Stochastic Stability of Queueing Networks*

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Abstract.

This paper investigates geometric stability and L_p-stability of discrete-time Markov chains associated with closed and open queueing networks with Markovian routing. By geometric stability (resp. L_p-stability) we mean that the chain is regenerative in the Harris-recurrent sense and that the times between the successive regeneration points have a bounded moment generating function (resp. a bounded pth moment). We show that the closed queueing networks are geometrically stable (resp. L_p-stable) if the service times have a bounded moment generation function (resp. a bounded pth moment), and that the open queueing networks are geometrically stable (resp. L_p-stable) if the service times have a bounded moment generating function (resp. a bounded $\max\{p, 2\}$-th moment) and the interarrival times have a bounded second moment (resp. a bounded pth moment).

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1 Introduction

Perturbation analysis and the score function (or likelihood ratio) method are two important sample path-based techniques for sensitivity analysis of stochastic discrete event systems. When these methods are applied to estimate the gradient of a steady-state average over an infinite horizon, the strong consistency of the resulting estimator is usually established under the assumption that the system is regenerative in a wide sense [1], and that the times between the successive regeneration points have a bounded moment generating function or a bounded pth moment for a certain value of $p \geq 1$ which depends on the context (see, e.g., [6] and [7]). Similar assumptions are often made to prove the strong consistency and to derive the convergence rates of stochastic approximation algorithms based on those gradient estimators to optimize continuous system parameters; see, e.g., [4, 12, 13, 26]. In [4], for instance, the pth moment ($p \geq 6$) of the regenerative cycle length is assumed to be finite. In the stochastic approximation case, the bound on the pth moment or on the moment generating function is typically assumed to hold uniformly with respect to the parameter which is to be optimized, and not only on the regenerative cycle length, but also on the total cost incurred during a regenerative cycle.

Our aim in this paper is to provide mild conditions under which closed and open Markovian queueing networks enjoy those stability properties. We call a random variable *exponentially stable* (resp., *L_p-stable*) if it has a bounded moment generation function in a neighborhood of zero (resp., a bounded pth moment). A Harris-recurrent Markov chain (HRMC) is *exponentially stable* or *geometrically stable* (resp., *L_p-stable*) if the lengths of the regenerative cycles are exponentially stable (resp., L_p-stable). (We normally use the word *geometric* instead of *exponential* in the discrete-time case.) A HRMC is *geometrically ergodic* (resp., *L_p-ergodic*) if it is *aperiodic* and geometrically stable (resp., L_p-stable). Geometric ergodicity of Harris chains is well-covered, e.g., in [16] and [18]. A closed or open queueing network can be represented as a Markov process with an embedded discrete-time chain $\mathcal{X} = \{X_n, n \geq 0\}$, and conditions for this chain to be Harris-recurrent have been given in the literature. In this paper, we say that a queueing network with embedded HRMC $\mathcal{X}$ is exponentially stable or L_p-stable if $\mathcal{X}$ is so.

Under some assumptions on the routing probabilities, we show that a closed queueing network is geometrically stable if the service times at all nodes are exponentially stable, and L_p-stable if the service times are L_p-stable, for $p \geq 1$. For an open queueing network with mutually independent interarrival and service times, we show that the chain is geometrically stable (resp., L_p-stable) if the interarrival times are L_2-stable (resp., L_p-stable) and the service times at all nodes are exponentially stable (resp., L_p-stable if $p \geq 2$, L_2-stable if $1 \leq p \leq 2$).

After these stability results are established, bounding the moments of the total cost incurred during a cycle can be achieved with little additional work. It is also easy to see from our proofs that if the queue evolution depends on a continuous parameter θ and if the stability conditions on the interarrival times, service times, or routing probabilities hold uniformly in θ, then the stability results also hold uniformly in θ, which is what we typically need in the context of stochastic optimization.

The earlier work on this topic goes back to Heyde [9], who gave necessary and sufficient conditions for the geometric stability of a single queue. Borovkov [2] gives sufficient conditions for the geometric stability of a queueing network, but these conditions are more restrictive than ours, and his proof of stability for the open queueing network case assumes a priori that the long-run departure rate of customers from each node is equal to the arrival rate (see page 418 of his paper). In other words, as noted by Meyn and Down [14, p.143], some form of stability has already been assumed before establishing the geometric stability results.

We do not make such an a priori assumption. Our proof relies on the representation of the queueing networks as Harris-recurrent Markov chains. This approach was pioneered by Nummelin [17], who found explicit regeneration points for tandem queues, and was pursued in [22, 23, 24, 25]. Explicit regeneration points are provided in [24] for the closed queueing network and in [25] for the open one. Proofs of L_1-stability are also given in these papers for two-nodes closed and open queueing networks. The proof of [24] for the closed case can be generalized to networks with an arbitrary number of nodes.

Chang, Thomas, and Kiang [3] and Meyn and Down [14] have analyzed the stability of the continuous-time queue length process associated with an open queueing network. They proved different forms of convergence for this process, under different sets of assumptions. Here, we study the embedded discrete-time chain instead.

Before stating our results, we need to recall some definitions and properties of Harris-recurrent Markov chains (HRMCs), which we do in Section 2. In Section 3, we describe the queueing models, express them as continuous-time Markov processes with an embedded Harris-recurrent discrete-time chain, and state our main results relative to the stochastic stability of that chain. The proofs for the closed case are given in Section 4, while the proofs for the open case are in Section 5. Some technical results related to our proofs are collected in appendices.

2 Harris Chains and Stability Properties

The theory of Harris-recurrent Markov chains in the context of queueing networks plays a major role in our approach. In this section, we quickly recall some basic definitions and facts. We refer the reader to [1, 15, 16, 18, 27] for more elaborate coverages.

Let $\mathcal{X} = \{X_n, n \geq 0\}$ be a Markov chain living on a state space $\mathcal{S}$, a separable metric space with Borel σ-field $\mathcal{B}(\mathcal{S})$. Let P_x denote the probability conditional on $X_0 = x$ and E_x the corresponding mathematical expectation. The chain is called *Harris-recurrent* if there exists a nontrivial σ-finite measure φ on $\mathcal{S}$ such that for all $A \in \mathcal{B}(\mathcal{S})$ with $\varphi(A) > 0$ and each $x \in \mathcal{S}$, $P_x[X_n \in A \text{ i.o.}] = 1$, where i.o. means *infinitely often*. This definition is known to be equivalent to the following: $\mathcal{X}$ is called Harris-recurrent, if for some integer $m^* \geq 0$, number $\delta^* \in (0, 1)$, probability measure ν on $\mathcal{B}$, and measurable set $A^* \in \mathcal{B}(\mathcal{S})$ with $\nu(A^*) > 0$,

$$P_x\{X_n \in A^* \text{ i.o. }\} = 1 \quad \text{for all } x \in \mathcal{S}, \tag{2.1}$$

$$P_x\{X_{m^*} \in \cdot\,\} \geq \delta^* \nu(\cdot) \qquad \text{for all } x \in A^*, \tag{2.2}$$

where A^* is called *a small set* [18]. From (2.1) and (2.2) it is seen that the set A^* is visited i.o. and that whenever such a visit occurs, the state of the chain m^* steps later is distributed according to ν with probability at least δ^*. If A^* consists of a single point of $\mathcal{S}$, one can take $m^* = 0$ and $\delta^* = 1$. This is regeneration in the "classical" sense, where a single state is visited i.o. a.s. (almost surely).

To define regeneration epochs for the Harris-recurrent chain, one can use the so-called *splitting technique*: if $X_n \in A^*$, then with probability δ^*, $n + m^*$ is a *regeneration epoch* for the chain and X_{n+m^*} has distribution ν. This distribution is independent of X_n. Let $R_0 = 0$ and $0 < R_1 < R_2 < \cdots$ be those regeneration epochs. Then $\{R_n, n \geq 1\}$ is a renewal process. Denote the regenerative cycle lengths by $\sigma_i = R_i - R_{i-1},\ i \geq 1,\ \sigma_0 = 0$. A HRMC $\mathcal{X}$ has a unique (up to a multiplicative constant) invariant measure π^*. If π^* is finite, then $\mathcal{X}$ is called *positive recurrent* in which case π^* is usually normalized to a probability measure. The positive recurrence of $\mathcal{X}$ is equivalent to $\mathsf{E}[\sigma_2] < \infty$, or $\sup_{x\in A^*} \mathsf{E}_x[\tau^*] < \infty$, where A^* is a small set and $\tau^* = \inf\{n > 0 :\ X_n \in A^*\}$. If $\mathcal{X}$ is also aperiodic, then we call $\mathcal{X}$ *Harris ergodic*. It is worth noticing that σ_1 may depend on the initial value $X_0 = x$, but $\{\sigma_i,\ i \geq 2\}$ is an i.i.d. sequence.

We say that a random variable X is exponentially stable if $\mathsf{E}[e^{uX}] < \infty,\ \forall\ u \in [0,\, u_0]$ for some $u_0 > 0$, and L_p-stable if $\mathsf{E}[X^p] < \infty$. We say that a HRMC $\mathcal{X}$ is geometrically stable or exponentially stable (resp. L_p-stable) if the regenerative cycle lengths are exponentially stable (resp. L_p-stable), i.e., there is a constant $u_1 > 0$ such that

$$\mathsf{E}[e^{u\sigma_2}] < \infty, \quad \forall\ u \in [0,\, u_1] \tag{2.3}$$

(resp. $\mathsf{E}[\sigma_2^p] < \infty$). We say that a queueing network with embedded HRMC $\mathcal{X}$ is geometrically stable or L_p-stable if $\mathcal{X}$ is. An aperiodic and geometrically stable (resp. L_p-stable) HRMC is called *geometrically ergodic* (resp., *L_p-ergodic*). When $p = 1$, the L_1-stability and L_1-ergodicity coincide with positive recurrence and Harris ergodicity, respectively. We note that geometric stability of a HRMC is equivalent to

$$\sup_{x\in A^*} \mathsf{E}_x[\exp(s\tau^*)] < \infty, \quad \forall\ u \in [0,\, u_1^*], \tag{2.4}$$

for some small set A^* and constant $u_1^* > 0$, where $\mathcal{X}$ satisfying (2.4) is called *geometrically recurrent* in [18] and A^* satisfying (2.4) is called *a Kendall set* in [16].

