ASYMPTOTIC EFFICIENCY OF PERTURBATION-ANALYSIS-BASED STOCHASTIC APPROXIMATION WITH AVERAGING

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Abstract. Central limit theorems are obtained for the PARMSR (perturbation analysis Robbins-Monro single run) algorithm updated either after every regenerative cycle or after every fixed-length observation period, and with averaging of the iterates, for one-dependent regenerative processes. When the convergence to the optimizer is expressed in terms of the total observation time of the system (or the total computing budget in the case of a simulation), the convergence rate and the limit covariance matrix turn out to be the same for all updating schemes, and are optimal within the class of stochastic approximation algorithms, under certain assumptions. A bound on the strong convergence rate of the usual PARMSR algorithm updated after every fixed length observation period is established using a limit theorem on double array martingales. This is the key step for obtaining central limit theorems for the algorithms with averaging and has interest in its own right.

Key words. perturbation analysis, asymptotic efficiency, central limit theorems, stochastic approximation, recursive estimation, queueing theory.

AMS subject classifications. 62L20, 93E12, 93E23, 93E30, 60F05, 60K25

1. Introduction.

Consider a discrete-time stochastic process $\{X_i,\ i \geq 0\}$ which follows the stochastic recurrence

$$X_{i+1} = h(X_i,\, \tilde{\theta}_i,\, U_i) \tag{1.1}$$

with initial state X_0, where h is a measurable function, each $\tilde{\theta}_i \in D$ is a *decision* (or control), D is a compact subset of the l-dimensional real space $\mathbb{R}^l$, and $\{U_i,\ i \geq 0\}$ is a sequence of i.i.d. random vectors defined on a probability space $\{\Omega,\, \mathcal{F},\, P\}$. In the case of a simulation model, $\{U_i,\ i \geq 0\}$ can be interpreted as the sequence of $U(0,1)$ random numbers that drive the simulation. A real-valued *cost process* $\{J_i,\ i \geq 0\}$ is defined by $J_i = \phi(X_i, \tilde{\theta}_i)$, for some measurable mapping ϕ. Denote (X_i, J_i) by $(X_i(\theta), J_i(\theta))$ when the control parameter $\tilde{\theta}_i$ is fixed at θ for all $i \geq 0$. The objective function is the steady-state average cost

$$\overline{J}(\theta) = \lim_{t\to\infty} \frac{1}{t} \sum_{i=1}^{t} \mathsf{E}[J_i(\theta)], \tag{1.2}$$

which is to be minimized with respect to θ, for $\theta \in D$.

To insure that (1.2) exists and to prove our results, we assume certain stability conditions on the process $\{J_i,\ i \geq 0\}$, expressed in the form of a regenerative structure. We suppose (roughly) that this process, as well as its stochastic gradient (perturbation analysis) process, are one-dependent regenerative in an extended sense

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(see the next section for details). One-dependent regenerative is a weakened version of the classical notion of a regenerative stochastic process, and covers a broad class of systems [1, 29, 36]. The weakening is that adjacent regenerative cycles are allowed to be dependent. The stochastic recursion model (1.1) is very general (it suffices to put enough information into the state X_i) and it will permit us to define the perturbation analysis gradient estimators (under some additional conditions).

We assume that no closed-form expression for $\bar{J}$ is available and we consider a stochastic approximation (SA) approach to find a root of $f(\theta) \triangleq \overline{J}_\theta(\theta) \triangleq d\overline{J}(\theta)/d\theta$ (in this paper, a θ subscript means the gradient with respect to θ). We suppose that the root is the unique optimizer θ^0 of $\overline{J}(\theta)$. The Robbins-Monro [33] SA algorithm has the general form

$$\theta_{n+1} = \Pi_D(\theta_n - a_n f_{n+1}) \tag{1.3}$$

with an initial value θ_0, where $\{a_n,\ n \geq 0\}$ is a deterministic sequence of matrices called the *step sizes*, $f_{n+1} = f(\theta_n) + \varepsilon_{n+1}$ is an unbiased estimator of $f(\theta_n)$, ε_{n+1} is the observation *noise* at the $(n+1)$-th step, $\Pi_D : \mathbb{R}^l \to D$ is a projection operator (which would disappear if D was $\mathbb{R}^l$, as in the original Robbins-Monro formulation), and θ_n is the nth estimate for the optimizer θ^0. See, e.g., [7, 22, 24, 28]. It is well-known that under certain conditions on the regression function $f(\theta)$, on the noise sequence $\{\varepsilon_n,\ n \geq 1\}$, and on the matrix A^* assuming that $a_n = A^*/n$, $\sqrt{n}(\theta_n - \theta^0)$ is asymptotically $N(0,\ S^*)$, where S^* is a limiting covariance matrix. The trace of S^* is minimized by taking $A^* = M_1^{-1}$, where $M_1 = f_\theta(\theta^0)$ is the Hessian matrix of $\overline{J}(\theta)$ at θ^0. This optimal covariance matrix is $S^* = M_1^{-1} S_0^* (M_1^{-1})'$, where the prime means "transpose" and S_0^* is the asymptotic covariance matrix of $(1/\sqrt{n}) \sum_{j=1}^n \varepsilon_j$.

However, since M_1 is generally unknown, this optimal SA scheme is usually impracticable. This has motivated the introduction of SA algorithms with averaging of the iterates, using a sequence of step sizes which decreases at a rate slower than $1/n$ (see [3, 5, 24, 31, 32, 43]). One of these algorithms uses (1.3) as usual, then retains the following estimator of θ^0 at step n:

$$\overline{\theta}_n = \frac{1}{n} \sum_{j=1}^{n} \theta_j = \frac{1}{n}(\theta_n + (n-1)\bar{\theta}_{n-1}). \tag{1.4}$$

Under certain conditions, $\sqrt{n}(\overline{\theta}_n - \theta^0)$ is asymptotically $N(0,\ S^*)$. The major advantage of this averaging algorithm is that there is no need to know M_1^{-1}.

Under appropriate smoothness conditions, *infinitesimal perturbation analysis* (IPA) offers viable means of estimating $f(\theta)$ by observing a single sample path of the system (see Section 2.2 and, e.g., [15, 21, 25] and other references therein). Going one step further, one can simultaneously optimize along that single sample path: At successive epochs along the path, a (generally biased) estimator of $f(\theta)$ for the current θ is put in the SA algorithm to recursively estimate θ^0 while the system is running (either on-line or via simulation). The general idea is as follows: Run the system for a short period of time, say L_{n+1} steps of (1.1) with θ fixed at θ_n, and use the collected information to compute the estimator f_{n+1} of $f(\theta_n)$; put f_{n+1} in (1.3) to get θ_{n+1}; continue running the system (from its current state) for another L_{n+2} steps with $\theta = \theta_{n+1}$ to compute f_{n+2}, and so on. Denoting $N_n = \sum_{j=1}^n L_j$ (with $N_0 = 0$), one has $\tilde{\theta}_i = \theta_n$ for $N_n \leq i < N_{n+1}$. In [38, 39], this is called the PARMSR (perturbation analysis Robbins-Monro single run) algorithm. Considerable effort has been devoted, in recent years, to study the convergence of the PARMSR algorithm in the field of discrete event dynamic systems (DEDSs); see, e.g., [8, 9, 10, 14, 23, 26, 27, 40, 41], among others. The convergence rate results mentioned previously for the SA algorithm, however, have been proved under conditions that do not hold in the PARMSR setup.

In this paper we study the convergence rate, in the sense of central limit theorems, for the PARMSR algorithm with averaging. Let N_n represent the cumulative *computing budget* for the first n steps of the PARMSR algorithm. We express the convergence speed of the algorithm in terms of N_n (as done, e.g., in [28]) instead of n. When $\sqrt{N_n}(\overline{\theta}_n - \theta^0)$ is asymptotically $N(0,\ S^*)$, with $S^* = M_1^{-1} S_0^* (M_1^{-1})'$, where S_0^* is the asymptotic covariance matrix of $(1/\sqrt{N_n}) \sum_{j=1}^{n} \varepsilon_j$, we say that the PARMSR algorithm is *asymptotically optimal* (see [3, 5, 43] for similar definitions in terms of n). This must be understood as optimal only within the class of SA algorithms and for the particular gradient estimator that is used. Another type of convergence rate, in the sense of large-deviation bounds on the probability of exit of a neighborhood of θ^0 by the tail process $\{\theta_n,\ n \geq n_0\}$ for large n_0, is studied in [12].

We analyze the following two cases:

- (R) The parameter θ_n is updated after each regenerative cycle of the process $\{J_i,\ i \geq 0\}$ (so, L_n is random and represents the length of the nth regenerative cycle). In this regenerative case, we use θ_{n-1} instead of θ_n to obtain f_{n+1}, as explained in Section 3.
- (F) θ_n is updated after every L steps of the recurrence (1.1), so $L_n = L$, a positive constant.

For case (R), the limiting behavior is relatively easy to analyze since the main part of the observation noise can be decomposed into two martingale difference sequences. Then, standard SA results are applicable. For case (F), the analysis is more difficult, primarily because the standard conditions on the observation noise, assumed in, e.g., [3, 24, 31, 32, 43], do not hold. These authors require the observation noise to satisfy the properties of martingale differences, or of stationary ϕ-mixing processes, or of the infinite sum of a martingale difference sequence. For the PARMSR algorithm with fixed-length observation period, the observation noise has a very complicated dynamic, as shown in previous convergence studies; see, e.g., [9, 10, 23, 26, 40, 41]. In this paper, we first obtain a bound on the almost sure (a.s.) convergence rate of the usual PARMSR algorithm (without averaging), using a limit theorem on double array martingales borrowed from [6] and [19]. We then apply this result to obtain a central limit theorem showing the asymptotic optimality of the PARMSR algorithm with averaging for the case (F).

Our results imply that the PARMSR algorithm with averaging has the same asymptotic efficiency, i.e., the same convergence rate and the same limiting covariance matrix in terms of N_n, for both (R) and (F). Moreover, for (F), this limit covariance is independent of the updating frequency L. Our emphasis in this paper is on the case (F). This case is more difficult to analyze than (R), but its implementation is much easier, because there is no need to recognize the regeneration points, so the implementation depends much less on the structure of the system. For case (R), the algorithm must identify the regeneration points explicitly, which is usually hard for complex systems. See [34]–[35] on the identification of regeneration points for closed and open queueing networks.

A third approach, not considered here, is to have $L_n \to \infty$ with n. This was studied in [28] for decreasing step sizes and in [13] for constant step sizes, without the averaging of the iterates.

The rest of the paper is organized as follows. In the next section we specify the regenerative structure of the model, We also define the IPA estimators and introduce some assumptions to make sure that the IPA process shares a common regenerative structure with the original process and is strongly consistent. The asymptotic efficiency of the PARMSR algorithm with averaging for cases (R) and (F) is analyzed in Sections 3 and 4, respectively. All the proofs are collected in the appendix.

2. The Model.

2.1. The Regenerative Structure. We now expand on the model introduced at the beginning of Section 1. The state process $\{X_i, i \geq 0\}$ lives in some Borel space $\mathcal{X}$ and the initial state $X_0 \in \mathcal{X}$ is a random variable. The stochastic process $\{X_i, \tilde{\theta}_i, J_i), i \geq 0\}$ is defined on the same probability space $\{\Omega, \mathcal{F}, P\}$ as $\{X_0, U_i, i \geq 0\}$, via (1.1) and (1.3). Let $\mathcal{F}_i = \sigma(X_0, \ldots, X_i)$, the sigma field generated by $X_0, \ldots, X_i$. We suppose that there is a sequence of integers $0 = k_0 < k_1 < k_2 < \cdots$ such that each k_m is a stopping time with respect to the filtration $\{\mathcal{F}_i, i \geq 0\}$, and such that the following condition (A0) is satisfied.

Suppose for a moment that until time k_m, the process is run as usual with $\tilde{\theta}_i$ varying according to the SA algorithm, but that from then on, the parameter is fixed at θ, i.e., $\tilde{\theta}_i = \theta$ for $i \geq k_m$. We denote this modified process starting at time k_m by $\{X_i^{(m+1)}(\theta), i \geq 0\}$, that is $X_i^{(m+1)}(\theta) = X_{k_m+i}$ for $i \geq 0$, and put $J_i^{(m+1)}(\theta) = \phi(X_i^{(m+1)}(\theta), \theta)$ for $i \geq 0$ and $k_r^{(m+1)}(\theta) = k_{m+r} - k_m$ for $r \geq 0$. We also denote $X_i(\theta) = X_i^{(1)}(\theta)$, $J_i(\theta) = J_i^{(1)}(\theta)$, and $k_m(\theta) = k_m^{(1)}(\theta)$, which correspond to a process where θ is fixed from the beginning.

