and $n = 4000$. Additional improvements are realized when integrating the Antithetic Variates and the Control Variates techniques.

Table 7: Hedging Partially a Call when $A = \left\{E \le S(T) \le a,\ M^S(T) \le b\right\}$

a	120	125	130	135	∞
u	3.61 $\pm .02$	4.46 $\pm .02$	4.97 $\pm .03$	5.21 $\pm .02$	5.39 $\pm .01$

4 Conclusion

A hedger may find an advantage in partially replicating a contingent claim if the lower cost of a partial hedge more than offsets the added default risk. Several partial replication strategies are possible. In this paper, the strategies analyzed use the final primitive asset price, its maximum over the trading period, and the time to maximum. The results show how the cost of a partial hedge and default risk vary depending on the replication event. These strategies are easy to implement and can be generalized to more complex contingent claims using more general evaluation models. Monte Carlo simulation, a flexible and robust tool, can be used to analyze such strategies. In addition, correlation induction techniques can be implemented easily with a great success.

Hedging contingent claims sometimes is not possible. For example, in constraint models, the super-replication cost is excessively high [see Cvitanić, Pham, and Touzi (1997) for some examples]. In such models, a partial super-replication may be an interesting solution.

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