Geometric ergodicity is a very strong form of stability. It implies that the distribution of X_n converges to an invariant measure at a geometric rate, as $n \to \infty$. See, e.g., [16, 18, 19] for other equivalent definitions and an extensive analysis of geometric ergodicity.

For the L_p-stability, we have the following equivalent condition; see [20] for the proof.

Proposition 2.1 *Let $p \geq 1$ be fixed. The chain is L_p*-stable *if and only if*

$$\sup_{x\in A^*} \mathsf{E}_x[(\tau^*)^p] < \infty, \tag{2.5}$$

for some small set A^*.

Checking geometric stability or L_p-stability directly from the definitions or from (2.4) and (2.5) is rarely practical. The following three stochastic Lyapunov function criteria are often easier to verify. The first two are proved, for example, in [11], [16], and [18]. They are usually stated with $m = 1$, but it is easily seen that the positive recurrence of the m-skeleton chain $\{X_{nm}, n \geq 0\}$ implies the positive recurrence of $\mathcal{X}$, and similarly for geometric or L_p-stability.

Proposition 2.2 *The chain $\mathcal{X}$ is positive recurrent if and only if there is a small set A^*, a function $V : \mathcal{S} \to [0, \infty)$, a fixed integer $m \geq 1$, and a constant $b > 0$ such that*

$$\mathsf{E}_x[V(X_m)] \leq V(x) - b, \quad \forall\, x \notin A^*, \tag{2.6}$$

$$\sup_{x \in A^*} \mathsf{E}_x[V(X_m)] < \infty. \tag{2.7}$$

Proposition 2.3 *The chain $\mathcal{X}$ is geometrically stable if and only if there is a small set A^*, a function $V : \mathcal{S} \to [0, \infty)$, a fixed integer $m \geq 1$, and two constants $\beta \in (0, 1)$ and $b > 0$ such that*

$$\mathsf{E}_x[V(X_m)] \leq \beta V(x) - b, \quad \forall\, x \notin A^*, \tag{2.8}$$

$$\sup_{x \in A^*} \mathsf{E}_x[V(X_m)] < \infty. \tag{2.9}$$

The next proposition follows from Theorem 5.2.2 of [11], p.116.

Proposition 2.4 *For $p \geq 1$, suppose that there is a small set A^*, a constant $b > 0$, a function $V : \mathcal{S} \to [0, \infty)$, a fixed integer $m \geq 1$, and a function $\Delta(x)$ defined on $\mathcal{S}$ such that*

(i) $\sup_{x \in A^*} V(x) < \infty$;

(ii) $P\{V(X_m) - V(x) \leq \Delta(x)\} = 1, \quad \forall\, x \in \mathcal{S}$;

(iii) $\sup_{x \notin A^*} \mathsf{E}_x \Delta(x) \leq -b$;

(iv) $\sup_{x \in \mathcal{S}} \mathsf{E}_x |\Delta(x)|^p < \infty$.

*Then, the chain $\mathcal{X}$ is L_p-*stable.

The function V in these propositions is called a *test function*, or *Foster-Lyapunov function*, or *stochastic Lyapunov function* in the literature. In Section 3, we will use these stochastic Lyapunov function criteria to establish stability properties for the closed and open queueing networks.

In simulation and optimization contexts, a cost (or reward) is usually incurred at each step of the process, and one is interested in the stability of the process of cumulated costs. Let $\{J_n,\ n \geq 0\}$ be a stochastic process adapted to the filtration $\{\mathcal{F}_n, n \geq 0\}$ (i.e., J_n is $\mathcal{F}_n$-measurable for each n), where $\mathcal{F}_n = \sigma(X_0, \ldots, X_n)$. Let

$$Z_i = \sum_{j=1}^{\sigma_i} J_{R_{i-1}+j} \tag{2.10}$$

be the total cumulated cost during the ith regenerative cycle. The following proposition gives stability results for $\{Z_i,\ i \geq 1\}$. Its proof is in Appendix A.

Proposition 2.5 *(i) Suppose that the chain $\mathcal{X}$ is geometrically stable and that*

$$\sup_{n\geq 1} \mathsf{E}[\exp(u_2|J_n|)|\mathcal{F}_{n-1}] < \infty \tag{2.11}$$

where $u_2 > 0$ is a constant. Then there is a constant $u_3 > 0$ such that

$$\sup_{i\geq 1} \sup_{x\in A^*} \mathsf{E}_x[\exp(u_3|Z_i|)] < \infty, \tag{2.12}$$

where A^* is a small set defined in (2.1)-(2.2).

(ii) Suppose that for some $p \geq 1$, the chain $\mathcal{X}$ is L_p-stable and

$$\sup_{n\geq 1} \mathsf{E}[|J_n|^p|\mathcal{F}_{n-1}] < \infty. \tag{2.13}$$

Then,

$$\sup_{i\geq 1} \sup_{x\in A^*} \mathsf{E}_x[|Z_i|^p] < \infty. \tag{2.14}$$

As a special case, suppose that $\mathcal{X}$ is a discrete-time Markov chain embedded in a continuous-time Markov process and that the continuous-time process regenerates when the discrete-time process does. If J_n represents the elapsed time in the continuous-time process between the $(n-1)$th and nth steps of the discrete-time chains, then Z_i becomes the length of a regenerative cycle for the continuous-time process. The previous proposition gives us sufficient conditions for this length to have bounded moments or a bounded moment generating function. Similarly, if J_n is the cost for the continuous-time process between the times of the $(n-1)$th and nth steps, then Z_i is the total cost for that process during a regenerative cycle.

3 The Queueing Models and Main Results

3.1 Closed and Open Networks

We consider a Jackson-type queueing network with c FIFO single-server nodes and a single class of customers. The service time of the nth customer at node i is denoted by $S_n(i)$. We assume that $\{S_n(i),\ n \geq 1,\ 1 \leq i \leq c\}$ are mutually independent and that for each node i, $\{S_n(i),\ n \geq 1\}$ are identically distributed with distribution G_i and mean $\mathsf{E}\,[S_1(i)] \triangleq 1/\mu_i$. To avoid triviality, we assume that $G_i(0) < 1,\ 1 \leq i \leq c$. We shall consider two different kinds of queueing networks: the closed ones and the open ones.

In the *closed* network, the total number of customers is fixed at N. Each customer goes from node to node, independently of the others, according to a Markov chain with state space $\{1,\ 2,\ ...,\ c\}$ and irreducible transition matrix Π^* with elements π_{ij}, $1 \leq i, j \leq c$.

For the *open* network, there is an exogenous source of arrivals and each customer, after completing its service at node i, leaves the system with probability π_{i0}, $1 \leq i \leq c$. The transition matrix Π^* is thus extended into a $(c+1) \times (c+1)$ matrix Π, with elements π_{ij}, $0 \leq i, j \leq c$, such that $\pi_{00} = 1$, and the corresponding Markov chain has 0 as a single absorbing state. Thus, every customer will visit at most a finite number of nodes. The exogenous arrival process is a renewal process, independent of the service times, and the times between successive arrivals have distribution F, with mean $1/\lambda$, so λ is the arrival rate. Each arriving customer joins queue i with probability p_i, so the exogenous arrival rate at node i is λp_i. The traffic equations for this system are

$$\lambda_i = \lambda p_i + \sum_{j=1}^{c} \lambda_j \pi_{ji}, \quad 1 \le i \le c. \tag{3.1}$$

We assume that at each node i, the traffic intensity $\rho_i \stackrel{\triangle}{=} \lambda_i/\mu_i$ is less than 1.

In the next subsections, we represent the evolution of both types of queuing systems by discrete-time Markov chains. These chains are defined differently for the two types of systems: The jumps occur at service completions in the closed case and at arrival times in the open case.

3.2 Markov Chain Representation for the Closed System

For the closed queueing network, let $Q_n = (Q_n(1),\ ...,\ Q_n(c))$ and $Y_n = (Y_n(1),\ ...,\ Y_n(c))$, where $Q_n(i)$ and $Y_n(i)$ are the queue size and the residual service time at queue i, respectively, immediately *before* the nth service completion (so for each n, at least one of the $Y_n(i)$ is zero). If $X_n = (Q_n,\ Y_n)$, then $\mathcal{X} = \{X_n, n \ge 0\}$ is a Markov chain with state space

$$\begin{aligned} \mathcal{S} \ \stackrel{\triangle}{=} \ \{(q_1, \ldots, q_c, y_1, \ldots, y_c) :\ & 0 \le q_i \le N \text{ and } 0 \le y_i < \infty \text{ for each } i, \\ & \text{and } q_1 + \cdots + q_c = N\}. \end{aligned}$$

We introduce the following condition, which is restrictive but similar to that of Theorem 1 in [24].

Assumption (A1) *There is a node i_0 such that for all nodes i, $\pi_{ii_0} > 0$.*

Without loss of generality, suppose that $i_0 = 1$. The successive indices $n \ge 1$, for $\mathcal{X}$, at which the N customers are at node 1 and one customer is just leaving node 1, are regeneration points for the Markov chain $\mathcal{X}$. Let $R_1 \le R_2 \le R_3 \le \cdots$ be these indices. Formally, $R_0 = 0$ and $R_i = \inf\{n > R_{i-1} \ :\ Q_n(1) = N,\ Y_n(1) = 0\}$ for $i > 0$. For any $b_0 > 0$, define the bounded rectangle

$$B(b_0) = \{(q_1, \ldots, q_c, y_1, \ldots, y_c) \in \mathcal{S} \ :\ y_i \le b_0 \text{ for all } i\}. \tag{3.2}$$

Under the assumption that $\pi_{ij} > 0$ for all i, j, and that the service times have finite expectations, [24] shows that there are three positive constants b_0, m^*, and δ^* such that whenever $X_n \in B(b_0)$, the probability that $\mathcal{X}$ will regenerate during the next m^* steps is at least δ^*. He goes on to show that the chain $\mathcal{X}$ regenerates infinitely often, so it is Harris-recurrent and satisfies (2.1)–(2.2) with $A^* = B(b_0)$. His proof still holds if his condition on the π_{ij}'s is replaced by the condition (A1). The following theorem establishes the geometric stability of the Harris chain, under the assumption that the service times are exponentially stable.