(A0). For any fixed $\theta \in D$ and $m \geq 0$, $\{J_i^{(m+1)}(\theta), i \geq 0\}$ and $\{k_r^{(m+1)}(\theta), r \geq 0\}$ have the same joint distribution as $\{J_i(\theta), i \geq 0\}$ and $\{k_r(\theta), r \geq 0\}$, and are independent of $\mathcal{F}_{k_{m-1}-1}$ and k_m.

When this condition (A0) is satisfied, the controlled process $\{J_i, i \geq 0\}$ is called (non-delayed) *one-dependent regenerative*. This extends the usual definition found, e.g., in [29, 36]: For the process $\{J_i(\theta), i \geq 0\}$, for which θ is fixed from the beginning, we get the usual definition of one-dependent regenerative, which we call in this paper *one-dependent regenerative for fixed* θ. The k_m's are called the *regeneration points*. Denote $\eta_m = k_m - k_{m-1}$, the length of the mth regenerative cycle, $\eta_r^{(m+1)}(\theta) = k_r^{(m+1)}(\theta) - k_{r-1}^{(m+1)}(\theta)$, and $\eta_m(\theta) = k_m(\theta) - k_{m-1}(\theta)$. If (A0) holds with $\mathcal{F}_{k_{m-1}-1}$ replaced by $\mathcal{F}_{k_m-1}$, the process is called *classically regenerative*. From the definition of J_i and by Proposition V.1.1 of [1], if (A0) holds for $\{X_i, i \geq 0\}$, it also holds for $\{J_i, i \geq 0\}$ with the same regeneration points.

It is well-known that Harris-recurrent Markov chains (HRMCs) are one-dependent regenerative processes for fixed θ; this can be seen by applying the splitting technique due to [2]. HRMCs cover a very large class of models. We refer the reader to [1, 29, 30, 36] for appropriate background.

If $\mathsf{E}[\eta_1(\theta)] < \infty$ and $\mathsf{E}\left[\sum_{i=1}^{\eta_1(\theta)} |J_i(\theta)|\right] < \infty$ for all $\theta \in D$, then $\overline{J}(\theta)$ in (1.2) is well defined on D and the renewal-reward theorem (see, e.g., [1] and [42]) says that

$$\overline{J}(\theta) = \frac{1}{\mathsf{E}[\eta_1(\theta)]} \mathsf{E}\left[\sum_{i=1}^{\eta_1(\theta)} J_i(\theta)\right]. \tag{2.1}$$

2.2. Gradient Estimation via IPA and the PARMSR Algorithm. We now explain how IPA is used to estimate the gradient in this context. For fixed θ, the

IPA estimators are constructed from the recurrence

$$\begin{aligned} J_{\theta,i}(\theta) &= \phi_x(X_i(\theta),\theta)X_{\theta,i}(\theta) + \phi_\theta(X_i(\theta),\theta), \\ X_{\theta,i+1}(\theta) &= h_x(X_i(\theta),\theta,U_i)X_{\theta,i}(\theta) + h_\theta(X_i(\theta),\theta,U_i), \end{aligned}$$

for $i \geq 0$, where $X_{\theta,0}(\theta) = 0$, ϕ_x and ϕ_θ denote the partial derivatives of ϕ with respect to its first and second components, and similarly for h. Assuming that these stochastic derivatives exist and under appropriate uniform integrability conditions (see [15, 25]), $J_{\theta,i}(\theta)$ is an unbiased IPA estimator of the gradient of $\mathsf{E}[J_i(\theta)]$.

Suppose now that θ is *not* fixed, but take the successive values $\{\tilde{\theta}_i,\, i \geq 0\}$ as in (1.1). This is what happens when the PARMSR algorithm is applied. We then define the IPA estimator as follows. Let

$$\begin{aligned} J_{\theta,i} &= \phi_x(X_i,\tilde{\theta}_i)X_{\theta,i} + \phi_\theta(X_i,\tilde{\theta}_i), \\ X_{\theta,i+1} &= h_x(X_i,\tilde{\theta}_i,U_i)X_{\theta,i} + h_\theta(X_i,\tilde{\theta}_i,U_i), \end{aligned}$$

The gradient estimator at the nth step of SA has the general form

$$f_{n+1} = K \sum_{i=1}^{L_{n+1}} J_{\theta,N_n+i} \tag{2.2}$$

for some constant K. In our context, we have $L_{n+1} = \eta_{n+1}$ and $K = 1$ for case (R), and $L_{n+1} = L$ and $K = 1/L$ for case (F).

Equations (1.3) and (2.2) give the general form of the PARMSR algorithm. In this paper, we define the projection operator Π_D of (1.3) as

$$\Pi_D(\theta_n - a_n f_{n+1}) = \begin{cases} \theta_n - a_n f_{n+1}, & \text{if } \theta_n - a_n f_{n+1} \in D; \\ \theta_n, & \text{if } \theta_n - a_n f_{n+1} \notin D, \end{cases}$$

If D is a convex set, we could also define $\Pi_D(x)$ to be the nearest boundary point of D whenever $x \notin D$.

For the analysis of the PARMSR algorithm, we need to assume that $\{J_{\theta,i},\, i \geq 0\}$ is also one-dependent regenerative with the same regeneration points as $\{J_i,\, i \geq 0\}$. Glasserman and his co-authors [15]—[18] give several sets of sufficient conditions for this property to hold for fixed θ, for classically regenerative systems (see, e.g., Theorem 8.3 in [15], Theorem 5.4 in [16], and Theorem 5.6 in [17]). The following conditions ensure, among other things, that the IPA estimators are *strongly consistent* for $f(\theta)$, in the sense that $\lim_{t\to\infty}(1/t)\sum_{i=1}^{t} J_{\theta,i}(\theta) = f(\theta)$ a.s. We denote $J_{\theta,i}^{(m+1)}(\theta) = dJ_i^{(m+1)}(\theta)/d\theta$. Later on, for our theorems, we will require values of ξ_0 larger than 2 in (A3).

(A1). For $i \geq 0$, $J_i(\theta)$ is absolutely continuous with respect to θ on D.

(A2). $\{J_i,\, i \geq 0\}$ and $\{J_{\theta,i},\ i \geq 0\}$ are one-dependent regenerative in the sense of (A0), with common regeneration points $\{k_m,\ m \geq 0\}$.

(A3). There are two sequences of one-dependent and identically distributed random variables (r.v.'s) $\{Z_m,\, m \geq 1\}$ and $\{\eta_m^*,\, m \geq 1\}$, and a constant $\xi_0 \geq 2$, such that $\mathsf{E}\left[Z_1^{\xi_0}\right] < \infty$ and $\mathsf{E}\left[(\eta_1^*)^{\xi_0}\right] < \infty$, and such that for all $m \geq 0$, $1 \leq i \leq \eta_{m+1}$, and $\theta \in D$, $\max(\|J_{\theta,k_m+i}\|,\ \|J_{\theta,k_m+i}(\theta)\|,\ \|J_{\theta,i}^{(m+1)}(\theta)\|)$ $\leq Z_{m+1}$ and $\max(\eta_{m+1},\ \eta_{m+1}(\theta),\ \eta_1^{(m+1)}(\theta)) \leq \eta_{m+1}^*$.

A classical example where (A1) to (A3) are verified in the literature is that of the $GI/G/1$ queue considered in [8, 9, 24, 26, 41].

LEMMA 2.1. *If conditions (A0)—(A3) hold, then for each $\theta \in D$, the IPA derivative estimator is strongly consistent for $f(\theta)$ and*

$$f(\theta) = \frac{1}{\mathsf{E}[\eta_1(\theta)]}\mathsf{E}\left[\sum_{i=1}^{\eta_1(\theta)} J_{\theta,i}(\theta)\right]. \tag{2.3}$$

Proof. The proof is along the same lines as in [15]—[18]. In fact (A2) and (A3) are needed only for fixed θ (uniformly in θ) to prove this lemma. ☐

3. The PARMSR Algorithm with Averaging for Case (R). Since $\eta_1(\theta) \geq 1$, finding a root of $f(\theta)$ is equivalent to finding a root of $f(\theta)\overline{\eta}(\theta)$, where $\overline{\eta}(\theta) = \mathsf{E}[\eta_1(\theta)]$, i.e., to finding a root of the second expectation in (2.3). If regeneration points for the model under consideration can be identified explicitly, one can use the sum of the $J_{\theta,i}(\theta)$ over any regenerative cycle as an unbiased estimator of $f(\theta)\overline{\eta}(\theta)$, where θ is the (fixed) value of the parameter during that cycle. This motivates taking $L_{n+1} = \eta_{n+1}$ and $K = 1$ in (2.2).

The PARMSR algorithm *without averaging* for case (R) is comprised of (1.3) and (2.2) (cf. [8, 14, 26, 27]), with $\tilde{\theta}_i = \theta_{n-1}$ for $k_n < i \leq k_{n+1}$, and where $\theta_{-1},\ \theta_0 \in D$ are initial values. Thus, f_{n+1} is viewed as an estimator of $f(\theta_{n-1})\overline{\eta}(\theta_{n-1})$. It is worth noticing that the decision parameter throughout the evolution of the $(n+1)$-th regenerative cycle is fixed at θ_{n-1} rather than θ_n. This is because of the one-dependent nature of our model, and simplifies the convergence analysis.

The *averaging* version of this PARMSR algorithm works the same, but retains $\overline{\theta}_n$ instead of θ_n as an estimator of θ^0. Our aim is to obtain a central limit theorem for $\{\bar{\theta}_n,\, n \geq 1\}$. For this, we introduce the following conditions on the step sizes a_n and the functions f and $\bar{\eta}$.

(A4). There are constants $\overline{a} > 0$ and $\nu \in (1/2,\ 1)$ such that $0 < a_n \leq \overline{a}n^{-\nu}$ for all $n \geq 1$; $a_j = \overline{a}$ for all $j \leq 0$; $\sum_{n=1}^{\infty} a_n = \infty$; $0 \leq a_{n+1}^{-1} - a_n^{-1} \xrightarrow[n\to\infty]{} 0$.

(A5). The set D is compact and convex, and $f(\theta)$ is bounded on D. The optimizer θ^0 is an interior point of D. There are a stable matrix $-M_1$ (all eigenvalues of M_1 have positive real parts) and positive constants r_0 and c_1 such that $\|f(\theta) - M_1(\theta - \theta^0)\| \leq c_1\|\theta - \theta^0\|^2$, whenever $\|\theta - \theta^0\| \leq r_0$.

(A6). There exists a continuously differentiable function $v : R^l \to R$ such that $v(\theta^0) = 0$ and for all $\Delta_1 > 0$, $\inf\{(f(\theta))' v_\theta(\theta) : \theta \in D,\ \|\theta - \theta^0\| \leq \Delta_1\} > 0$.

(A7). $f(\theta)$ is Lipschitz with modulus B_1 on D, i.e., $\|f(x_1) - f(x_2)\| \leq B_1\|x_1 - x_2\|,\quad \forall\, x_1,\ x_2 \in D$.

(A8). $\overline{\eta}(\theta)$ and $H(\theta)$ are continuous at θ^0, where $H(\theta)$ is defined as

$$H(\theta) = \mathsf{E}\left[\left(\sum_{i=1}^{\eta_1(\theta)} J_{\theta,i}(\theta)\right)\left(\sum_{i=1}^{\eta_1(\theta)} J_{\theta,i}(\theta)\right)'\right].$$

Let us comment on conditions (A4)–(A8). Observe that $\nu < 1$ in (A4), so the classical choice of $a_n = O(1/n)$ is not allowed. This (A4) is the usual condition of *slowly decreasing step sizes* (e.g., as in [3, 24, 32]). Conditions (A5) and (A6) are standard in the context of SA. (A5) asks the objective function to be locally quadratic around the optimizer θ^0, while (A6) says that θ^0 should be an attractor point from every other point of D. In the context of queueing systems, for example, the set D may represent a stability region where the standard load conditions are fulfilled. In the case where D is unbounded, an SA procedure with randomly varying truncations can be employed (see, e.g., [3]–[5], and [7]). Condition (A7) requires that the regression function $f(\theta)$ is sufficiently smooth. We note that if the model is classically regenerative, then the Lipschitz conditions on $f(\theta)$ and $\overline{\eta}(\theta)$ required in Theorem 3.1 can be dropped since in this situation, $\tilde{\theta}_i = \theta_{n-1}$ can be replaced by $\tilde{\theta}_i = \theta_n$ for $k_n < i \leq k_{n+1}$, and we do not need to decompose the noise ε_{n+1} into two parts (see the proof of Theorem 3.1 (i)). Condition (A8) requires the continuity of $\overline{\eta}(\theta)$ and $H(\theta)$ at the optimizer θ^0. This is a mild condition. See [26] for the verification in the context of a GI/G/1 queue.