Assumption (A2) *The service times $\{S_n(i)\}$ are exponentially stable, that is, there are positive constants a_1 and b_1 such that for all $s \in [0, a_1]$ and $1 \le i \le c$,*

$$\mathsf{E}\left[\exp(sS_1(i))\right] \le b_1.$$

Theorem 3.1 *Under conditions (A1) and (A2), the Markov chain $\mathcal{X}$ associated with the closed queueing network is geometrically stable.*

Related results have been obtained by Borovkov [2]. He considered the continuous-time process $\{Q(t), t \ge 0\}$, where $Q(t)$ is the number of customer in each queue at time t, and studied the stability of this process under the assumption that either (i) the service time at node 1 is unbounded ($G_1(x) < 1$ for all $x > 0$), or (ii) the service times at all nodes $i > 1$ are not bounded away from zero ($G_i(x) > 0$ for all $x > 0$ and all $i > 1$). These assumptions play a similar role as ours, i.e., they ensure the recurrence of regeneration epochs. Theorem 3.1 still holds if (A1) is replaced by the above conditions (i) or (ii).

With slightly weaker assumptions, we get L_p-stability.

Assumption (A3) *The service times $\{S_n(i)\}$ have bounded pth moments, i.e.,*

$$\sup_{1 \le i \le c} \mathsf{E}[S_1(i)^p] < \infty.$$

Theorem 3.2 *Under conditions (A1) and (A3), the Markov chain $\mathcal{X}$ associated with the closed queueing network is L_p-stable.*

Note that when $\mathcal{X}$ regenerates, the continuous-time queue size process $\{Q(t), t \ge 0\}$ also regenerates.

3.3 Regenerative Structure for the Open System

For the open queueing network, for $n \ge 1$, let $Q_n = (Q_n(1),\ ...,\ Q_n(c))$ and $Y_n = (Y_n(1),\ ...,\ Y_n(c))$, where $Q_n(i)$ and $Y_n(i)$ are the queue size and the residual service time at queue i, respectively, immediately *before* the nth exogenous arrival (that is, at the nth arrival time, but not counting the arriving customer). If $X_n = (Q_n,\ Y_n)$, then $\mathcal{X}$ is a Markov chain with state space $\mathcal{S} \triangleq \mathbb{N}^c \times [0, \infty)^c$.

For any positive constants k_0 and b_0, define the bounded rectangle

$$B(k_0,\, b_0) = \{(q_1, \ldots, q_c, y_1, \ldots, y_c) \in \mathcal{S} :\ q_i \le k_0 \text{ and } y_i \le b_0 \text{ for all } i\}. \tag{3.3}$$

Sigman [25] has shown that if the service times at all nodes have finite expectation, then the Markov chain $\{X_n\}$ is Harris-recurrent and there exist positive constants k_0, b_0, m^*, and δ^* such that whenever $X_n \in B(k_0, b_0)$, the chain regenerates within the next m^* steps with probability at least δ^*. Therefore, (2.1)–(2.2) are satisfied with $A^* = B(k_0, b_0)$. The next theorem establish the geometric stability of the Harris chain under the assumption of exponential stability of the service times and finite variance of the interarrival times.

Assumption (A4) *The interarrival-times distribution F has a bounded second moment.*

Theorem 3.3 *Under conditions (A2) and (A4), the Markov chain $\mathcal{X}$ associated with the open queueing network is geometrically stable.*

Again, we get L_p-stability under a slightly weaker assumption on the service times, but a slightly stronger condition on the interarrival times.

Assumption (A5) *The interarrival times distribution F has a bounded qth moment, where $q = \max(p, 2)$.*

Theorem 3.4 *Under conditions (A3) and (A5) for some $p \geq 1$, the Markov chain $\mathcal{X}$ associated with the open queueing network is L_p-stable.*

The condition (A5) in Theorem 3.4 appears stronger than necessary. This is because the stochastic Lyapunov function criterion in Proposition 2.4 for the L_p-stability is only a sufficient condition. Borovkov [2] and Meyn and Down [14] used the following conditions (B1) and (B2), respectively, on the service times, instead of our condition (A2):

(B1). There are constants $b > 0$ and $\beta \in (0, 1)$ such that

$$P[S_1(i) > t + b | S_1(i) > t] \leq 1 - \beta, \quad \forall\, t \geq 0,\, 1 \leq i \leq c.$$

(B2). There is a constant $b > 0$ such that

$$\mathsf{E}[S_1(i) - t | S_1(i) > t] \leq b, \quad \forall\, t \geq 0,\, 1 \leq i \leq c.$$

Conditions (B1) and (B2) are equivalent, and either of them implies (A2). We conclude that (A2) is weaker than (B1) or (B2) by providing a counterexample, in Appendix E, which satisfies (A2) but violates (B1) and (B2). Condition (A2) is reasonable for the geometric stability of queueing networks, since Heyde [9] has shown that it is a necessary and sufficient condition on the service times in the case of a $GI/GI/1$ queue.

4 Proofs of Stochastic Stability for Closed Queueing Networks

In this section we prove Theorems 3.1 and 3.2.

Proof of Theorem 3.1:

We construct a test function V that satisfies the conditions of Proposition 2.3, with $m = c$ and $A^* = B(b_0)$ for some b_0. Denote:

$$D_1 = \min_{1 \leq i \leq c} S_1(i) \quad \text{and} \quad D_2 = \max_{1 \leq i \leq c} \sum_{j=1}^{c} S_j(i).$$

Choose $a_2 \in [0,\, a_1]$, with a_1 as in (A2), and a positive constant $\gamma_1 < 1$, such that $\mathsf{E}\,[\exp(-a_2 D_1)] \leq \gamma_1$. The assumption $G_i(0) < 1$ for $1 \leq i \leq c$ ensures that such a γ_1 exists. For $x = (q_1, ..., q_c, y_1, ..., y_c) \in \mathcal{S}$, define

$$g(x) = \max_{1 \leq i \leq c} y_i \tag{4.1}$$

and

$$V(x) = e^{a_2 g(x)}. \tag{4.2}$$

Observe that $V(x) \leq e^{a_2 b_0}$ for $x \in B(b_0)$ and $V(x) > e^{a_2 b_0}$ for $x \notin B(b_0)$. Define the event $A(b_0) = \{b_0 < D_1 + D_2\}$. By (A2), one has

$$\mathsf{E}_x\,[\exp(a_2 D_2) I[A(b_0)]] \;\leq\; \sum_{i=1}^{c} \prod_{j=1}^{c} \mathsf{E}\,[\exp(a_2 S_j(i)) I[A(b_0)]] \xrightarrow[b_0 \to \infty]{} 0.$$

Choose b_0 large enough so that $b \triangleq e^{a_2 b_0}(\gamma_2 - \gamma_1) - 1 > 0$ and so that

$$\mathsf{E}\,[\exp(a_2 D_2) I[A(b_0)]] \;\leq\; 1. \tag{4.3}$$

Each residual time $Y_c(i)$ at step c is either one of the new service times generated so far, minus the time since the beginning of that service, or the initial y_i minus the time t_c^* of the cth jump of the chain. In the first case, one has $Y_c(i) \leq D_2$, while in the second case, $Y_c(i) = y_i - t_c^* \leq y_i - D_1$, because there is at least one node where two service completions have occured so far. Therefore, $\forall\, x \in \mathcal{S}$,

$$\begin{aligned} g(X_c) \;&\leq\; \max(D_2, g(x) - D_1) \\ &=\; (g(x) - D_1)\; I\{g(x) \geq D_1 + D_2\} +\; D_2\; I\{g(x) < D_1 + D_2\}. \end{aligned} \tag{4.4}$$

For any $x \notin B(b_0)$, by (4.4) it follows that

$$\begin{aligned} \mathsf{E}_x[V(X_c)] \;&=\; \mathsf{E}_x[\exp(a_2 g(X_c))] \\ &\leq\; \mathsf{E}_x[\exp(a_2(g(x) - D_1))\; I\{g(x) \geq D_1 + D_2\}] \\ &\quad +\, \mathsf{E}_x[\exp(a_2 D_2)\; I\{g(x) < D_1 + D_2\}] \\ &\leq\; V(x)\, \mathsf{E}_x[\exp(-a_2 D_1)] + \mathsf{E}_x[\exp(a_2 D_2) I[A(b_0)]] \\ &\leq\; \gamma_1 V(x) + 1 \\ &\leq\; \gamma_2 V(x) - (\gamma_2 - \gamma_1) e^{a_2 b_0} + 1 \\ &=\; \gamma_2 V(x) - b. \end{aligned}$$

This verifies (2.8). By condition (A2), (2.9) follows easily from (4.4).

□

Proof of Theorem 3.2:

We distinguish between the cases $p = 1$ and $p > 1$.

(i). For $p = 1$, we show that the conditions of Proposition 2.2 are satisfied with $V(x) = g(x)$ defined in (4.1), $m = c$, $A^* = B(b_0)$ for some b_0, and $b = \mathsf{E}[D_1]/2$. Condition (2.7) follows easily from (4.4). We now check (2.6). Similar to (4.3), we can choose b_0 large enough so that

$$\mathsf{E}\left[(D_1 + D_2)I[A(b_0)]\right] \leq b. \tag{4.5}$$

Then, for all $x \notin B(b_0)$, by (4.5) it follows from (4.4) that

$$\begin{aligned} \mathsf{E}_x[g(X_c)] &\leq g(x) - \mathsf{E}\left[D_1\right] + \mathsf{E}\left[D_1 I[A(b_0)]\right] + \mathsf{E}\left[D_2 I[A(b_0)]\right] \\ &\leq g(x) - b. \end{aligned} \tag{4.6}$$

(ii). For $p > 1$, from (4.4) we have

$$g(X_c) - g(x) \leq \Delta(x) \triangleq -D_1 + (D_1 + D_2)I\{g(x) < D_1 + D_2\}.$$

Under our current assumptions, it is easy to see that

$$\sup_{x \in \mathcal{S}_0} \mathsf{E}_x[|\Delta(x)|^p] < \infty.$$

The desired result follows from Proposition 2.4.