The following theorem contains our main results for case (R).

THEOREM 3.1.

(i). Suppose that the conditions (A0)–(A7) hold with $\xi_0 \geq 4$ in (A3), and that $\overline{\eta}(\theta)$ is Lipschitz with modulus B_2. Then, $\|\theta_n - \theta^0\| = o(a_n^\delta)$ a.s. for all $\delta \in [0,\ 1 - 1/(2\nu))$, where ν is given by condition (A4).

(ii). Suppose that conditions (A0)–(A8) hold with some $\xi_0 > 4/\nu$ and that $\overline{\eta}(\theta)$ is Lipschitz on D. Then $(\theta_n - \theta^0)/\sqrt{a_n} \to N(0,\ S_1)$ in distribution as $n \to \infty$, where

$$S_1 = \int_0^\infty e^{-\overline{\eta}(\theta^0)M_1 t} S e^{-\overline{\eta}(\theta^0)M_1' t} dt, \quad S = H(\theta^0).$$

(iii). Under the same conditions as in (ii) with $\xi_0 > 4/\nu$ replaced by $\xi_0 > 4$, the PARMSR algorithm with averaging for case (R) satisfies $\sqrt{n}(\overline{\theta}_n - \theta^0) \to N(0,\ S_2)$ in distribution as $n \to \infty$, where $S_2 = \overline{\eta}(\theta^0)^{-2} M_1^{-1} S (M_1^{-1})'$.

For the first n iterations of the PARMSR algorithm for case (R), the total computing budget is $N_n = k_n = \sum_{i=1}^n \eta_i$. By the continuity of $\overline{\eta}(\theta)$ at θ^0 and the law of large numbers for martingales, it is seen that $N_n/n \to \overline{\eta}(\theta^0)$ a.s. as $n \to \infty$. Combining this with Theorem 3.1 we obtain:

COROLLARY 3.2. *Under the assumptions of Theorem 3.1 (iii), one has $\sqrt{N_n}(\overline{\theta}_n - \theta^0) \to N(0,\ S^*)$ in distribution as $n \to \infty$, where $S^* = \overline{\eta}(\theta^0)^{-1} M_1^{-1} S (M_1^{-1})'$.*

4. Convergence of the PARMSR Algorithm for Case (F). The PARMSR algorithm *without averaging*, with updating period L, is composed of (1.3) and (2.2) with $L_n = L$, $K = 1/L$, and $\tilde{\theta}_{nL+i} = \theta_n$ for $0 \leq i \leq L-1$ and $n \geq 0$. For $L = 1$, we have $\tilde{\theta}_i = \theta_i$ for all $i \geq 0$. The *averaging* version uses $\overline{\theta}_n$ as a final estimate of θ^0.

(A9). There exist a sequence of one-dependent and identically distributed r.v.'s $\{Z_{m+1}^{(0)},\ m \geq 0\}$, with $\mathsf{E}[(Z_1^{(0)})^{6/\nu}] < \infty$, such that for all $m \geq 0$ and $1 \leq i \leq \eta_{m+1}$, $\|J_{\theta,k_m+i} - J_{\theta,i}^{(m+1)}(\tilde{\theta}_{k_m})\| \leq a_{k_m} Z_{m+1}^{(0)}$ a.s.

(A10). There are two positive constants α_0 and γ_1 such that

$$P\{\eta_{m+1} \neq k_1^{(m+1)}(\tilde{\theta}_{k_m}) \mid \mathcal{F}_{k_m}\} \leq \alpha_0 a_{k_m}^{\gamma_1}.$$

(A11). The constants ξ_0 in (A3) and ν in (A4) satisfy $\xi_0 \geq \max\{2/\zeta,\ 4/\nu,\ 2p_1\}$, where $p_1 > 1$ and $\zeta > 0$ are other constants such that for γ_1 in (A10),

$$\nu\,(1 + \gamma_1(1 - 1/p_1)) > 1,\ 0 < \zeta < \delta'\nu,\ \gamma_1(1 - 1/p_1) \geq 1/2,\ \delta' \in (0,\ 1/2].$$

Chong and Ramadge [10] have checked conditions (A9) and (A10) for certain classically regenerative systems, though the convergence rates of the PARMSR algorithms have not been studied there. For example, (A9) and (A10) have been verified in [9, 10, 40, 41] in the context of a $GI/G/1$ queue, with some conditions on the service time and the interarrival time distributions.

THEOREM 4.1. *For the PARMSR algorithm with fixed updating period $L_n = L$, we have:*

(i). If conditions (A0)–(A10) hold with $\xi_0 \geq \max\{4,\ 2p_1\}$ and $\nu(1+\gamma_1(1-1/p_1)) > 1$, where $p_1 > 1$ is some constant and γ_1 is given in (A10), then $\theta_n \to \theta^0$ a.s. as $n \to \infty$.

(ii). If (A0)–(A11) hold and the constant δ' satisfies (A11), one has $\|\theta_n - \theta^0\| = o(a_n^{1/2-\delta'})$ a.s.

(iii). If (A0)–(A11) hold with $\nu\gamma_1(1-1/p_1) > 1/2$ and $0 < \delta' < 1/2(1-1/(2\nu))$, then for the algorithm with averaging, one has $\sqrt{n}(\overline{\theta}_n - \theta^0) \to N(0,\ S_3)$ in distribution as $n \to \infty$, where $S_3 = L^{-1}\overline{\eta}(\theta^0)^{-1}M_1^{-1}S(M_1^{-1})'$.

We note that δ' in Theorem 4.1 (ii) is independent of ν. Thus $1/2 - \delta'$ can be arbitrarily close to $1/2$ if condition (A3) is fulfilled with a very large ξ_0. This is different from our Theorem 3.1 and from Theorem 2.2 in [41], where $\|\theta_n - \theta^0\| = o(a_n^{\delta})$ a.s. with $\delta \in [0,\ 1-1/(2\nu))$ depending on ν. In the latter setup, if ν is close to $1/2$, δ must be close to zero.

For the first n SA iterations, the total computing budget of the PARMSR algorithm with updating period L is $N_n = nL$. The following corollary provides a central limit theorem in terms of N_n.

COROLLARY 4.2. *Suppose that the conditions of Theorem 4.1 (ii) are satisfied. Then, the PARMSR algorithm with averaging and with updating period L satisfies $\sqrt{N_n}(\overline{\theta}_n - \theta^0) \to N(0,\ S^*)$ in distribution as $n \to \infty$, where S^* is defined in Corollary 3.2.*

It follows from Corollaries 3.2 and 4.2 that the PARMSR algorithms with averaging, updated either after every regenerative cycle or after every L steps of the process $\{J_i,\ i \geq 0\}$, have the same convergence rate and the same limit covariance matrix, for arbitrary $L \geq 1$, as a function of the computing budget. It should be pointed out that the small-sample or transient behavior of these algorithms generally differ. We also recall that our analysis of the asymptotic behavior is based on what happens after no projection of θ_n on D occurs anymore. Thus, our results do not apply if the optimizer θ^0 lies on the boundary of D. If θ^0 is very close to the boundary, it may take a long while before no projection occurs, and this may affect the convergence speed.

The choices of $K = 1$ for (R) and $K = 1/L$ for (F) in (2.2) appear natural, but are not necessary for the convergence results to hold. The two theorems and their corollaries are still valid for an arbitrary constant $K > 0$ in (2.2).

5. Appendix.

Proof of Theorem 3.1. We use the notation $\mathcal{F}^{(m)} = \mathcal{F}_{k_m}$ for $m > 0$ and $\mathcal{F}^{(m)} = \sigma\{X_0, \theta_0,\ \theta_{-1}\}$ for $m \leq 0$. For case (R), the observation noise is $\varepsilon_{n+1} = f_{n+1} - f(\theta_n)\overline{\eta}(\theta_n)$.

(i). We first decompose ε_{n+1} as $\varepsilon_{n+1} = \varepsilon_{n+1}^{(1)} + \varepsilon_{n+1}^{(2)}$, where

$$\varepsilon_{n+1}^{(1)} = \sum_{i=1}^{\eta_{n+1}} J_{\theta,k_n+i} - f(\theta_{n-1})\overline{\eta}(\theta_{n-1}), \tag{5.1}$$

$$\varepsilon_{n+1}^{(2)} = (f(\theta_{n-1}) - f(\theta_n))\overline{\eta}(\theta_{n-1}) + f(\theta_n)(\overline{\eta}(\theta_{n-1}) - \overline{\eta}(\theta_n)). \tag{5.2}$$

By Lemma 2.1 and the one-dependence assumption, it is seen that $\{\varepsilon_{2n}^{(1)},\ \mathcal{F}^{(2n)},\ n \geq$

$1\}$ and $\{\varepsilon^{(1)}_{2n-1},\, \mathcal{F}^{(2n-1)},\, n \ge 1\}$ are martingale difference sequences. By (5.1) and condition (A3),

$$\|\varepsilon^{(1)}_{n+1}\| \le \eta^*_{n+1} Z_{n+1} + \sup_{\theta\in D} \|f(\theta)\| \sup_{\theta\in D} \overline{\eta}(\theta), \tag{5.3}$$

which yields

$$\sup_n \mathsf{E}\left[\|\varepsilon^{(1)}_{n+1}\|^2 \mid \mathcal{F}^{(n-1)}\right] < \infty \tag{5.4}$$

by Schwarz inequality and (A3). Then, using the local convergence theorem of martingales (see, e.g., [6, 20, 37]), it is seen that $\sum_{n=1}^{\infty} a^{(1-\delta)}_{2n-1}\varepsilon^{(1)}_{2n} < \infty$ and $\sum_{n=1}^{\infty} a^{(1-\delta)}_{2n-2}\varepsilon^{(1)}_{2n-1} < \infty$ a.s., which implies that

$$\sum_{i=1}^{\infty} a^{1-\delta}_{i-1}\varepsilon^{(1)}_i < \infty, \quad a.s. \tag{5.5}$$

By Lemma 2 of [40] and condition (A3), one derives

$$a^{(1-\delta)/2}_{n-1} Z_n \xrightarrow[n\to\infty]{} 0 \text{ and } a^{(1-\delta)/2}_{n-1}\eta^*_n \xrightarrow[n\to\infty]{} 0 \;a.s. \quad \forall\, \delta \in [0,\, 1-1/(2\nu)). \tag{5.6}$$

Using condition (A3) and (5.6), it follows from (1.3) and (2.2) that

$$\|\theta_n - \theta_{n-1}\| \le a_{n-1}\|f_n\| \le a^{\delta}_{n-1} a^{1-\delta}_{n-1}\eta^*_n Z_n = o(a^{\delta}_n) \quad a.s.,$$

which gives

$$\begin{aligned} \|\varepsilon^{(2)}_{n+1}\| &\le \|\theta_n - \theta_{n-1}\| \left(B_1 \sup_{\theta\in D}\overline{\eta}(\theta) + B_2 \sup_{\theta\in D}\|f(\theta)\|\} \right) \\ &= O(\|\theta_n - \theta_{n-1}\| = o(a^{\delta}_n) \quad a.s. \end{aligned} \tag{5.7}$$

via (5.2) and the Lipschitz conditions.

By (5.5) and (5.7), the a.s. convergence of the algorithm follows from Theorem 3.1 in [4] (see also Theorem 2.4.1 in [7]). Note that we are using a projection algorithm in (1.3), rather than a randomly varying truncation procedure as in [4, 7]. But the convergence analysis works the same way, since θ^0 is assumed in (A5) to be in the interior of D. After establishing the convergence, by (5.5), (5.7), and Theorem 3.2.1 of [7], it follows that $\|\theta_n - \theta\| = o(a^{\delta}_n)$ a.s.

(ii). Since $\theta_n \to \theta^0$ a.s. as $n\to\infty$, there is a finite time n_0 (which may depend on the sample path) such that

$$\theta_{n+1} = \theta_n - a_n f_{n+1} \quad \forall\, n \ge n_0. \tag{5.8}$$

Define $\varphi_{0,\,0} = I$, $\varphi_{n,n+1} = I$, and

$$\varphi_{n,\,k} = (I + a_n A_n)\cdots(I + a_k A_k) \quad \forall\, n \ge k, \tag{5.9}$$

where $\{A_n,\, n\ge 0\}$ is a sequence of deterministic matrices such that $\lim_{n\to\infty} A_n = -\overline{\eta}(\theta^0)M_1$. Then it is standard to derive that

$$\|\varphi_{n,\,k}\| \le c_0 \exp\left(-c\sum_{j=k}^{n} a_j\right) \text{ and } \sup_n \sum_{i=1}^{n} a_i \|\varphi_{n,\,i+1}\|^{r_1} < \infty \quad \forall\, r_1 > 0, \tag{5.10}$$

where c_0 and c are some positive constants. From (5.8) one derives

$$\begin{aligned}(5.11)\quad & \frac{\theta_{n+1}-\theta^0}{\sqrt{a_{n+1}}} \\ &= \varphi_{n,n_0}\frac{\theta_{n_0}-\theta^0}{\sqrt{a_{n_0}}} - \sum_{i=n_0}^{n}\varphi_{n,i+1}\sqrt{a_i}\varepsilon_{i+1}^{(1)} - \sum_{i=n_0}^{n}\varphi_{n,i+1}\sqrt{a_i}\varepsilon_{i+1}^{(2)} \\ &\quad - \sum_{i=n_0}^{n}\varphi_{n,i+1}o(a_i)\sqrt{a_i}\varepsilon_{i+1} - \sum_{i=n_0}^{n}\varphi_{n,i+1}\sqrt{a_i}(1+o(a_i))O(\|\theta_i-\theta^0\|^2).\end{aligned}$$

In the proof of Theorem 2 of [3], it is shown that $\mathsf{E}[\|\theta_n-\theta^0\|^2]=O(a_n)$, so the last term on the right side of (5.11) converges to zero in probability via (5.10). By the conditions (A2), (A3), and Lemma 2 of [40], it is derived that $\sqrt{a_n}\varepsilon_{n+1}\to 0$ a.s. as $n\to\infty$, which yields that the fourth term on the right side of (5.11) converges to zero a.s.. Similarly, we can prove that the first and the third terms converge to zero a.s.. Then, the result follows from standard martingale arguments.

(iii). To prove (iii), we verify that all conditions of Theorem 2 in [3] are fulfilled. Using (2.9) in [3], it is easy to see that condition (A4) on the step sizes is equivalent to (1.5) and (1.6) in [3]. The condition on the existence of the Lyapunov function $v(\theta)$ is put in (A6), which is weaker than A1) in [3] but is consistent with A2 in [4] and A 4.3.3 in [7]. Note that we use a projection algorithm in (1.3) rather than a randomly varying truncation procedure as in [3, 4, 7]. Condition A2) in [3] is implied by our condition (A5).

We now check conditions A3) and A4) in [3], concerning the observation noise $\varepsilon_n=\varepsilon_n^{(1)}+\varepsilon_n^{(2)}$. Letting $\delta=0$ in (5.5), by the Kronecker lemma it follows that as $n\to\infty$, $a_n\sum_{i=1}^n\varepsilon_{i+1}^{(1)}\to 0$ a.s., which is (2.5) in [3]. Since $\{\varepsilon_{2n}^{(1)},\ \mathcal{F}^{(2n)},\ n\ge 1\}$ and $\{\varepsilon_{2n-1}^{(1)},\ \mathcal{F}^{(2n-1)},\ n\ge 1\}$ are martingale difference sequences, by (5.3) and (5.4) it is easy to derive that (2.6) and (2.7) in [3] are satisfied. From (5.7) it follows that $\|\varepsilon_{n+1}^{(2)}\|=O(\|\theta_n-\theta_{n-1}\|)=O(a_{n-1}\|f_n\|)$. Then we derive that $\mathsf{E}[\|\varepsilon_{n+1}^{(2)}\|^2]=O(a_n^2)$, which satisfies A4) in [3]. We can thus apply Theorem 2 in [3] and this concludes the proof. ∎

Proof of Theorem 4.1. For Theorem 4.1, we first give the proof for $L=1$ for simplicity of writing, then extend the results to $L>1$. As in the proof of Theorem 3.1, for the convergence of $\sum_{n=1}^\infty a_n^{1-\delta}\varepsilon_{n+1}^{(1)}$, it suffices to prove the convergence of $\sum_{n=1}^\infty a_{2n-1}^{1-\delta}\varepsilon_{2n}^{(1)}$ and $\sum_{n=1}^\infty a_{2n-2}^{1-\delta}\varepsilon_{2n-1}^{(1)}$, where $\{\varepsilon_{2n}^{(1)},\ \mathcal{F}^{(2n)},\ n\ge 1\}$ and $\{\varepsilon_{2n-1}^{(1)},\ \mathcal{F}^{(2n-1)},\ n\ge 1\}$ are martingale difference sequences. Such a technique is standard when one wants to extend some results on classically regenerative processes to the one-dependent regenerative processes. Hence, no generality is lost by supposing that $\{J_n(\theta),\ n\ge 1\}$ is a classically regenerative process, which we now do for simplicity of writing. We first give two lemmas.

LEMMA 5.1. *Suppose that $\{z_i\}$ is a sequence of r.v.s with the same distribution. Then for any $r>0$, $\mathsf{E}|z_1|^r<\infty$ implies*

$$\lim_{n\to\infty} n^{-1/r}z_n = 0 \quad a.s.$$

Furthermore, if $\{z_i,\ i\ge 1\}$ are mutually independent, then the converse is true.

Proof. The proof follows from the Borel-Cantelli lemma and Corollary 4.1.3 in [11], pp. 90-91. See also Lemma 2 in [40]. ∎

LEMMA 5.2. *Suppose that $\{z_n,\ \mathcal{B}_n^*\}$ is an l-dimensional martingale difference sequence satisfying*

$$\sup_n \mathsf{E}[\|z_{n+1}\|^2|\mathcal{B}_n^*] < \infty,\quad \|z_n\| = o(h(n)),\ a.s.,\ h(n) \le h(n+1),\qquad \forall\ n \ge 0,$$

and that $g_{n,i}$ is an $\mathcal{B}_i^*$-measurable $l \times l$-dimensional random matrix, for $1 \le i \le n$, which satisfies

$$\sum_{i=1}^n \|g_{n,i}\|^2 \le \overline{g} < \infty,\quad a.s.,\ \forall\ n \ge 1,$$

where $h(n)$ and $\overline{g}$ are positive constants. Then, as $n \to \infty$,

$$\max_{1\le i\le n} \|\sum_{j=1}^i g_{n,j} z_{j+1}\| = o(h(n+1)\log n)\quad a.s.$$

Proof. The proof is in Guo, Huang, and Hannan [19]. See also [6]. ◻

Proof of Theorem 4.1 (i) for $L = 1$. The key idea lies in verifying that $\sum_{n=1}^\infty a_n\varepsilon_{n+1}$ converges a.s.. Then the desired result follows from Theorem 3.1 in [4] (see also Theorem 2.4.1 in [7]). Recall that the observation noise here is

$$\varepsilon_{n+1} = f_{n+1} - f(\theta_n). \tag{5.12}$$

Denote $D_{m,i} = J_{\theta,i}^{(m+1)}(\theta_{k_m})$, the value of f_{k_m+i} obtained if θ_{k_m+j} is fixed at θ_{k_m} for $j \ge 0$.

(a). We first show that $\sum_{m=1}^\infty \sum_{i=1}^{\eta_{m+1}} a_{k_m+i-1}\varepsilon_{k_m+i}$ converges a.s.. From (2.2) and (5.12) it is easy to see that

$$\begin{aligned}
&\sum_{m=1}^\infty \sum_{i=1}^{\eta_{m+1}} a_{k_m+i-1}\varepsilon_{k_m+i} \\
&= \sum_{m=1}^\infty \sum_{i=1}^{\eta_{m+1}} (a_{k_m+i-1} - a_{k_m})\varepsilon_{k_m+i} + \sum_{m=1}^\infty a_{k_m} \sum_{i=1}^{\eta_1^{(m+1)}(\theta_{k_m})} (D_{m,i} - f(\theta_{k_m})) \\
&\quad + \sum_{m=1}^\infty a_{k_m} \sum_{i=1}^{\eta_{m+1}} (f(\theta_{k_m}) - f(\theta_{k_m+i-1})) + \sum_{m=1}^\infty a_{k_m} \sum_{i=1}^{\eta_{m+1}} (f_{k_m+i} - D_{m,i}) \\
&\quad + \sum_{m=1}^\infty a_{k_m} \sum_{i=\eta_1^{(m+1)}(\theta_{k_m})+1}^{\eta_{m+1}} (D_{m,i} - f(\theta_{k_m}))I\{\eta_{m+1} > \eta_1^{(m+1)}(\theta_{k_m})\} \\
&\quad - \sum_{m=1}^\infty a_{k_m} \sum_{i=\eta_{m+1}+1}^{\eta_1^{(m+1)}(\theta_{k_m})} (D_{m,i} - f(\theta_{k_m}))I\{\eta_{m+1} < \eta_1^{(m+1)}(\theta_{k_m})\}.
\end{aligned} \tag{5.13}$$

As in Lemma 3.4 of [41], we can prove that each term on the right-hand side of (5.13) converges a.s.

(b). From the result of step (a), analogous to Lemma 3.5 of [41], we obtain the almost sure convergence of $\sum_{n=1}^\infty a_n\varepsilon_{n+1}$. ◻

Proof of Theorem 4.1 (ii) for $L = 1$. We need the following definitions and lemma. For $n \geq 0$, let

$$\psi_{n,i} = \begin{cases} (I - a_n M_1) \cdots (I - a_i M_1) & \text{for } i \leq n; \\ I & \text{for } i = n+1; \\ 0 & \text{for } i \geq n+2; \end{cases} \tag{5.14}$$

$$\sigma(n) = \max\{m : \ k_m \leq n\}, \quad \tau(n) = \sigma(n) + 1. \tag{5.15}$$

LEMMA 5.3. *If conditions (A0)–(A11) are satisfied, with δ' satisfying (A11), then*

$$a_{n+1}^{\delta'-1/2} \sum_{j=0}^{n} a_j \psi_{n,j+1} \varepsilon_{j+1} \xrightarrow[n \to \infty]{} 0, \quad a.s.$$

Proof. Since $-M_1$ is stable, it is standard to derive that (see, e.g., [3] and [7]):

$$\|\psi_{n,i}\| \leq c_0 \exp\left(-c \sum_{j=i}^{n} a_j\right), \quad \frac{a_i}{a_n} = \exp\left(o(1) \sum_{s=i}^{n-1} a_s\right), \quad \forall\, n \geq i, \tag{5.16}$$

$$\sup_n \sum_{i=1}^{n} a_i \exp\left(-rc \sum_{j=i+1}^{n} a_j\right) < \infty, \quad \sup_n \sum_{i=1}^{n} a_i \|\psi_{n,i+1}\|^r < \infty, \quad \forall\, r > 0, \tag{5.17}$$

where c_0 and c are some constants and $o(1) \to 0$ as $i \to \infty$.

Using (2.2), (5.12), and (5.15), we have

$$\begin{aligned} & a_{n+1}^{\delta'-1/2} \sum_{j=0}^{n} a_j \psi_{n,j+1} \varepsilon_{j+1} \\ &= a_{n+1}^{\delta'-1/2} \sum_{j=k_{\sigma(n)}+1}^{n} a_j \psi_{n,j+1} \varepsilon_{j+1} \\ &\quad + a_{n+1}^{\delta'-1/2} \sum_{m=0}^{\sigma(n)-1} \sum_{i=1}^{\eta_{m+1}} a_{k_m+i-1} \psi_{n,k_m+i} (f_{k_m+i} - f(\theta_{k_m+i-1})) \\ &= a_{n+1}^{\delta'-1/2} \sum_{j=k_{\sigma(n)}+1}^{n} a_j \psi_{n,j+1} \varepsilon_{j+1} \\ &\quad + a_{n+1}^{\delta'-1/2} \sum_{m=0}^{\sigma(n)-1} \sum_{i=1}^{\eta_{m+1}} a_{k_m+i-1} \psi_{n,k_m+i} (f(\theta_{k_m}) - f(\theta_{k_m+i-1})) \\ &\quad + a_{n+1}^{\delta'-1/2} \sum_{m=0}^{\sigma(n)-1} \sum_{i=1}^{\eta_{m+1}} a_{k_m+i-1} \psi_{n,k_m+i} (f_{k_m+i} - D_{m,i}) \\ &\quad + a_{n+1}^{\delta'-1/2} \sum_{m=0}^{\sigma(n)-1} \sum_{i=1}^{\eta_{m+1}} (a_{k_m+i-1} - a_{k_m}) \psi_{n,k_m+i} (D_{m,i} - f(\theta_{k_m})) \end{aligned} \tag{5.18}$$

$$+a_{n+1}^{\delta'-1/2}\sum_{m=0}^{\sigma(n)-1}\sum_{i=1}^{\eta_{m+1}} a_{k_m}(\psi_{n,k_m+i}-\psi_{n,k_m})(D_{m,i}-f(\theta_{k_m}))$$

$$+a_{n+1}^{\delta'-1/2}\sum_{m=0}^{\sigma(n)-1} a_{k_m}\psi_{n,k_m}w_{m+1}$$

$$+a_{n+1}^{\delta'-1/2}\sum_{m=0}^{\sigma(n)-1} a_{k_m}\psi_{n,k_m}\sum_{i=\eta_1^{(m+1)}(\theta_{k_m})+1}^{\eta_{m+1}}(D_{m,i}-f(\theta_{k_m}))I\{\eta_{m+1}>\eta_1^{(m+1)}(\theta_{k_m})\}$$

$$-a_{n+1}^{\delta'-1/2}\sum_{m=0}^{\sigma(n)-1} a_{k_m}\psi_{n,k_m}\sum_{i=\eta_{m+1}+1}^{\eta_1^{(m+1)}(\theta_{k_m})}(D_{m,i}-f(\theta_{k_m}))I\{\eta_{m+1}<\eta_1^{(m+1)}(\theta_{k_m})\},$$

where

$$w_{m+1}=\sum_{i=1}^{\eta_1^{(m+1)}(\theta_{k_m})}(D_{m,i}-f(\theta_{k_m})). \tag{5.19}$$

We will show that each term on the right hand side of the second equality in (5.18) converges to zero a.s.. Then, the desired result follows from (5.18).