□

5 Proofs of Stochastic Stability for Open Queueing Networks

We now turn to the Markov chain $\mathcal{X}$, where X_n is the state of the open queueing network at time t_n^-, i.e., immediately before the nth arrival epoch t_n. At time t_n^-, we know that an arrival is coming, but we don't know at which node.

Let $T_{s,t}(i)$ denote the amount of time that node i is busy during the time interval $(s,\, t]$, $1 \leq i \leq c$. Let $W_t(i)$ denote the total service time that node i will use to serve all of the customers that are in the network (in any queue or at any node) at time t. Define

$$W_t = \sum_{i=1}^{c} W_t(i),$$

the total work in the network at time t. Let $\Omega^l_{s,t}(i)$ denote the work to be completed by node i for those customers that join node l after arriving from the exogenous source in the interval $(s,\, t]$, i.e.,

$$\Omega^l_{s,t}(i) \triangleq \sum_{j=N_s(l)+1}^{N_t(l)} \sum_{k=1}^{M^l_j(i)} S^l_{j,k}(i),$$

where $N_0(l) = Q_0(l)$ and $N_s(l)$ is the sum of $Q_0(l)$ and the number of external arrivals to node l during $(0,\, s]$ for $s > 0$; $M^l_j(i)$ is the number of times that the jth customer, among those arriving to node l from outside or already at node l at the beginning, visits node i; and $S^l_{j,k}(i)$ is the service time of this customer at its kth visit to node i. Then, $\Omega_{s,t}(i) \triangleq \sum_{l=1}^{c} \Omega^l_{s,t}(i)$ is the total work to be completed by node i, arriving from outside during the time interval $(s, t]$, and

$$v_n \triangleq W_{t_n} - W_0 = \sum_{i=1}^{c} (\Omega_{0,t_n}(i) - T_{0,t_n}(i)) \tag{5.1}$$

is the net change in the total work in the network from time 0 to time t_n. For $n \geq 1$, we have

$$\mathsf{E}[\Omega_{0,t_n}(i)|\mathcal{F}_0] \quad = \quad n\rho_i/\lambda, \qquad 1 \leq i \leq c. \tag{5.2}$$

We are now in a position to give the following geometric stability results, from which Theorem 3.3 follows.

Theorem 5.1 *Under conditions (A2) and (A4), one has*

(i) (2.4) holds with $A^* = B(k_0, b_0)$ for sufficiently large k_0 and b_0;

(ii) the queueing network is geometrically stable;

(iii) if $X_0 = 0$, then $\lim_{n\to\infty} \mathsf{E}[v_n]/n = 0$.

Proof. We prove (i)–(iii) by induction on c. It is known that (i) and (ii) are equivalent. By (ii) and the regenerative structure of the Markov chain $\mathcal{X}$ we can prove (iii), i.e., that in the long run, the network processes work at the same rate as it arrives. The proof that (ii) implies (iii) is given in Appendix C. The proof of (i) is relatively easy in the case of $c = 1$, while in the case of arbitrary $c > 1$, the proof needs the result (iii) in the last induction.

Define

$$g(s, X_n) \triangleq \mathsf{E}[\exp(sW_{t_n})|X_n] = \mathsf{E}[\exp(s(W_0 + v_n))|\mathcal{F}_n], \quad n \geq 1, \tag{5.3}$$

where the last equality holds a.s. because the total workload in the system at time t_n has the same distribution conditional on X_n as conditional on $\mathcal{F}_n$. By (5.3), one has

$$\mathsf{E}_x[g(s, X_m)] \quad = \quad \mathsf{E}_x[\exp(sW_0 + sv_m)] \quad = \quad \mathsf{E}_x\left[\exp(sW_0)\mathsf{E}[\exp(sv_m)|\mathcal{B}_0]\right], \tag{5.4}$$

where

$$\mathcal{B}_0 = \sigma\{Y_0(i),\ S^l_{j,k}(i),\ \ 1 \leq k \leq M^l_j(i),\ 1 \leq j \leq Q_0(l),\ 1 \leq l \leq c,\ 1 \leq i \leq c\}$$

represents the information on the residual times and the future service times of all the customers already in the network at time $t_0 = 0$. To prove (i), we show that the conditions of Proposition 2.3 hold with $A^* = B(k_0, b_0)$ for some constants k_0 and b_0, and $V(\cdot) = g(s, \cdot)$ for some constant $s > 0$.

We first check (2.9). For any $x \in B(k_0, b_0)$ and any fixed m, from (5.1) and (5.4) we derive

$$\mathsf{E}_x[g(s, X_m)] \quad \leq \quad \mathsf{E}_x\left[\exp\left(sW_0 + s\sum_{i=1}^{c} \Omega_{0,t_m}(i)\right)\right]$$

$$
\begin{aligned}
&\leq \ \mathsf{E}_x\left[\exp\left(scb_0 + s\sum_{i=1}^{c}\sum_{l=1}^{c}\sum_{j=1}^{k_0+m}\sum_{k=1}^{M_j^l(i)} S_{j,\,k}^l(i)\right)\right] \\
&\leq \ e^{scb_0}\prod_{i=1}^{c}\prod_{l=1}^{c}\left(\mathsf{E}\left[\exp\left(s\sum_{k=1}^{M_j^l(i)} S_{j,\,k}^l(i)\right)\right]\right)^{k_0+m},
\end{aligned}
$$

which gives (2.9) by Lemma B.3 (see Appendix B).

For the moment, we prove (2.8) under the following condition (A6) on the service times. The proof of Theorem 5.1 without Condition (A6) is given in Appendix D.

(A6). There exists a small constant H_0 such that

$$P\{S_1(i) \geq H_0\} = 1, \quad \forall\ 1 \leq i \leq c.$$

Let $X_0 = x = (q_1, q_2, ..., q_c;\ y_1, y_2, ..., y_c)$. Denote by A_0 the event $\{t_m < \sum_{i=1}^{k_0} S_i(1)\}$ and by A_0^c the complementary event. If $x \notin B(k_0, b_0)$, then either $\max_{1\leq i\leq c} Q_0(i) > k_0$ or $\max_{1\leq i\leq c} Y_0(i) > b_0$. We prove (2.8) by induction on c, beginning with the case where $c = 1$.

a). Suppose $Q_0(1) > k_0$. When A_0 occurs, node 1 is busy during the time interval $[0, t_m]$. So we have

$$
\begin{aligned}
\mathsf{E}[v_m \mid \mathcal{B}_0] &= m\rho_1/\lambda - \mathsf{E}\left[T_{0,t_m}(1)I[A_0] + T_{0,t_m}(1)I[A_0^c] \mid \mathcal{B}_0\right] \\
&\leq m\rho_1/\lambda - \mathsf{E}[t_m \mid \mathcal{B}_0] + \mathsf{E}\left[t_m I[A_0^c] \mid \mathcal{B}_0\right] \\
&= -m(1-\rho_1)/\lambda + \mathsf{E}\left[t_m I[A_0^c] \mid \mathcal{B}_0\right] \\
&\leq -3 + \mathsf{E}\left[t_m I[A_0^c] \mid \mathcal{B}_0\right], \qquad (5.5)
\end{aligned}
$$

if we take $m \geq 3\lambda/(1-\rho_1)$. Now, choose a_0 large enough such that

$$H(x) \stackrel{\triangle}{=} \mathsf{E}[t_m I\{t_m \geq x\}] \leq 1, \quad \forall\ x \geq a_0,$$

and let $k_0 \geq 1 + a_0/H_0$. It follows from (A6) that

$$
\begin{aligned}
\mathsf{E}\left[t_m\, I[A_0^c] \mid \mathcal{B}_0\right] &= \mathsf{E}\left[t_m\, I[A_0^c] \,\middle|\, \sum_{i=1}^{k_0} S_i(1)\right] \\
&= H\left(\sum_{i=1}^{k_0} S_i(1)\right) \leq H(a_0) \leq 1,
\end{aligned}
$$

which, together with (5.5), yields

$$\mathsf{E}[v_m \mid \mathcal{B}_0] \leq -2. \qquad (5.6)$$

Note that

$$
\begin{aligned}
& \mathsf{E}[\exp(sv_m)|\mathcal{B}_0] \\
= \;& 1 + s\mathsf{E}[v_m|\mathcal{B}_0] + \frac{s^2}{2}\mathsf{E}[v_m^2 \exp(\xi v_m)|\mathcal{B}_0] \\
\leq \;& 1 + s\mathsf{E}[v_m|\mathcal{B}_0] + \frac{s^2}{2}\mathsf{E}\left[\left(\sum_{i=1}^{m}\Omega_{0,\,t_m}(i) + ct_m\right)^2 \exp\left(s\sum_{i=1}^{m}\Omega_{0,\,t_m}(i)\right)\right] \qquad (5.7)
\end{aligned}
$$

for some $\xi \in (0,\, s)$. By condition (A2) and Lemma B.4 (see Appendix B), we have

$$
\mathsf{E}\left[\left(\sum_{i=1}^{m}\Omega_{0,\,t_m}(i) + ct_m\right)^2 \exp\left(s\sum_{i=1}^{m}\Omega_{0,\,t_m}(i)\right)\right] < \infty, \quad \forall\, s \in [0,\, \overline{s}_1],
$$

for some constant $\overline{s}_1 > 0$. Combining (5.6) and (5.7) leads to

$$
\begin{aligned}
\mathsf{E}[\exp(sv_m)|\mathcal{B}_0] &\leq 1 - 2s + \frac{s^2}{2}\mathsf{E}\left[\left(\sum_{i=1}^{c}\Omega_{0,\,t_m}(i) + ct_m\right)^2 \exp\left(s\sum_{i=1}^{c}\Omega_{0,\,t_m}(i)\right)\right] \\
&\leq 1 - s, \qquad (5.8)
\end{aligned}
$$