(i). By condition (A4), there is a constant α_1 such that

$$\max_{1\le i\le\eta_{m+1}}\left(\frac{a_{k_m}}{a_{k_m+i}}\right)\le 1+\alpha_1 a_{k_m}\eta^*_{m+1} \text{ and } \max_{1\le i\le\eta_{m+1}}|a_{k_m+i}-a_{k_m}|\le\alpha_1 a_{k_m}^2\eta^*_{m+1}. \tag{5.20}$$

Using (A3), it follows from (5.12) that

$$\|\varepsilon_{k_m+i}\|\le W_{m+1}^{(0)}\triangleq\max_{\theta\in D}\|f(\theta)\|+Z_{m+1},\quad \text{for}\quad 1\le i\le\eta_{m+1}. \tag{5.21}$$

By Lemma 5.1 and conditions (A3) and (A11), one has

$$\frac{1}{\sqrt{m^v}}\eta^*_{m+1}W^{(0)}_{m+1}\xrightarrow[m\to\infty]{}0\quad a.s., \tag{5.22}$$

which yields

$$\sqrt{a_{k_{\sigma(n)}}}\eta_{\sigma(n)+1}(\theta^*)W^{(0)}_{\sigma(n)+1}\xrightarrow[n\to\infty]{}0\quad a.s. \tag{5.23}$$

By (5.16), (5.20) and (5.21), we have

$$\frac{a_{n+1}^{\delta'}}{\sqrt{a_{n+1}}}\left\|\sum_{j=k_{\sigma(n)}+1}^{n}a_j\psi_{n,j+1}\varepsilon_{j+1}\right\|\le\frac{a_{n+1}^{\delta'}}{\sqrt{a_{n+1}}}c_0a_{k_{\sigma(n)}}W^{(0)}_{\sigma(n)+1}\eta_{\sigma(n)+1}(\theta^*)$$

$$\le c_0a_{n+1}^{\delta'}\left(\frac{a_{k_{\sigma(n)}}}{a_{k_{\sigma(n)+1}}}\right)^{1/2}a^{1/2}_{k_{\sigma(n)}}\eta_{\sigma(n)+1}(\theta^*)W^{(0)}_{\sigma(n)+1}$$

$$\le c_0a_{n+1}^{\delta'}(1+\alpha_1a_{k_{\sigma(n)}}\eta_{\sigma(n)+1}(\theta^*))^{1/2}\sqrt{a_{k_{\sigma(n)}}}\eta_{\sigma(n)+1}(\theta^*)W^{(0)}_{\sigma(n)+1}\xrightarrow[n\to\infty]{}0\quad \text{a.s.},$$

which implies that the first term on the right hand side of the second equality in (5.18) converges to zero as $n \to \infty$, a.s.

(ii). Using (A3), it follows from (1.3) and (2.2) that

$$\|\theta_{k_m+i} - \theta_{k_m}\| \le \sum_{j=1}^{i} \|\theta_{k_m+j} - \theta_{k_m+j-1}\| \le a_{k_m} W_{m+1} \quad \text{for } \ 1 \le i \le \eta_{m+1}, \tag{5.24}$$

where $W_{m+1} = \eta^*_{m+1} Z_{m+1}$.

By (A7), (5.20) and (5.24), one has

$$\begin{aligned} (5.25) \qquad & a_{n+1}^{\delta'-1/2} \left\| \sum_{m=0}^{\sigma(n)-1} \sum_{i=1}^{\eta_{m+1}} a_{k_m+i-1} \psi_{n,k_m+i} (f(\theta_{k_m}) - f(\theta_{k_m+i-1})) \right\| \\ & \le B_1 a_{n+1}^{\delta'} \sum_{m=0}^{\sigma(n)-1} \sum_{i=1}^{\eta_{m+1}} a_{k_m+i-1} \|\psi_{n,k_m+i}\| \frac{1}{\sqrt{a_{n+1}}} a_{k_m} W_{m+1} \\ & \le B_1 a_{n+1}^{\delta'} \sum_{m=0}^{\sigma(n)-1} \sum_{i=1}^{\eta_{m+1}} a_{k_m+i-1} \|\psi_{n,k_m+i}\| \left(\frac{a_{k_m+i}}{a_{n+1}} \right)^{1/2} \\ & \qquad \cdot \max_{1 \le i \le \eta_{m=1}} \left(\frac{a_{k_m}}{a_{k_m+i}} \right)^{1/2} \sqrt{a_{k_m}} W_{m+1} \\ & \le B_1 c_0 a_{n+1}^{\delta'} \sum_{m=0}^{\sigma(n)-1} \sum_{i=1}^{\eta_{m+1}} a_{k_m+i-1} \exp\left(-c \sum_{j=k_m+i}^{n} a_j \right) \exp\left(o(1) \sum_{j=k_m+i}^{n} a_j \right) \\ & \qquad \cdot (1 + \alpha_1 a_{k_m} \eta^*_{m+1})^{1/2} \sqrt{a_{k_m}} W_{m+1} \\ & \xrightarrow[n \to \infty]{} 0 \quad a.s., \end{aligned}$$

since by (5.16)-(5.17),

$$\begin{aligned} & \sum_{m=0}^{\sigma(n)-1} \sum_{i=1}^{\eta_{m+1}} a_{k_m+i-1} \exp\left(-c \sum_{j=k_m+i}^{n} a_j \right) \exp\left(o(1) \sum_{j=k_m+i}^{n} a_j \right) \\ & \le \sum_{i=0}^{n} a_i \exp\left(-c \sum_{j=i+1}^{n} a_j \right) \exp\left(o(1) \sum_{j=i+1}^{n} a_j \right) \quad = \ O(1) \quad a.s., \end{aligned}$$

and by (5.22)

$$(1 + \alpha_1 a_{k_m} \eta^*_{m+1})^{1/2} \sqrt{a_{k_m}} W_{m+1} \xrightarrow[m \to \infty]{} 0 \quad a.s.$$

Thus, the second term on the right hand side of the second equality in (5.18) converges to zero a.s., as $n \to \infty$.

(iii). By Condition (A9) one has

$$\|f_{k_m+i} - D_{m,i}\| \le a_{k_m} Z^{(0)}_{m+1} \qquad \text{a.s.} \tag{5.26}$$

for $1 \le i \le \eta_{m+1}$. Similar to (5.22), by Lemma 5.1 it follows that

$$\sqrt{a_{k_m}} Z^{(0)}_{m+1} \xrightarrow[m \to \infty]{} 0 \quad \text{a.s.}. \tag{5.27}$$

By the same argument as in (5.25), it is seen that

$$\begin{aligned}
& a_{n+1}^{\delta'-1/2} \left\| \sum_{m=0}^{\sigma(n)-1} \sum_{i=1}^{\eta_{m+1}} a_{k_m+i-1} \psi_{n,k_m+i} (f_{k_m+i} - D_{m,i}) \right\| \\
\le\ & a_{n+1}^{\delta'} \sum_{m=0}^{\sigma(n)-1} \sum_{i=1}^{\eta_{m+1}} a_{k_m+i-1} \|\psi_{n,k_m+i}\| \left(\frac{a_{k_m+i}}{a_{n+1}} \right)^{1/2} \max_{1 \le i \le \eta_{m+1}} \left(\frac{a_{k_m}}{a_{k_m+i}} \right)^{1/2} \\
& \cdot \sqrt{a_{k_m}} Z^{(0)}_{m+1} \\
\le\ & c_0 a_{n+1}^{\delta'} \sum_{m=0}^{\sigma(n)-1} \sum_{i=1}^{\eta_{m+1}} a_{k_m+i-1} \exp\left(-c \sum_{j=k_m+i}^{n} a_j \right) \exp\left(o(1) \sum_{j=k_m+i}^{n} a_j \right) \\
& \cdot (1 + \alpha_1 a_{k_m})^{1/2} \sqrt{a_{k_m}} Z^{(0)}_{m+1} \\
\xrightarrow[n \to \infty]{} & 0 \quad a.s.,
\end{aligned} \tag{5.28}$$

which implies that the third term on the right hand side of the second inequality in (5.18) converges to zero as $n \to \infty$, a.s.

(iv). Analogous to (5.25), by (5.20) it is derived that the fourth term on the right hand side of the second inequality in (5.18) converges to zero as $n \to \infty$, a.s.

(v). By the definition (5.14), we have

$$\begin{aligned}
\psi_{n,k_m+i} - \psi_{n,k_m} &= \sum_{j=1}^{i} a_{k_m+j-1} \psi_{n,k_m+j} M_1 \\
&= \psi_{n,k_m+i} \sum_{j=1}^{i} a_{k_m+j-1} \psi_{k_m+i-1,k_m+j} M_1.
\end{aligned} \tag{5.29}$$

Similar to (5.25), we arrive at the conclusion that the fifth term on the right hand side of the second inequality in (5.18) converges to zero as $n \to \infty$, a.s.

(vi). We now consider the convergence of the sixth term. First, by the definition (5.14),

$$\sum_{m=0}^{\tau(n)} \psi_{n,k_m} a_{k_m} w_{m+1} = \sum_{m=0}^{n+1} a_{k_m} \psi_{n,k_m} w_{m+1}. \tag{5.30}$$

Let $h(m) = m^{\zeta}, \forall\, m \ge 1$, where ζ is some constant satisfying $0 < \zeta < \delta' \nu$. Then by condition (A11) and Lemma 2.1 we have

$$\frac{\|w_{m+1}\|}{h(m)} \le \frac{W^{(0)}_{m+1} \eta^*_{m+1}}{m^{\zeta}} \xrightarrow[m \to \infty]{} 0, \quad a.s. \tag{5.31}$$

By (5.16)–(5.17), it is derived that

$$\begin{aligned}\frac{1}{a_{n+1}}\sum_{m=0}^{n+1}a_{k_m}^2\|\psi_{n,k_m}\|^2 &\le \frac{1}{a_{n+1}}\sum_{m=0}^{n+1}a_{k_m-1}a_{k_m}\|\psi_{n,k_m}\|^2 \\ \le \frac{1}{a_{n+1}}\sum_{i=0}^{n+1}a_{i-1}\|\psi_{n,i}\|^2 a_i \\ \le c_0^2\sum_{i=0}^{n+1}\exp\left(-2c\sum_{j=i}^{n}a_j\right)a_{i-1}\exp\left(o(1)\sum_{j=i}^{n}a_j\right) = O(1),\end{aligned} \tag{5.32}$$

which, incorporating with (5.31) and Lemma 5.2, leads to

$$\frac{1}{\sqrt{a_{n+1}}}\sum_{m=0}^{n+1}\psi_{n,k_m}a_{k_m}w_{m+1} = O(1)+o(h(n)\log n). \tag{5.33}$$

By (5.33), it follows from (5.30) that

$$\begin{aligned}a_{n+1}^{\delta'-1/2}\sum_{m=0}^{\tau(n)}a_{k_m}\psi_{n,k_m}w_{m+1} &= O(a_{n+1}^{\delta'})+o\left(a_{n+1}^{\delta'}h(n)\log n\right)\\ &= o\left(\frac{1}{n^{\delta'\nu-\zeta}}\log n\right) = o(1) \quad \text{a.s.},\end{aligned}$$

which implies that $a_{n+1}^{\delta'-1/2}\sum_{m=0}^{\sigma(n)-1}a_{k_m}\psi_{n,k_m}w_{m+1} \to 0$ a.s. as $n\to\infty$, since by (5.16), (5.20), and (5.22)–(5.23) we have

$$\begin{aligned}&\frac{a_{n+1}^{\delta}}{\sqrt{a_{n+1}}}\|\psi_{n,k_{\sigma(n)}}a_{k_{\sigma(n)}}w_{\sigma(n)+1}+\psi_{n,k_{\tau(n)}}a_{k_{\tau(n)}}w_{\tau(n)+1}\|\\ &\le c_0 a_{n+1}^{\delta}\Big((1+\alpha_1 a_{k_{\sigma(n)}}\eta^*_{\sigma(n)+1})^{1/2}\sqrt{a_{k_{\sigma(n)}}}\eta^*_{\sigma(n)+1}W^{(0)}_{\sigma(n)+1}\\ &\qquad +(\frac{a_n}{a_{n+1}})^{1/2}\sqrt{a_{k_{\tau(n)}}}\eta^*_{\tau(n)+1}W^{(0)}_{\tau(n)+1}\Big)\xrightarrow[n\to\infty]{}0, \quad a.s.\end{aligned}$$