if s is small enough, which we assume henceforth. Then, by (5.4) and (5.8), since $Q_0(1) > k_0$, we have

$$
\begin{aligned}
\mathsf{E}_x[g(s, X_m)] &\leq (1-s)g(s,x) \\
&= (1-s/2)g(s,x) - (s/2)g(s,x) \\
&\leq (1-s/2)g(s,x) - \frac{s}{2}\mathsf{E}\left[\exp\left(s\sum_{i=1}^{k_0}S(i)\right)\right]. \qquad (5.9)
\end{aligned}
$$

b). Suppose $Y_0(1) > b_0$. We choose the same m as in step a). Similar to (5.5), we have

$$
\begin{aligned}
\mathsf{E}[v_m|\mathcal{B}_0] &= \frac{m}{\lambda}\rho_1 - \mathsf{E}[t_m I\{t_m < b_0\}] - \mathsf{E}[T_{0,\,t_m}(1) I\{t_m \geq b_0\}|\mathcal{B}_0] \\
&\leq -\frac{m}{\lambda}(1-\rho_1) + \mathsf{E}[t_m I\{t_m \geq b_0\}] \\
&\leq -3 + \mathsf{E}[t_m I\{t_m \geq b_0\}] \\
&\leq -2 \qquad (5.10)
\end{aligned}
$$

if we choose b_0 large enough so that $\mathsf{E}[t_m I\{t_m \geq b_0\}] \leq 1$. Analogous to (5.7)–(5.9), we can choose the same constant $s > 0$ as in (5.8) such that

$$
\mathsf{E}_x[g(s, X_m)] \;\leq\; (1-s)g(s,x) \;\leq\; (1-s/2)g(s,x) - (s/2)\mathsf{E}[\exp(sb_0)], \qquad (5.11)
$$

via the assumption $Y_0(1) > b_0$.

Let $\beta = 1 - s/2$ and $b = (s/2)\min\{e^{sb_0},\ \mathsf{E}[\exp(s\min_{1\leq i\leq c}\sum_{j=1}^{k_0}S_j(i))]\}$. Then (2.8) follows from (5.9) and (5.11).

We now proceed with the induction step: we assume that the results of Theorem 5.1 (i)-(iii) hold for networks with less than c nodes and show that they hold for networks with c nodes. Define $g(s,x)$ as in (5.3) and let $x \notin B(k_0,\, b_0)$. Again, we consider two cases.

a). Suppose $\max_{1\le i\le c} Q_0(i) > k_0$. Without loss of generality we assume that $Q_0(1) > k_0$. Note that if the event A_0 occurs, node 1 is busy during the entire time interval $[0,\, t_m]$. So we have

$$\begin{aligned}
\mathsf{E}[v_m|\mathcal{B}_0] &= \mathsf{E}\left[\sum_{i=1}^{c}(\Omega_{0,\,t_m}(i) - T_{0,\,t_m}(i))\,\middle|\, \mathcal{B}_0\right] \\
&= \mathsf{E}[\Omega_{0,\,t_m}(1)] - \mathsf{E}[t_m I[A_0]|\mathcal{B}_0] \\
&\qquad - \mathsf{E}[T_{0,\,t_m}(1) I[A_0^c]|\mathcal{B}_0] + \mathsf{E}\left[\sum_{i=2}^{c}(\Omega_{0,\,t_m}(i) - T_{0,\,t_m}(i))\,\middle|\, \mathcal{B}_0\right] \\
&\le m\rho_1/\lambda - \mathsf{E}[t_m] + \mathsf{E}[t_m I[A_0^c]|\mathcal{B}_0] + \mathsf{E}\left[\sum_{i=2}^{c}\Omega_{0,\,t_m}(i)\,\middle|\, \mathcal{B}_0\right] \\
&\qquad - \mathsf{E}\left[\sum_{i=2}^{c} T_{0,\,t_m}(i) I[A_0]\,\middle|\, \mathcal{B}_0\right] - \mathsf{E}\left[\sum_{i=2}^{c} T_{0,\,t_m}(i) I[A_0^c]\,\middle|\, \mathcal{B}_0\right] \\
&\le -m(1-\rho_1)/\lambda + \mathsf{E}[t_m I[A_0^c]|\mathcal{B}_0] + \sum_{i=2}^{c}\frac{m\rho_i}{\lambda} - \mathsf{E}\left[\sum_{i=2}^{c} T_{0,\,t_m}(i) I[A_0]\,\middle|\, \mathcal{B}_0\right]. \qquad (5.12)
\end{aligned}$$

As in Meyn and Down [14], we introduce a slow system with $(c-1)$ nodes, labeled 2, 3, ..., c. Customers arrive at rate r_i from outside to node i. Customers also arrive from an independent source at rate λ_1, and each customer arriving from this source joins node i with probability π_{1i}, $2 \le i \le c$, independently of the others. The initial workload in the slow system is $W_0^{\rm slow} = 0$. The traffic equations for this slow system are

$$\lambda_i' = r_i + \lambda_1 \pi_{1i} + \sum_{j=2}^{c} \lambda_j' \pi_{ji}, \quad 2 \le i \le c.$$

It is easy to see that $\lambda_i = \lambda_i'$ for $2 \le i \le c$.

We now reduce the output of node 1 in the original system to obtain a slow system as just described. Each time a customer leaves node 1, an independent Bernoulli random variable η is drawn, with $P[\eta = 1] = \rho_1$. If $\eta = 1$, this customer is routed to node i with probability π_{1i}, $2 \le i \le c$, otherwise this customer is removed from the system. Let $\Omega_{0,t}^{\rm slow}(i)$ and $T_{0,t}^{\rm slow}(i)$ denote the work entering node i and the total busy time at node i in the slow system during the time interval $[0,\, t]$, respectively. Then it is easy to verify that there is a constant $L_0^* > 0$ such that

$$\mathsf{E}\left[\Omega_{0,\,t_m}^{\rm slow}(i)\right] \in \left[\frac{m}{\lambda}\rho_i - L_0^*,\, \frac{m}{\lambda}\rho_i + L_0^*\right]. \qquad (5.13)$$

It is shown in Meyn and Down [14] that $T_{0,\,t}(i) \ge T_{0,t}^{\rm slow}(i)$ a.s. for $t \ge 0$ if node 1 never empties. We now continue (5.12) as follows.

$$\mathsf{E}[v_m|\mathcal{B}_0] \;\le\; -\frac{m}{\lambda}(1-\rho_1) + \mathsf{E}[t_m I[A_0^c]|\mathcal{B}_0] + \mathsf{E}\left[\sum_{i=2}^{c}(\Omega_{0,\,t_m}^{\rm slow}(i) + L_0^*)\,\middle|\, \mathcal{B}_0\right]$$

$$
\begin{aligned}
& - \mathsf{E}\left[\sum_{i=2}^{c} T^{\text{slow}}_{0,t_m}(i) I[A_0] \,\middle|\, \mathcal{B}_0 \right] \\
= \;& m\left(-\frac{1}{\lambda}(1-\rho_1) + \frac{c}{m}L_0^* + \frac{1}{m}\mathsf{E}\left[\sum_{i=2}^{c} (\Omega^{\text{slow}}_{0,t_m}(i) - T^{\text{slow}}_{0,t_m}(i)) \right] \right) \\
& + \mathsf{E}\left[\sum_{i=2}^{c} T^{\text{slow}}_{0,t_m}(i) I[A_0^c] \,\middle|\, \mathcal{B}_0 \right] \; + \; \mathsf{E}\left[t_m I[A_0^c] | \mathcal{B}_0 \right]. \qquad (5.14)
\end{aligned}
$$

For the slow system, we have from the induction hypothesis that

$$
\frac{1}{m}\mathsf{E}\left[\sum_{i=2}^{c} (\Omega^{\text{slow}}_{0,t_m}(i) - T^{\text{slow}}_{0,t_m}(i)) \right] \xrightarrow[m \to \infty]{} 0.
$$

Thus, from (5.14), we can choose m sufficiently large such that

$$
\mathsf{E}[v_m|\mathcal{B}_0] \leq -3 + c\,\mathsf{E}[t_m I[A_0^c]|\mathcal{B}_0]. \qquad (5.15)
$$

As in the case of $c = 1$, this, with condition (A6), implies that $\mathsf{E}[v_m|\mathcal{B}_0] \leq -2$ for some k_0. Analogous to (5.7)–(5.9), we have

$$
\mathsf{E}_x[g(s, X_m)] \; \leq \; (1 - \frac{s}{2})g(s,x) - \frac{s}{2}\mathsf{E}\left[\exp\left(s\sum_{i=1}^{k_0} S_i(1) \right) \right], \qquad (5.16)
$$

for some $s > 0$.

b). Now, suppose that $\max_{1 \leq i \leq c} Y_0(i) > b_0$. Without loss of generality, assume $Y_0(1) > b_0$. Similar to (5.12), we have

$$
\mathsf{E}[v_m|\mathcal{B}_0] \leq -\frac{m}{\lambda}(1-\rho_1) + \mathsf{E}[t_m I\{b_0 \leq t_m\}] + \sum_{i=2}^{c}\frac{m}{\lambda}\rho_i - \mathsf{E}\left[\sum_{i=2}^{c} T_{0,t_m}(i) I\{b_0 > t_m\} \,\middle|\, \mathcal{B}_0 \right]. \qquad (5.17)
$$

Introduce a slow system as in a). Similar to (5.14) and (5.15), we obtain

$$
\begin{aligned}
\mathsf{E}[v_m|\mathcal{B}_0] \; & \leq \; m\left(-\frac{1}{\lambda}(1-\rho_1) + \frac{c}{m}L_0^* + \frac{1}{m}\mathsf{E}\left[\sum_{i=2}^{c} (\Omega^{\text{slow}}_{0,t_m}(i) - T^{\text{slow}}_{0,t_m}(i)) \right] \right) \\
& \quad + c\,\mathsf{E}[t_m I\{b_0 \leq t_m\}|\mathcal{B}_0] \\
& \leq \; -3 + c\,\mathsf{E}[t_m I\{t_m > b_0\}] \; \leq \; -2, \qquad (5.18)
\end{aligned}
$$

if we choose b_0 sufficiently large and the same m as in a).

As in the case of $c = 1$, we can choose $s > 0$ such that

$$
\mathsf{E}_x[g(s, X_m)] \; \leq \; (1-s)g(s,x) \; \leq \; (1 - \frac{s}{2})g(s,x) - \frac{s}{2}\mathsf{E}[\exp(sb_0)]. \qquad (5.19)
$$

Let $\beta = 1 - s/2$ and $b = (s/2)\min\{e^{sb_0},\ \mathsf{E}[\exp(s\min_{1\leq i\leq c}\sum_{j=1}^{k_0} S_j(i))]\}$. Then by (5.16) and (5.19) we obtain (2.8).

The proof of Theorem 5.1 with Condition (A6) is thus completed. The proof of Theorem 5.1 without this condition is provided in Appendix E.

$\square$

Based on the analysis in the proof of Theorem 5.1, we easily arrive at the following L_p−stability result for the open queueing networks, from which Theorem 3.4 follows.

Theorem 5.2 *Suppose that Conditions (A3) and (A5) are satisfied. Then*

(i) (2.5) holds with $A^* = B(k_0, b_0)$ *for sufficiently large* k_0 *and* b_0*;*

(ii) the queueing network is L_p*-stable;*

(iii) if $X_0 = 0$*, then* $\lim_{n\to\infty} \mathsf{E}[v_n]/n = 0$.

Proof. We use induction on c. As in the proof of Theorem 5.1, the key step is to verify (i). Define

$$V(X_n) \stackrel{\triangle}{=} \mathsf{E}[W_{t_n}|X_n] = \mathsf{E}[W_0 + v_n|\mathcal{F}_n], \qquad n \geq 0, \tag{5.20}$$

where the last equality holds a.s.. Then we have

$$\mathsf{E}_x[V(X_m)] = V(x) + \mathsf{E}_x[v_m]. \tag{5.21}$$

We distinguish between the cases $p = 1$ and $p > 1$.

(i). Let $p = 1$. We will show that the conditions of Proposition 2.2 hold with $A^* = B(k_0, b_0)$ for some constants k_0 and b_0 and V as just defined. This implies Theorem 5.2 (i). The verification of (2.7) is trivial. To check (2.6), we use the same argument as in the proof of Theorem 5.1. To help the reader, we sketch the proof for $c = 1$. If $x \notin B(k_0, b_0)$, then we have either $Q_0(1) > k_0$ or $Y_0(1) > b_0$.

a). Let $Q_0(1) > k_0$. Similar to (5.5), for $m \geq 3\lambda/(1-\rho_1)$, we have

$$\mathsf{E}_x[v_m] \leq -3 + \mathsf{E}\left[t_m I\{t_m \geq \sum_{i=1}^{k_0} S_i(1)\}\right]. \tag{5.22}$$

Let b_2 and k_0 be large enough such that $\mathsf{E}[t_m I\{t_m \geq b_2\}] \leq 1/2$ and $\mathsf{E}[t_m] P\{\sum_{i=1}^{k_0} S_i(1) < b_2\} \leq 1/2$. Then, (5.22) gives

$$\mathsf{E}_x[v_m] \;\leq\; -3 + \mathsf{E}[t_m I\{t_m \geq b_2\}] + \mathsf{E}[t_m] P\left\{\sum_{i=1}^{k_0} S_i(1) < b_2\right\} \;\leq\; -2, \tag{5.23}$$

which, together with (5.21), yields (2.6) with $b = 2$.

b). Suppose that $Y_0(1) > b_0$. As in (5.10) we can choose m and b_0 such that

$$\mathsf{E}_x[v_m] \leq -2. \tag{5.24}$$

From (5.23) and (5.24), it follows that (2.6) holds in both cases with $b = 2$.

For arbitrary $c > 1$, to obtain (5.13), our argument assumes that the service times have a bounded second moment.

(ii). For the case $p > 1$, define $\Delta(x) = V(X_m) - V(x)$, where m is chosen as in (i). It suffices to verify the four conditions of Proposition 2.4 and Theorem 5.2 (i) will follow. The first two conditions of that proposition are trivial to verify and we have proved the third one in (i). To check the fourth condition, observe that by (5.1), for $x \in \mathcal{S}$, we have

$$\begin{aligned}\mathsf{E}_x[|\Delta(x)|^p] &= \mathsf{E}_x[|\mathsf{E}[W_{t_m} - W_0|\mathcal{F}_m]|^p] \\ &\leq \mathsf{E}_x[|v_m|^p] \leq \mathsf{E}\left[\left(\sum_{i=1}^{c} \Omega_{0,\,t_m}(i)\right)^p\right] + c\mathsf{E}[t_m^p] < \infty.\end{aligned}$$

□

Acknowledgement

We are grateful to L. Guo and his students for providing us the counterexample in Appendix E.

Appendix A

Proof of Proposition 2.5:

By our regenerative scheme, $X_{R_i - m^*} \in A^*$ for $i \geq 2$. We only prove (2.12) and (2.14) for $i = 1$. The proof for arbitrary i works the same way.

(i). By (2.11) and the geometric stability of $\mathcal{X}$, there are positive constants a_1^*, b_1^*, and b_2^* such that for all $s \in [0,\, a_1^*]$,

$$\begin{cases} \sup\limits_{x \in A^*} \mathsf{E}_x[\exp(s\sigma_1)] \leq b_1^*, \\ \sup\limits_{n \geq 1} \mathsf{E}[\exp(s|J_n|)|\mathcal{F}_{n-1}] \leq b_2^*. \end{cases} \tag{A.1}$$

One has

$$\begin{aligned}\mathsf{E}_x\left[\exp(s\sum_{i=1}^{\sigma_1}|J_i|)\right] &= \mathsf{E}_x\left[\sum_{k=1}^{\infty}\exp(s\sum_{i=1}^{k}|J_i|)I\{\sigma_1 = k\}\right] \\ &= \mathsf{E}_x\left[\sum_{k=2}^{\infty}\left(\exp(s\sum_{i=1}^{k}|J_i|) - \exp(s\sum_{i=1}^{k-1}|J_i|)\right)I\{\sigma_1 \geq k\}\right] + \mathsf{E}_x\left[\exp(s|J_1|)\right] \\ &\leq \mathsf{E}_x\left[\sum_{k=2}^{\infty}\exp(s\sum_{i=1}^{k}|J_i|)I\{\sigma_1 \geq k\}\right] + \mathsf{E}_x\left[\exp(s|J_1|)\right].\end{aligned}$$

By (A.1) and Schwarz's inequality, it follows that for all $s \in [0,\, a_1^*]$,

$$\begin{aligned}\mathsf{E}_x\left[\exp(s|\sum_{i=1}^{\sigma_1}|J_i|)\right] &\leq b_2^* + \sum_{k=2}^{\infty}\sqrt{P_x\{\sigma_1 \geq k\}}\sqrt{\mathsf{E}_x\left[\exp(2s\sum_{i=1}^{k}|J_i|)\right]} \\ &\leq b_2^* + \sqrt{b_1^*}\sum_{k=2}^{\infty}\exp(-\frac{a_1^*}{2}k)\sqrt{\mathsf{E}_x\left[\exp(2s\sum_{i=1}^{k}|J_i|)\right]}.\end{aligned} \tag{A.2}$$

Notice that

$$\begin{aligned}
\mathsf{E}_x\left[\exp(2s\sum_{i=1}^{k}|J_i|)\right] &= \mathsf{E}_x\left[\exp(2s\sum_{i=1}^{k-1}|J_i|)\mathsf{E}[\exp(2s|J_k|)|\mathcal{F}_{k-1}]\right] \\
&\le (\sup_{n\ge1}\mathsf{E}[\exp(2s|J_n|)|\mathcal{F}_{n-1}])\mathsf{E}_x\left[\exp(2s\sum_{i=1}^{k-1}|J_i|)\right] \\
&\le \left(\sup_{n\ge1}\mathsf{E}[\exp(2s|J_n|)|\mathcal{F}_{n-1}]\right)^k. \qquad \text{(A.3)}
\end{aligned}$$

Choosing $s_1^* \in (0,\, a_1^*/2)$, for any $s \in [0,\, s_1^*]$ and $b^* > 0$, using (A.1) we derive

$$\begin{aligned}
& \sup_{n\ge1}\mathsf{E}[\exp(2s|J_n|)|\mathcal{F}_{n-1}] \\
\le\ & \sup_{n\ge1}(\exp(2sb^*) + \mathsf{E}[\exp(2s|J_n|)I\{|J_n| > b^*\}|\mathcal{F}_{n-1}]) \\
\le\ & \exp(2sb^*) + \sup_{n\ge1}\mathsf{E}[\exp(a_1^*|J_n|)I\{|J_n| > b^*\}|\mathcal{F}_{n-1}]\exp(-(a_1^* - 2s)b^*) \\
\le\ & \exp(2sb^*) + b_2^*\exp(-(a_1^* - 2s_1^*)b^*). \qquad \text{(A.4)}
\end{aligned}$$

Let ε_0, $a_2^*(< s_1^*)$ be small enough and b^* be large enough such that

$$\begin{cases} \rho_0 \triangleq \sqrt{1+2\varepsilon_0}\exp(-a_1^*/2) < 1, \\ b_2^*\exp(-(a_1^* - 2s_1^*)b^*) \le \varepsilon_0, \quad e^{2b^*s} \le 1+\varepsilon_0, \quad \forall\ s \in [0,\, a_2^*]. \end{cases} \qquad \text{(A.5)}$$