(vii). Set

$$V^{(0)}_{m+1} = \sum_{i=\eta_1^{(m+1)}(\theta_{k_m})+1}^{\eta_{m+1}}(D_{m,i}-f(\theta_{k_m}))I\{\eta_{m+1}>\eta_1^{(m+1)}(\theta_{k_m})\}, \quad \forall\, m\ge 0.$$

By (5.31), we get

$$\frac{V^{(0)}_m-\mathsf{E}[V^{(0)}_m|\mathcal{F}^{(m-1)}]}{h(m)}\xrightarrow[m\to\infty]{}0, \quad a.s., \tag{5.34}$$

which incorporating with (5.32) and Lemma 5.2 yields

$$\begin{aligned}&\frac{1}{\sqrt{a_{n+1}}}\sum_{m=0}^{n+1}\psi_{n,k_m}a_{k_m}(V^{(0)}_{m+1}-\mathsf{E}[V^{(0)}_{m+1}|\mathcal{F}^{(m)}])\\ &= O(1)+o(h(n)\log n) = o(h(n)\log n), \quad a.s.\end{aligned} \tag{5.35}$$

By Hölder inequality and condition (A10) for any $p_1 > 1$ we have

$$
\begin{aligned}
(5.36)\qquad & \|\mathsf{E}[V^{(0)}_{m+1}|\mathcal{F}^{(m)}]\| \\
&\le \mathsf{E}[\eta^*_{m+1}W^{(0)}_{m+1}I\{\eta_{m+1} > \eta_1^{(m+1)}(\theta_{k_m})\}|\mathcal{F}^{(m)}] \\
&\le \left(\mathsf{E}\left[(W^{(0)}_{m+1}\eta^*_{m+1})^{p_1}\right]\right)^{1/p_1} P\{\eta_{m+1} > \eta_1^{(m+1)}(\theta_{k_m})|\mathcal{F}^{(m)}\}^{1-1/p_1} \\
&\le \alpha_0^{1-1/p_1}\left(\mathsf{E}\left[(W^{(0)}_{m+1}\eta^*_{m+1})^{p_1}\right]\right)^{1/p_1} a_{k_m}^{\gamma_1(1-1/p_1)},
\end{aligned}
$$

which leads to

$$
\begin{aligned}
(5.37)\qquad & \frac{1}{a_{n+1}^{1/2-\delta'}}\|\sum_{m=0}^{n+1}\psi_{n,k_m}a_{k_m}\mathsf{E}[V^{(0)}_{m+1}|\mathcal{F}^{(m)}]\| \\
&\le \alpha_0^{1-1/p_1}\sup_m\left\{\left(\mathsf{E}\left[(W^{(0)}_{m+1}\eta^*_{m+1})^{p_1}\right]\right)^{1/p_1}\right\} \\
&\quad\cdot\frac{1}{a_{n+1}^{1/2-\delta'}}\sum_{m=0}^{n+1}\|\psi_{n,k_m}\|a_{k_m-1}a_{k_m}^{\gamma_1(1-1/p_1)} \\
&\le O(1)a_{n+1}^{\delta'}\sum_{i=-1}^{n+1}a_i\exp\left(-c\sum_{j=i+1}^{n}a_j\right)\left(\frac{a_{i+1}}{a_{n+1}}\right)^{1/2}a_{i+1}^{\gamma_1(1-1/p_1)-1/2} \\
&= o(1)\quad a.s.,
\end{aligned}
$$

provided that $\gamma_1(1-1/p_1) \ge 1/2$. Combining (5.35) and (5.37) gives

$$
\frac{1}{a_{n-1}^{1/2-\delta'}}\|\sum_{m=0}^{n+1}\psi_{n,k_m}a_{k_m}V^{(0)}_{m+1}\| = o(1) + o(a_{n+1}^{\delta'}h(n)\log n) = o(1)\quad a.s.,
$$

where $0 < \zeta < \delta'\nu$.

(viii). In the same way as in (vii), one shows that the last term on the right hand side of the second equality in (5.18) converges to zero, a.s. □

Proof of Theorem 4.1 (ii) for $L = 1$. By Theorem 4.1 (i), there is an a.s. finite r.v. n_0^* such that no projection occurs after step n_0^*, i.e.,

$$
(5.38)\qquad \theta_{n+1} = \theta_n - a_n(f(\theta_n) + \varepsilon_{n+1}),\quad \forall\ n \ge n_0^*.
$$

Then, for a deterministic integer n_0, on the event $\{n \ge n_0 \ge n_0^*\}$, one has

$$
\begin{aligned}
(5.39)\qquad \frac{\theta_{n+1}-\theta^0}{a_{n+1}^{1/2-\delta'}} &= a_{n+1}^{\delta'-1/2}\psi_{n,n_0}(\theta_{n_0}-\theta^0) - a_{n+1}^{\delta'-1/2}\sum_{j=n_0}^{n}\psi_{n,j+1}a_j\varepsilon_{j+1} \\
&\quad - a_{n+1}^{\delta'-1/2}\sum_{j=n_0}^{n}\psi_{n,j+1}a_j(f(\theta_j) - M_1(\theta_j-\theta^0)).
\end{aligned}
$$

Let $r \le r_0$, where r_0 is given by condition (A5). Define

$$
(5.40)\qquad \sigma^* = \begin{cases} 0 & \text{if } \|\theta_{n_0}-\theta^0\| \ge r, \\ \inf\{j:\ \ j > n_0,\ \|\theta_j-\theta^0\| \ge r\} & \text{otherwise.}\end{cases}
$$

By (5.16), it is easy to see that

$$
\begin{aligned}
\tag{5.41} &\|a_{n+1}^{\delta'-1/2}\psi_{n,n_0}(\theta_{n_0}-\theta^0)\| \\
&\le c_0 a_{n_0}^{\delta'-1/2}\left(\frac{a_{n_0}}{a_{n+1}}\right)^{1/2-\delta'}\exp\left(-c\sum_{j=n_0}^{n}a_j\right)\|\theta_{n_0}-\theta^0\| \\
&\le O(1)\exp\left(-c\sum_{j=n_0}^{n}a_j\right)\exp\left(o(1)\sum_{j=n_0}^{n}a_j\right) \\
&= o(1) \quad \text{as} \quad n\to\infty.
\end{aligned}
$$

Let c_0^* be a constant such that

$$
\sup_{j\ge 1}\left\{\left(\frac{a_j}{a_{j+1}}\right)^{1/2-\delta'}\right\}\le c_0^*.
$$

By condition (A5) and (5.16) we have

$$
\begin{aligned}
\tag{5.42} & a_{n+1}^{\delta'-1/2}\left\|\sum_{j=n_0}^{n}\psi_{n,j+1}a_j(f(\theta_j)-M_1(\theta_j-\theta^0))I\{\sigma^*>n+1,\ n_0^*\le n_0\}\right\| \\
&\le c_0c_1a_{n+1}^{\delta'-1/2}\sum_{j=n_0}^{n}\exp\left(-c\sum_{s=j+1}^{n}a_s\right)a_j\|\theta_j-\theta^0\|^2 I\{\sigma^*>j,\ n_0^*\le n_0\} \\
&\le c_0c_1r\sum_{j=n_0}^{n}\exp\left(-c\sum_{s=j+1}^{n}a_s\right)a_j\left(\frac{a_j}{a_{n+1}}\right)^{1/2-\delta'}\frac{\|\theta_j-\theta^0\|I\{\sigma^*>j,\ n_0^*\le n_0\}}{a_j^{1/2-\delta'}} \\
&\le c_0^*c_0c_1r\sum_{j=n_0}^{n}\exp\left(-\frac{c}{2}\sum_{s=j+1}^{n}a_s\right)a_j\frac{\|\theta_j-\theta^0\|I\{\sigma^*>j,\ n_0^*\le n_0\}}{a_j^{1/2-\delta'}},
\end{aligned}
$$

if n_0 is large enough.

By Lemma 5.3 and (5.41)—(5.42), it follows from (5.39) that

$$
\begin{aligned}
&\|\theta_{n+1}-\theta^0\|I\{\sigma^*>n+1,\ n_0^*\le n_0\}a_{n+1}^{\delta'-1/2}, \\
&\le o(1)+c_0^*c_0c_1r\sum_{j=n_0}^{n}\exp\left(-\frac{c}{2}\sum_{s=j+1}^{n}a_s\right)a_j \\
\tag{5.43} &\cdot\|\theta_j-\theta^0\|^2 I\{\sigma^*>j,\ n_0^*\le n_0\}a_j^{\delta'-1/2},
\end{aligned}
$$

which, incorporating with the Bellman-Gronwall inequality, leads to

$$
\begin{aligned}
\tag{5.44} &\|\theta_n-\theta^0\|I\{\sigma^*>n,\ n_0^*\le n_0\}a_n^{\delta'-1/2} \\
&\le o(1)+o(1)\sum_{j=n_0}^{n-1}\exp\left(-\frac{c}{2}\sum_{s=j+1}^{n}a_s\right)a_j\exp\left(c_0^*c_0c_1r\sum_{s=j+1}^{n-1}a_s\right)
\end{aligned}
$$

$$\leq o(1) + o(1) \sum_{j=n_0}^{n-1} a_j \exp\left(-\frac{c}{4} \sum_{s=j+1}^{n-1} a_s\right) \; = \; o(1),$$

where r is sufficiently small such that $c_0^* c_0 c_1 r \leq c/4$. From (5.44), it is readily seen that $\|\theta_n - \theta^0\| = o(a_n^{1/2-\delta'})$ as $n \to \infty$, a.s. □

Proof of Theorem 4.1 (iii) for $L = 1$. In order to prove Theorem 4.1 (iii), we need several lemmas. From (5.14), we have

$$\psi_{n,j} = \psi_{n-1,j} - a_n M_1 \psi_{n-1,j} = I - M_1 \sum_{i=j}^{n} a_i \psi_{i-1,j},$$

which gives

$$a_{j-1} \sum_{i=j}^{n} \psi_{i-1,j} = \sum_{i=j}^{n} a_i \psi_{i-1,j} + \sum_{i=j}^{n} (a_{j-1} - a_i)\psi_{i-1,j} = M_1^{-1} + G_{n,j}, \tag{5.45}$$

where

$$G_{n,j} \triangleq -M_1^{-1}\psi_{n,j} + \sum_{i=j}^{n} (a_{j-1} - a_i)\psi_{i-1,j}, \quad \forall\, n \geq j. \tag{5.46}$$

LEMMA 5.4. *Let the conditions of Theorem 4.1 (iii) be satisfied. Then*

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \sum_{j=1}^{n-1} (M_1^{-1} + G_{n,j+1})(f(\theta_j) - M_1(\theta_j - \theta^0)) = 0, \quad a.s.$$

Proof. For any $\delta' \in (0,\ 1/2(1 - 1/(2\nu)))$, by condition (A4) and Theorem 3.2 it follows that

$$\sum_{i=1}^{\infty} \frac{1}{\sqrt{i}} \|\theta_i - \theta^0\|^2 = \sum_{i=1}^{\infty} \frac{1}{\sqrt{i}} o(a_i^{1-2\delta'}) = \sum_{i=1}^{\infty} \frac{1}{\sqrt{i}}\, o\left(\frac{1}{i^{\nu(1-2\delta')}}\right) < \infty \text{ a.s.},$$

which gives

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \sum_{j=1}^{n-1} \|\theta_j - \theta^0\|^2 = 0 \quad \text{a.s.}, \tag{5.47}$$

via the Kronecker lemma (see, e.g., [11]).

It is shown in Lemma 1 of [3] that for all $n \geq j \geq 1$, $G_{n,j}$ defined by (5.46) are bounded. By Theorem 4.1, condition (A5), and (5.47), the lemma follows easily. □

LEMMA 5.5. *Suppose that conditions (A0)—(A11) are satisfied. Then $1/\sqrt{n}\sum_{i=1}^{n} \varepsilon_i \to N(0,\ S_4)$ in distribution as $n \to \infty$, where $S_4 = \overline{\eta}(\theta^0)^{-1} S$.*

Proof. As in (5.13), we decompose

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \varepsilon_i \tag{5.48}$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{n}} \sum_{j=k_{\sigma(n)}+1}^{n} \varepsilon_j + \frac{1}{\sqrt{n}} \sum_{m=0}^{\sigma(n)-1} \sum_{i=1}^{\eta_{m+1}} (f_{k_m+i} - D_{m,i}) \\
&+ \frac{1}{\sqrt{n}} \sum_{m=0}^{\sigma(n)-1} \sum_{i=1}^{\eta_{m+1}} (f(\theta_{k_m}) - f(\theta_{k_m+i-1})) \\
&+ \frac{1}{\sqrt{n}} \sum_{m=0}^{\sigma(n)-1} \sum_{i=\eta_1^{(m+1)}(\theta_{k_m})+1}^{\eta_{m+1}} (D_{m,i} - f(\theta_{k_m})) I\{\eta_{m+1} > \eta_1^{(m+1)}(\theta_{k_m})\} \\
&- \frac{1}{\sqrt{n}} \sum_{m=0}^{\sigma(n)-1} \sum_{i=\eta_{m+1}+1}^{\eta_1^{(m+1)}(\theta_{k_m})} (D_{m,i} - f(\theta_{k_m})) I\{\eta_{m+1} < \eta_1^{(m+1)}(\theta_{k_m})\} \\
&+ \frac{1}{\sqrt{n}} \sum_{m=0}^{\sigma(n)-1} w_{m+1},
\end{aligned}
$$

where $\sigma(n)$ is defined by (5.15) and w_{m+1} is defined by (5.19).

We first show that each of the first five terms on the right hand side of (5.48) converges to zero a.s., as $n \to \infty$. Then we show that the last term converges to a $N(0,\ S_4)$ in distribution. Hence the desired result follows.

(i). Similar to (5.23), it is easy to show that the first term on the right hand side of (5.48) converges to zero as $n \to \infty$, a.s.

(ii). By (5.26) it is derived that

$$
\begin{aligned}
\sum_{m=1}^{\infty} \frac{1}{\sqrt{m}} \mathsf{E}\left[\left\| \sum_{i=1}^{\eta_{m+1}} (f_{k_m+i} - D_{m,i}) \right\| \,\middle|\, \mathcal{F}^{(m)} \right] &\le \sum_{m=1}^{\infty} \frac{1}{\sqrt{m}} a_{k_m} \mathsf{E}\left[\eta_{m+1}^* Z_{m+1}^* Z_{m+1}^{(0)} \right] \\
&\le \sum_{m=1}^{\infty} \frac{\overline{a}}{m^{\nu+1/2}} \mathsf{E}\left[\eta_{m+1}^* Z_{m+1}^* Z_{m+1}^{(0)} \right] < \infty, \quad a.s.,
\end{aligned}
$$

which implies

$$
\sum_{m=1}^{\infty} \frac{1}{\sqrt{m}} \sum_{i=1}^{\eta_{m+1}} (f_{k_m+i} - D_{m,i}) < \infty \quad a.s., \tag{5.49}
$$

by the local convergence theorem of martingales (see, e.g., [6, 20, 37]). By the Kronecker lemma, it follows from (5.49) that

$$
\lim_{n\to\infty} \frac{1}{\sqrt{n}} \sum_{m=0}^{n} \sum_{i=1}^{\eta_{m+1}} (f_{k_m+i} - D_{m,i}) = 0 \quad a.s.,
$$

which gives

$$
\lim_{n\to\infty} \frac{1}{\sqrt{n}} \sum_{m=0}^{\sigma(n)-1} \sum_{i=1}^{\eta_{m+1}} (f_{k_m+i} - D_{m,i}) = 0 \quad a.s., \tag{5.50}
$$

since $\sigma(n) \le n,\ \forall\ n \ge 1$.

(iii). By (5.24), similar to (5.50) we can prove that the third term on the right hand side of (5.48) converges to zero as $n \to \infty$, a.s.

(iv). By (5.36) and condition (A4), we derive that

$$\mathsf{E}\left\|\sum_{m=1}^{\infty}\frac{1}{\sqrt{m}}\sum_{i=\eta_1^{(m+1)}(\theta_{k_m})+1}^{\eta_{m+1}}(D_{m,i}-f(\theta_{k_m}))I\{\eta_{m+1}>\eta_1^{(m+1)}(\theta_{k_m})\}\right\| \tag{5.51}$$

$$\le \sum_{m=1}^{\infty}\frac{1}{\sqrt{m}}\alpha_0^{1-1/p_1}\left(\mathsf{E}\left[(W_{m+1}^{(0)}\eta_{m+1}^{*})^{p_1}\right]\right)^{1/p_1}\overline{a}^{\gamma_1(1-1/p_1)}m^{-\gamma_1(1-1/p_1)\nu}<\infty,$$

if $\nu\gamma_1(1-1/p_1) > 1/2$. By (5.51), it is easy to prove that the fourth term on the right hand side of (5.48) converges to zero as $n \to \infty$, a.s.

(v). Similar to (iv), we can prove that the fifth term on the right hand side of (5.48) converges to zero a.s., as $n \to \infty$.

(vi). By the central limit theorem for martingales (see, e.g., [11] and [20]),

$$\frac{1}{\sqrt{n}}\sum_{m=0}^{n} w_{m+1}\xrightarrow[n\to\infty]{d}N(0,\ S). \tag{5.52}$$

We now show that

$$\frac{\sigma(n)}{n}\xrightarrow[n\to\infty]{}\frac{1}{\overline{\eta}(\theta^0)},\quad a.s. \tag{5.53}$$

One can decompose

$$\frac{1}{n}\sum_{i=1}^{n}\eta_i=\frac{1}{n}\sum_{i=1}^{n}(\eta_i-\eta_1^{(i)}(\theta_{k_{i-1}}))+\frac{1}{n}\sum_{i=1}^{n}(\eta_1^{(i)}(\theta_{k_{i-1}})-\overline{\eta}(\theta_{k_{i-1}})) \tag{5.54}$$

$$+\frac{1}{n}\sum_{i=1}^{n}(\overline{\eta}(\theta_{k_{i-1}})-\overline{\eta}(\theta^0))+\overline{\eta}(\theta^0).$$

By conditions (A9) and (A10), we have

$$\mathsf{E}\left[\sum_{i=1}^{\infty}\frac{1}{i}\|\eta_i-\eta_1^{(i)}(\theta_{k_{i-1}})\|\right]\le\sum_{i=1}^{\infty}\frac{1}{i}\mathsf{E}\left[\eta_i^{*}I\{\eta_i\ne\eta_1^{(i)}(\theta_{k_{i-1}})\}\right]$$

$$\le\sum_{i=1}^{\infty}\frac{1}{i}\sqrt{\mathsf{E}[(\eta_1^{*})^2]}\sqrt{P\{\eta_i\ne\eta_1^{(i)}(\theta_{k_{i-1}})\}}\le O(1)\sum_{i=1}^{\infty}\frac{1}{i}i^{-\gamma_1\nu/2}<\infty,$$

which, combining with the Kronecker lemma, yields $\lim_{n\to\infty} 1/n\sum_{i=1}^{n}(\eta_i-\eta_1^{(i)}(\theta_{k_{i-1}}))<\infty$ a.s. By the local convergence theorem of martingales and the Kronecker lemma, it is derived that $\lim_{n\to\infty}\ 1/n\ \sum_{i=1}^{n}\ (\eta_1^{(i)}(\theta_{k_{i-1}})-\overline{\eta}(\theta_{k_{i-1}}))=0$ a.s. By Theorem 3.1, $\theta_n\to\theta^0$ a.s. as $n\to\infty$, which implies $1/n\sum_{i=1}^{n}(\overline{\eta}(\theta_{k_{i-1}})-\overline{\eta}(\theta^0))\to 0$ as $n\to\infty$ a.s. via the continuity of $\overline{\eta}(\theta)$ at θ^0. Thus it follows from (5.54) that

$$\frac{1}{n}\sum_{i=1}^{n}\eta_i\xrightarrow[n\to\infty]{}\overline{\eta}(\theta^0),\quad a.s. \tag{5.55}$$

From the definition (5.15), we have $n < \sum_{i=1}^{\sigma(n)+1} \eta_i \le n + \eta_{\sigma(n)+1}$, which gives

$$1 < \left(\frac{\sigma(n)}{n} + \frac{1}{n}\right) \frac{1}{\sigma(n)+1} \sum_{i=1}^{\sigma(n)+1} \eta_i \le 1 + \frac{\sigma(n)+1}{n} \frac{1}{\sigma(n)+1} \eta^*_{\sigma(n)+1}. \tag{5.56}$$

This yields (5.53) by (5.55) and Lemma 5.1.

To prove

$$\frac{1}{\sqrt{n}} \sum_{m=0}^{\sigma(n)-1} w_{m+1} \xrightarrow[n \to \infty]{} N(0,\ S_4), \tag{5.57}$$

we need a central limit theorem for stopped martingales. We note that if the Kolmogorov inequality is replaced by Doob's inequality (see, e.g., [11]), the proof of Theorem 9.4.1 in [11] goes through for the stopped martingales. Then, by (5.52) and (5.53), (5.57) follows. □

LEMMA 5.6. *If the conditions of Theorem 4.1 (iii) are fulfilled, then* $1/\sqrt{n} \sum_{j=1}^{n} G_{n,j}\varepsilon_j \to 0$ *in probability as* $n \to \infty$.

Proof. Similar to (5.48), we have

$$\begin{aligned}
&\frac{1}{\sqrt{n}} \sum_{j=1}^{n} G_{n,j}\varepsilon_j \\
&= \frac{1}{\sqrt{n}} \sum_{j=k_{\sigma(n)}+1}^{n} G_{n,j}\varepsilon_j + \frac{1}{\sqrt{n}} \sum_{m=0}^{\sigma(n)-1} \sum_{i=1}^{\eta_{m+1}} G_{n,k_m+i}(f_{k_m+i} - D_{m,i}) \\
&+ \frac{1}{\sqrt{n}} \sum_{m=0}^{\sigma(n)-1} \sum_{i=1}^{\eta_{m+1}} G_{n,k_m+i}(f(\theta_{k_m}) - f(\theta_{k_m+i-1})) \\
&+ \frac{1}{\sqrt{n}} \sum_{m=0}^{\sigma(n)-1} \sum_{i=\eta_1^{(m+1)}(\theta_{k_m})+1}^{\eta_{m+1}} G_{n,k_m+i}(D_{m,i} - f(\theta_{k_m})) I\{\eta_{m+1} > \eta_1^{(m+1)}(\theta_{k_m})\} \\
&- \frac{1}{\sqrt{n}} \sum_{m=0}^{\sigma(n)-1} \sum_{i=\eta_{m+1}+1}^{\eta_1^{(m+1)}(\theta_{k_m})} G_{n,k_m+i}(D_{m,i} - f(\theta_{k_m})) I\{\eta_{m+1} < \eta_1^{(m+1)}(\theta_{k_m})\} \\
&+ \frac{1}{\sqrt{n}} \sum_{m=0}^{\sigma(n)-1} \sum_{i=1}^{\eta_1^{(m+1)}(\theta_{k_m})} G_{n,k_m}(D_{m,i} - f(\theta_{k_m})) \\
&+ \frac{1}{\sqrt{n}} \sum_{m=0}^{\sigma(n)-1} \sum_{i=1}^{\eta_1^{(m+1)}(\theta_{k_m})} (G_{n,k_m+i} - G_{n,k_m})(D_{m,i} - f(\theta_{k_m})).
\end{aligned} \tag{5.58}$$

Since $G_{n,j}$ is bounded for $n \ge j \ge 1$, it follows from the proof of Lemma 5.5 that each of the first five terms on the right side of (5.58) converges to zero a.s. as $n \to \infty$. In what follows we prove that as $n \to \infty$ the sixth term converges to zero in probability, while the last term converges to zero a.s.

(i). By the definition (5.46), we get

$$\mathsf{E}\left[\left\|\frac{1}{\sqrt{n}}\sum_{m=0}^{\tau(n)} G_{n,k_m} w_{m+1}\right\|^2\right] \le \frac{1}{n}\mathsf{E}\left[\sum_{m=0}^{\tau(n)} \|G_{n,k_m}\|^2 \mathsf{E}[\|w_{m+1}\|^2|\mathcal{F}^{(m)}]\right] \tag{5.59}$$

$$\le \mathsf{E}\left[(\eta_1^* W_1^{(0)})^2\right] \frac{1}{n}\sum_{i=0}^{n+1} \|G_{n,i}\|^2 \xrightarrow[n\to\infty]{} 0$$

via Lemma 1 of [3]. By (5.59), it is easy to derive that $1/\sqrt{n}\sum_{m=0}^{\sigma(n)-1} G_{n,k_m} w_{m+1} \to 0$ in probability as $n \to \infty$.