From (A.2)–(A.5), for all $s \in [0,\ a_2^*]$ we have

$$\begin{aligned}
\sup_{x\in A^*}\mathsf{E}_x\left[\exp(s\sum_{i=1}^{\sigma_1}|J_i|)\right] &\le b_2^* + \sqrt{b_1^*}\sum_{k=2}^{\infty}\exp(-a_1^*k/2)\cdot(1+2\varepsilon_0)^{k/2} \\
&\le b_2^* + \sqrt{b_1^*}\sum_{k=2}^{\infty}\rho_0^k \\
&\le b_2^* + \sqrt{b_1^*}\rho_0^2/(1-\rho_0).
\end{aligned}$$

(ii). Since $\mathcal{X}$ is L_p-stable, it is easy to see that

$$\sup_{x\in A^*}\mathsf{E}_x[\sigma_1^p] < \infty. \qquad \text{(A.6)}$$

We will prove that there exists a constant c_p^* such that

$$\sup_{x\in A^*}\mathsf{E}_x\left[(\sum_{i=1}^{\sigma_1}|J_i|)^p\right] \le c_p^*(\sup_{x\in A^*}\mathsf{E}_x[(\sigma_1)^p])(\sup_{n\ge1}\mathsf{E}[|J_n|^p|\mathcal{F}_{n-1}]). \qquad \text{(A.7)}$$

By the C_r-inequality, we have, for all $x \in A^*$,

$$\mathsf{E}_x[(\sum_{i=1}^{\sigma_1}|J_i|)^p]$$

$$
\begin{aligned}
&= \mathsf{E}_x\left[\left(\sum_{i=1}^{\sigma_1}(|J_i| - \mathsf{E}[|J_i|\,|\mathcal{F}_{i-1}]) + \sum_{i=1}^{\sigma_1}\mathsf{E}[|J_i|\,|\mathcal{F}_{i-1}]\right)^p\right] \\
&\le 2^{p-1}\mathsf{E}_x\left[\left|\sum_{i=1}^{\sigma_1}(|J_i| - \mathsf{E}[|J_i|\,|\mathcal{F}_{i-1}])\right|^p\right] + 2^{p-1}(\sup_{n\ge1}\mathsf{E}[|J_n|^p|\mathcal{F}_{n-1}])(\sup_{x\in A^*}\mathsf{E}_x[\sigma_1^p]). \qquad \text{(A.8)}
\end{aligned}
$$

We now estimate the first term on the right hand side of the inequality in (A.8). Let $\sigma_1^{(n)} = \min\{\sigma_1,\, n\}$. We distinguish between the two cases $1 \le p \le 2$ and $p > 2$.

1). Let $1 \le p \le 2$. By the Burkholder inequality (see, e.g., [8] and [10]) and the C_r-inequality, there exists a constant c_p such that for all $x \in A^*$,

$$
\begin{aligned}
\mathsf{E}_x\left[\left|\sum_{i=1}^{\sigma_1^{(n)}}(|J_i| - \mathsf{E}[|J_i|\,|\mathcal{F}_{i-1}])\right|^p\right] &\le c_p\mathsf{E}_x\left[\left|\sum_{i=1}^{\sigma_1^{(n)}}(|J_i| - \mathsf{E}[|J_i|\,|\mathcal{F}_{i-1}])^2\right|^{p/2}\right] \\
&\le c_p\mathsf{E}_x\left[\sum_{i=1}^{\sigma_1^{(n)}}|(|J_i| - \mathsf{E}[|J_i|\,|\mathcal{F}_{i-1}])|^p\right] \\
&\le c_p\mathsf{E}_x\left[\sum_{i=1}^{\sigma_1}|(|J_i| - \mathsf{E}[|J_i|\,|\mathcal{F}_{i-1}])|^p\right] \\
&= c_p\mathsf{E}_x\left[\sum_{i=1}^{\infty}\mathsf{E}[(|\,|J_i| - \mathsf{E}[|J_i|\,|\mathcal{F}_{i-1}])|^p|\mathcal{F}_{i-1}]I\{\sigma_1 \ge i\}\right] \\
&\le c_p\mathsf{E}_x\left[\sum_{i=1}^{\infty}(\mathsf{E}[|J_i|^p|\mathcal{F}_{i-1}] + (\mathsf{E}[|J_i||\mathcal{F}_{i-1}])^p)I\{\sigma_1 \ge i\}\right] \\
&\le 2c_p\left(\sup_{n\ge1}\mathsf{E}[|J_n|^p|\mathcal{F}_{n-1}]\right)\left(\sup_{x\in A^*}\mathsf{E}_x[\sigma_1]\right), \qquad \text{(A.9)}
\end{aligned}
$$

which gives, by Fatou's lemma,

$$
\sup_{x\in A^*}\mathsf{E}_x\left[\left(\sum_{i=1}^{\sigma_1}|J_i|\right)^p\right] \le 2c_p\left(\sup_{n\ge1}\mathsf{E}[|J_n|^p|\mathcal{F}_{n-1}]\right)\left(\sup_{x\in A^*}\mathsf{E}_x[\sigma_1]\right). \qquad \text{(A.10)}
$$

Combining (A.8) and (A.10) leads to (A.7) with $c_p^* = 2^pc_p + 2^{p-1}$, for $1 \le p \le 2$.

2). Let $p > 2$. By the Burkholder inequality, there exists a constant c_p such that

$$
\begin{aligned}
&\mathsf{E}_x\left[\left|\sum_{i=1}^{\sigma_1^{(n)}}(|J_i| - \mathsf{E}[|J_i|\,|\mathcal{F}_{i-1}])\right|^p\right] \\
&\le c_p\mathsf{E}_x\left[\left(\sum_{i=1}^{\sigma_1^{(n)}}\mathsf{E}[(|J_i| - \mathsf{E}[|J_i|\,|\mathcal{F}_{i-1}])^2|\mathcal{F}_{i-1}]\right)^{p/2}\right] + c_p\mathsf{E}_x\left[\sup_{1\le i\le\sigma_1^{(n)}}|(|J_i| - \mathsf{E}[|J_i|\,|\mathcal{F}_{i-1}])|^p\right] \\
&\le c_p\mathsf{E}_x[(\sigma_1^{(n)}\sup_{n\ge1}\mathsf{E}[|J_n|^2\,|\mathcal{F}_{n-1}])^{p/2}] + c_p\mathsf{E}_x\left[\sum_{i=1}^{\sigma_1}|(|J_i| - \mathsf{E}[|J_i|\,|\mathcal{F}_{i-1}])|^p\right]
\end{aligned}
$$

$$\leq \quad c_p \mathsf{E}_x[\sigma_1^{p/2}] \sup_{n\geq 1} \mathsf{E}[|J_n|^p \,|\mathcal{F}_{n-1}] + c_p \mathsf{E}_x \left[\sum_{i=1}^{\infty} \mathsf{E}[|J_i|^p + (\mathsf{E}[|J_i|\,|\mathcal{F}_{i-1}])^p |\mathcal{F}_{i-1}\} I\{\sigma_1 \geq i\} \right]$$

$$\leq \quad c_p \mathsf{E}_x[\sigma_1^{p/2}] \sup_{n\geq} \mathsf{E}[|J_n|^p \,|\mathcal{F}_{n-1}] + 2c_p \mathsf{E}_x[\sigma_1] \sup_{n\geq 1} \mathsf{E}[|J_n|^p \,|\mathcal{F}_{n-1}]$$

$$\leq \quad 3c_p(\sup_{x\in A^*} \mathsf{E}_x[\sigma_1^p])(\sup_{n\geq 1} \mathsf{E}[|J_n|^p \,|\mathcal{F}_{n-1}]),$$

which gives, by Fatou's lemma,

$$\mathsf{E}_x \left[\left| \sum_{i=1}^{\sigma_1} (|J_i| - \mathsf{E}[|J_i|\,|\mathcal{F}_{i-1}]) \right|^p \right] \;\leq\; 3c_p(\sup_{x\in A^*} \mathsf{E}_x[\sigma_1^p])(\sup_{n\geq 1} \mathsf{E}[|J_n|^p \,|\mathcal{F}_{n-1}]). \tag{A.11}$$

Then, (A.7) follows from (A.8) and (A.11) with $c_p^* = 2^{p-1}(3c_p + 1)$ for all $p > 2$.

□

Appendix B

In this appendix, we will study customers' routing in the open queueing network defined in Section 2. Let C^1 be a customer that arrives to the system from outside, first visits node I_0, then visits nodes $I_1, I_2, ...$ After visiting a finite number of nodes, this customer will leave the system, since we assume that 0 is the only absorbing state of the transition matrix $\Pi = (\pi_{ij},\ 0 \leq i, j \leq c)$. If C^1 is at node l, then there exists a sequence of nodes $i_0^{(l)},\ i_1^{(l)},\ ..., i_{n_l}^{(l)}$ such that

$$i_0^{(l)} = l,\ i_{n_l}^{(l)} = 0,\ \prod_{j=1}^{n_l} \pi_{i_{j-1}^{(l)},\ i_j^{(l)}} > 0, \quad 1 \leq l \leq c.$$

Set

$$n^* = \max_{1\leq l\leq c} n_l \quad \text{ and } \quad \delta_* = \min_{1\leq l\leq c} \prod_{j=1}^{n_l} \pi_{i_{j-1}^{(l)},\ i_j^{(l)}}. \tag{B.1}$$

Let $I_0 = l$ and define

$$N_1^l = \inf\{k:\ I_k = 0\}, \tag{B.2}$$

the number of nodes that customer C^1 visits before departing from the system. By (B.1), we have

$$P\{N_1^l \leq k + n^* | I_k = j\} \geq \delta_* > 0, \quad \text{for } \ 1 \leq j \leq c. \tag{B.3}$$