(ii). By (5.14) and (5.45), we have

$$\begin{aligned} &\|G_{n,j} - G_{n,j-1}\| \\ &= \|(a_{j-1} - a_{j-2})\sum_{s=j}^{n} \psi_{s-1,j} + a_{j-2}\sum_{s=j}^{n}(\psi_{s-1,j} - \psi_{s-1,j-1}) - a_{j-2}\psi_{j-2,j-1}\| \\ &\le \alpha_1 a_{j-1} a_{j-2}\sum_{s=j}^{n}\|\psi_{s-1,j}\| + a_{j-2}\sum_{s=j}^{n} a_{j-1}\|M_1\|\ \|\psi_{s-1,j}\| + a_{j-2} \\ &\le a_{j-1}\left((\alpha_1 + \|M_1\|)\sup_j\{\frac{a_{j-2}}{a_{j-1}}\}\ \sup_j\{a_{j-1}\sum_{s=j}^{n}\|\psi_{s-1,j}\|\} + \sup_j\{\frac{a_{j-2}}{a_{j-1}}\}\right) \\ &\le c_3 a_{j-1}, \qquad \forall\, n \ge j \ge 1, \end{aligned} \tag{5.60}$$

where α_1 and c_3 are some constants. By (5.60) and (5.26), it follows that

$$\begin{aligned} &\frac{1}{\sqrt{n}}\left\|\sum_{m=0}^{\sigma(n)-1}\sum_{i=1}^{\eta_1^{(m+1)}(\theta_{k_m})}(G_{n,k_m+i} - G_{n,k_m})(D_{m,i} - f(\theta_{k_m}))\right\| \\ &\le \frac{1}{\sqrt{n}}\sum_{m=0}^{\sigma(n)-1}\sum_{i=1}^{\eta_1^{(m+1)}(\theta_{k_m})}\|\sum_{j=1}^{i}(G_{n,k_m+j} - G_{n,k_m+j-1})\|\cdot\|D_{m,i} - f(\theta_{k_m})\| \\ &\le \frac{c_3}{\sqrt{n}}\sum_{m=0}^{\sigma(n)-1} a_{k_m}(\eta_{m+1}^*)^2 W_{m+1}^{(0)} \xrightarrow[n\to\infty]{} 0, \quad a.s., \end{aligned}$$

via similar arguments as for (5.50). Thus, the proof of Lemma 5.6 is completed. □

Proof of Theorem 4.1 (iii) for $L = 1$. By (5.45), from (5.38) it follows that

$$\begin{aligned} \sqrt{n}(\overline{\theta}_n - \theta^0) &= o(1) + \frac{1}{\sqrt{n}}\sum_{i=n_0^*}^{n}(\theta_i - \theta^0) \\ &= o(1) + \frac{1}{\sqrt{n}}\frac{1}{a_{n_0^*-1}}(M_1^{-1} + G_{n,n_0^*})(\theta_{n_0^*} - \theta^0) \\ &\quad - \frac{1}{\sqrt{n}}\sum_{j=n_0^*}^{n-1}(M_1^{-1} + G_{n,j+1})(f(\theta_j) - M_1(\theta_j - \theta^0)) \end{aligned} \tag{5.61}$$

$$-\frac{M_1^{-1}}{\sqrt{n}}\sum_{j=n_0^*}^{n-1}\varepsilon_{j+1}-\frac{1}{\sqrt{n}}\sum_{j=n_0^*}^{n-1}G_{n,j+1}\varepsilon_{j+1}.$$

Since the $G_{n,j}$ are bounded uniformly for $n \geq j \geq 1$, one has

$$\frac{1}{\sqrt{n}}\frac{1}{a_{n_0^*-1}}(M_1^{-1}+G_{n,n_0^*})(\theta_{n_0^*}-\theta^0)\xrightarrow[n\to\infty]{}0, \quad a.s. \tag{5.62}$$

By (5.62) and Lemmas 5.4—5.6, the desired result follows from (5.61). ☐

Proof of Theorem 4.1 for $L > 1$. Let $\widetilde{\varepsilon}_n = \beta_n - f(\widetilde{\theta}_{n-1})$, $\forall\, n \geq 1$. The proof for the case $L = 1$ applies to the present setting if we replace $\theta_n,\ \varepsilon_n,\ f_n$ by $\widetilde{\theta}_n,\ \widetilde{\varepsilon}_n$, and β_n, respectively (cf. [40] and [41]). We only mention the key point for Theorem 4.1 (iii). By the definitions (5.12) and (2.2),

$$\begin{aligned}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\varepsilon_i &= \frac{1}{\sqrt{n}}\sum_{i=0}^{n}\left(\frac{1}{L}\sum_{j=1}^{L}\beta_{iL+j}-f(\theta_{i-1})\right)\\ &= \frac{1}{L\sqrt{n}}\sum_{i=1}^{(n+1)L}(\beta_i - f(\widetilde{\theta}_{i-1})) \;=\; \frac{1}{L\sqrt{n}}\sum_{i=1}^{(n+1)L}\widetilde{\varepsilon}_i.\end{aligned} \tag{5.63}$$

As in the proof of Lemma 5.5, one can show that $1/\sqrt{n}\sum_{i=1}^{n}\widetilde{\varepsilon}_i \to N(0,\ S_4)$ in distribution as $n \to \infty$. Combined with (5.63), this gives $1/\sqrt{n}\sum_{i=1}^{n}\varepsilon_i \to N(0,\ S_4^*)$ in distribution as $n \to \infty$, where $S_4^* = L^{-1}S_4 = L^{-1}\overline{\eta}(\theta^0)^{-1}S$. The rest of the proof works the same way as for $L = 1$. ☐

REFERENCES

[1] S. Asmussen, *Applied Probability and Queues*, John Wiley and Sons, New York, 1987.

[2] K. B. Athreya and P. Ney, *A new approach to the limit theorems of recurrent Markov chains*, Trans. Amer. Math. Soc., 245 (1978), pp. 493–501.

[3] H. F. Chen, *Asymptotic efficient stochastic approximation*, Stochastics and Stochastics Reports, 45 (1993), pp. 1–16.

[4] H. F. Chen, *Stochastic approximation and its new applications*, Proc. 1994 Hong Kong International Workshop on New Directions in Control and Manufacturing, pp. 2–12, 1994.

[5] H. F. Chen, *Continuous-time stochastic approximation: convergence and asymptotic efficiency*, Stochastics and Stochastics Reports, 51 (1994), pp. 111–132.

[6] H. F. Chen and L. Guo, *Identification and Stochastic Adaptive Control*, Birkhäuser, 1991.

[7] H. F. Chen and Y. M. Zhu, *Stochastic Approximation*, Shanghai Scientific and Technological Publishers, 1996.

[8] E. K. P. Chong and P. J. Ramadge, *Convergence of recursive optimization algorithms using infinitesimal perturbation analysis estimates*, Discrete Event Dynamic Systems: Theory and Applications, 1 (1992), pp. 339–372.

[9] E. K. P. Chong and P. J. Ramadge, *Optimization of queues using an infinitesimal perturbation analysis-based stochastic algorithm with general updates times*, SIAM J. Contr. Optim., 31 (1993), pp. 698–732.

[10] E. K. P. Chong and P. J. Ramadge, *Stochastic optimization of regenerative systems using infinitesimal perturbation analysis*, IEEE Trans. Automat. Contr., 39 (1994), pp. 1400–1410.

[11] Y. S. Chow and H. Teicher, *Probability Theory: Independence, Interchangeability, Martingale*, Second Edition, Springer-Verlag, 1988.

[12] P. Dupuis and H. J. Kushner, *Stochastic approximations via large deviations: Asymptotic properties*, SIAM Journal on Control and Optimization, 23 (1985), pp. 675–696.

[13] P. Dupuis and R. Simha, *On Sampling Controlled Stochastic approximation*, IEEE Transactions on Automatic Control, 36 (1991), pp. 915–924.

[14] M. C. Fu, *Convergence of a stochastic approximation algorithm for the GI/G/1 queue using infinitesimal perturbation analysis*, J. of Optim. Theory and Appl., 65 (1990), pp. 149–160.

[15] P. Glasserman, *Gradient Estimation via Perturbation Analysis*, Kluwer Academic, Boston, 1991.

[16] P. Glasserman, *Stationary waiting time derivatives*, QUESTA, 12 (1992), pp. 369–390.

[17] P. Glasserman, *Regenerative derivatives of regenerative sequences*, Adv. Appl. Prob., 25 (1993), pp. 116–139.

[18] P. Glasserman, J. Q. Hu, and S. G. Strickland, *Strongly consistent steady-state derivative estimates*, Probability in the Engineering and Informational Sciences, 5 (1991), pp. 391–413.

[19] L. Guo, D. W. Huang, and E. J. Hannan, *On ARX (∞) approximation*, J. Multivariate Analysis, 32 (1990), pp. 17–47.

[20] P. Hall and C. C. Heyde, *Martingale Limit Theory and Its Application.* Academic Press, New York, 1980.

[21] Y. C. Ho and X. R. Cao, *Perturbation Analysis of Discrete Event Dynamic Systems*, Kluwer Academic, Boston, 1991.

[22] H. J. Kushner and D. S. Clark, *Stochastic Approximation Methods for Constrained and Unconstrained Systems*, Springer-Verlag, New York, 1978.

[23] H. J. Kushner and F. J. Vázquez-Abad, *Stochastic approximation methods for systems of interest over an infinite horizon*, SIAM J. on Control and Optim., 34 (1996), pp. 712–756.

[24] H. J. Kushner and G. G. Yin *Stochastic Approximation Algorithms and Applications*, Springer, New York, 1997.

[25] P. L'Ecuyer, *A Unified View of the IPA, SF, and LR Gradient Estimation Techniques*, Management Science, 36 (1990), 1364–1383.

[26] P. L'Ecuyer and P. W. Glynn, *Stochastic optimization by simulation: convergence proofs for the GI/G/1 queue in steady-state*, Management Sci., 40 (1994), pp. 1562–1578.

[27] P. L'Ecuyer, N. Giroux, and P. W. Glynn, *Stochastic optimization by simulation: numerical experiments with the M/M/1 queue in steady-state*, Management Sci., 40 (1994), pp. 1245–1261.

[28] P. L'Ecuyer and G. Yin, *Budget-dependent convergence rate of stochastic approximation*, SIAM J. on Optim., 8 (1998), pp. 217–247.

[29] S. P. Meyn and R. L. Tweedie, *Markov Chains and Stochastic Stability*, Springer-Verlag, New York, 1993.

[30] E. Nummelin, *General Irreducible Markov Chains and Non-negative Operators*, Cambridge Univ. Press, New York, 1984

[31] B. T. Polyak, *New method of stochastic approximation type procedure*, Automation and Remote Control, 51 (1990), pp. 937–946.

[32] B. T. Polyak and A. B. Juditsky, *Acceleration of Stochastic Approximation by Averaging*, SIAM J. Control and Optim., 30 (1992), pp. 838–855.

[33] H. Robbins and S. Monro, *A stochastic approximation method*, Ann. Math. Statist., 22 (1951), pp. 400–407.

[34] K. Sigman, *Notes on the stability of closed queueing networks*, J. Appl. Probab., 26 (1989), pp. 678–682; *Corrections*, J. Appl. Probab., 27 (1990), p. 735.

[35] K. Sigman, *The stability of open queueing networks*, Stoch. Processes and their Appls., 35 (1990), pp. 11–25.

[36] K. Sigman and R. W. Wolff, *A review of regenerative processes*, SIAM Review, 35 (1993), pp. 269–288.

[37] W. F. Stout, *Almost Sure Convergence*, Academic Press, New York, 1974.

[38] R. Suri, *Perturbation analysis: the state of the art and research issues explained via the GI/G/1 queue*, Proc. IEEE, 77 (1989), pp. 114–137.

[39] R. Suri and Y. T. Leung, *Single run optimization of discrete event simulations—An empirical study using the M/M/1 queue*, IIE Transactions, 21 (1989), pp. 35–49.

[40] Q. Y. Tang and H. F. Chen, *Convergence of perturbation analysis based optimization algorithm with fixed number of customers period*, Discrete Event Dynamic Systems: Theory and Appl., 4 (1994), pp. 359–375.

[41] Q. Y. Tang, H. F. Chen, and Z. J. Han, *Convergence rates of Perturbation-Analysis-Robbins-Monro-Single-Run algorithms for single server queues*, IEEE Trans. on Autom. Contr., 42 (1997), pp. 1442–1447.

[42] R. Wolff, *Stochastic Modeling and the Theory of Queues*, Prentice-Hall, Englewood Cliffs, NJ, 1989.

[43] G. Yin, *On extensions of Polyak's averaging approach to stochastic approximation*, Stochastics and Stochastics Reports, 36 (1991), pp. 245–264.