Lemma B.1 $\;N_1^l,\ 1 \leq l \leq c$ *are exponentially stable.*

Proof. By (B.3) we have, for $0 \leq r \leq \lfloor k/n^* \rfloor$,

$$\begin{aligned} P\{N_1^l > k\} &= P\{I_1 \neq 0,\ ...,\ I_k \neq 0\} \\ &= P\{I_1 \neq 0,\ ...,\ I_{k-n^*} \neq 0\} P\{I_k \neq 0,\ ...,\ I_{k-n^*+1} \neq 0 | I_1 \neq 0,\ ...,\ I_{k-n^*} \neq 0\} \\ &= P\{I_1 \neq 0,\ ...,\ I_{k-n^*} \neq 0\} P\{I_k \neq 0,\ ...,\ I_{k-n^*+1} \neq 0 | I_{k-n^*} \neq 0\} \\ &\leq (1-\delta_*) P\{I_1 \neq 0,\ ...,\ I_{k-n^*} \neq 0\} \\ &\leq (1-\delta_*)^r P\{I_1 \neq 0,\ ...,\ I_{k-rn^*} \neq 0\}, \end{aligned}$$

Thus, for $k \geq 1$

$$P\{N_1^l > k\} \leq (1-\delta_*)^{k/n^*-1}. \tag{B.4}$$

Choosing s_1^* small enough such that $e^{s_1^*}(1-\delta_*)^{1/n^*} < 1$, we then obtain that for all $s \in (0,\, s_1^*]$,

$$\begin{aligned}
\mathsf{E}[\exp(sN_1^l)] &= \sum_{k=1}^{\infty} e^{sk} P\{N_1^l = k\} \\
&= e^s + \sum_{k=1}^{\infty} P\{N_1^l > k\}(e^{s(k+1)} - e^{sk}) \\
&\leq e^s + (e^s - 1)\sum_{k=1}^{\infty} e^{sk}(1-\delta_*)^{k/n^*-1} \\
&< \infty.
\end{aligned}$$

□

Recall that $M_j^l(i)$ is the number of times that the jth customer, arriving to node l from outside or already at node l at time $t_0 = 0$, visits node i. Let $I_0 = l$ and assume that $M_1^l(i)$ is the number of times that C^1 visits node i.

Lemma B.2 *The $M_1^l(i)$, for $1 \leq i \leq c$ and $1 \leq l \leq c$, are exponentially stable.*

Proof. Noticing that $M_1^l(i) < N_1^l$ for $1 \leq i \leq c$ and $1 \leq l \leq c$, this follows from Lemma B.1.

□

Lemma B.3 *If (A2) holds, then for fixed $i,\ j$ and l, $\sum_{k=1}^{M_j^l(i)} S_{j,k}^l(i)$ is exponentially stable.*

Proof. Since $\{M_j^l(i),\ j \geq 1\}$ for fixed l and i are i.i.d. random variables, Lemma B.2 implies that $M_j^l(i)$ is exponentially stable, $\forall\ j \geq 1,\ 1 \leq i \leq c,\ 1 \leq l \leq c$. By condition (A2) and Proposition 2.5 (i), the result follows.

□

Lemma B.4 *For fixed m, $\sum_{i=1}^{c} \Omega_{0,t_m}(i)$ is exponentially stable.*

Proof. From the definition of $\Omega_{0,t_m}(i)$, we have

$$\begin{aligned}
\sum_{i=1}^{c} \Omega_{0,t_m}(i) &= \sum_{i=1}^{c}\sum_{l=1}^{c} \Omega_{0,t_m}^l(i) = \sum_{i=1}^{c}\sum_{l=1}^{c}\sum_{j=N_0(l)+1}^{N_{t_m}(l)}\sum_{k=1}^{M_j^l(i)} S_{j,\,k}^l(i) \\
&\leq \sum_{i=1}^{c}\sum_{l=1}^{c}\sum_{j=N_0(l)+1}^{N_0(l)+m}\sum_{k=1}^{M_j^l(i)} S_{j,\,k}^l(i),
\end{aligned}$$

which, incorporating with Lemma B.3, gives the desired result.

□

Appendix C

Proof of Theorem 5.1 (iii):

We prove that (ii) implies (iii) in Theorem 5.1. Recall that $\{X_n,\ n \geq 0\}$ is a regenerative process with regeneration points $\{R_n,\ n \geq 1\}$. Define

$$\nu(n) = \inf\{k: \quad R_k > n\}, \quad \alpha(n) = \nu(n) - 1, \quad \forall\, n \geq 1. \tag{C.1}$$

For an initially empty system, $W_0 = 0$. By (5.1) we derive

$$\begin{aligned}
\mathsf{E}[W_{t_n}] &= \mathsf{E}\left[\sum_{i=1}^{c}(\Omega_{0,\,t_n}(i) - T_{0,\,t_n}(i))\right] \\
&= \mathsf{E}\left[W_{t_{R_{\alpha(n)}}} + \sum_{i=1}^{c}(\Omega_{t_{R_{\alpha(n)}},\ t_n}(i) - T_{t_{R_{\alpha(n)}},\ t_n}(i))\right] \\
&\leq \mathsf{E}[W_{t_{R_{\alpha(n)}}}] + \mathsf{E}\left[\sum_{i=1}^{c}\Omega_{t_{R_{\alpha(n)}},\ t_{R_{\nu(n)}}}(i)\right],
\end{aligned}$$

which means that in order to prove Theorem 5.1 (iii), it suffices to prove that

$$\lim_{n\to\infty}\frac{1}{n}\mathsf{E}[W_{t_{R_{\alpha(n)}}}] = 0 \quad \text{and} \quad \lim_{n\to\infty}\frac{1}{n}\mathsf{E}\left[\sum_{i=1}^{c}\Omega_{t_{R_{\alpha(n)}},\ t_{R_{\nu(n)}}}(i)\right] = 0. \tag{C.2}$$

1). Define

$$u_i^* = \sum_{i=1}^{c}\Omega_{t_{R_{i-1}},\ t_{R_i}}(i), \quad \forall\, i \geq 1.$$

We first prove that $\lim_{n\to\infty}\mathsf{E}[u_{\nu(n)}^*]/n = 0$. Since $\mathsf{E}[\sigma_2] < \infty$, by the Wald equation we can prove $\mathsf{E}[u_2^*] < \infty$. By the fact that $\{u_i^*,\ i \geq 2\}$ is an i.i.d. sequence, it is readily seen that

$$\lim_{n\to\infty}\frac{1}{n}u_n^* = 0 \quad a.s.,$$

which implies

$$\lim_{n\to\infty}\frac{1}{\nu(n)}u_{\nu(n)}^* = 0 \quad a.s. \tag{C.3}$$

By (C.3) and the elementary renewal theorem we get

$$\frac{1}{n}u_{\nu(n)}^* = \frac{\nu(n)}{n}\frac{u_{\nu(n)}^*}{\nu(n)} \xrightarrow[n\to\infty]{} 0 \quad a.s.$$

If we can show that $\{u_{\nu(n)}^*/n,\ n \geq 1\}$ is uniformly integrable, then the assertion follows. By Theorem II.5.1 of Gut [8], we know that $\{(\nu(n)/n)^r,\ n \geq 1\}$ is uniformly integrable for any $r > 0$, which implies the uniform integrability of $\{(1/n)\sum_{i=1}^{\nu(n)}u_i^*,\ n \geq 1\}$ by Theorem I.6.1 of Gut [8]. Since

$$\frac{1}{n}u_{\nu(n)}^* \leq \frac{1}{n}\sum_{i=1}^{\nu(n)}u_i^*,$$

the uniform integrability of $\{u_{\nu(n)}^*/n,\ n \geq 1\}$ follows. Thus the second part of (C.2) holds.

2). We now prove that $\lim_{n\to\infty}(1/n)\mathsf{E}[W_{t_{R_{\alpha(n)}}}] = 0$. Define

$$u_k = \sum_{i=1}^{c}(\Omega_{t_{R_{k-1}},t_{R_k}}(i) - T_{t_{R_{k-1}},t_{R_k}}(i)), \quad \forall\ k \geq 1.$$

Since $W_{t_{R_k}}$ and $W_{t_{R_{k-1}}}$ have the same distribution for $k \geq 2$, and since

$$W_{t_{R_k}} = W_{t_{R_{k-1}}} + u_k, \tag{C.4}$$

$\{u_k,\ k \geq 2\}$ are identically distributed, with $\mathsf{E}[u_k] = 0$. Since $\{X_{R_n+k},\ k \geq 1\}$ are independent of $\{X_i,\ 1 \leq i \leq R_{n-1}\}$, u_k is independent of u_{k-2} for $k \geq 3$. From (C.4), we derive

$$\begin{aligned}
\mathsf{E}[W_{t_{R_{\alpha(n)}}}] &= \mathsf{E}[W_{t_{R_1}}] + \mathsf{E}\left[\sum_{k=2}^{\alpha(n)} u_k\right] \\
&= \mathsf{E}[W_{t_{R_1}}] + \mathsf{E}\left[\sum_{k=2}^{\nu(n)+1} u_k\right] - \mathsf{E}[u_{\nu(n)}] - \mathsf{E}[u_{\nu(n)+1}] \\
&\leq \mathsf{E}[W_{t_{R_1}}] + \mathsf{E}\left[\sum_{k=2}^{\nu(n)+1} u_k\right] + \sum_{i=1}^{c}(\mathsf{E}[T_{t_{R_{\nu(n)-1}},\, t_{R_{\nu(n)}}}(i)] + \mathsf{E}[T_{t_{R_{\nu(n)}},\, t_{R_{\nu(n)+1}}}(i)]) \\
&\leq \mathsf{E}[W_{t_{R_1}}] + \mathsf{E}\left[\sum_{k=2}^{\nu(n)+1} u_k\right] + c\mathsf{E}[t_{R_{\nu(n)+1}} - t_{R_{\nu(n)-1}}].
\end{aligned} \tag{C.5}$$

We now prove that $\mathsf{E}[\sum_{k=2}^{\nu(n)+1} u_k] = 0$. Since

$$\begin{aligned}
\mathsf{E}\left[\sum_{k=2}^{\infty} |u_k| I\{k \leq \nu(n)+1\}\right] &= \sum_{k=2}^{\infty} \mathsf{E}[|u_k| I\{\nu(n) \geq k-1\}] \\
&= \mathsf{E}[|u_2|] \sum_{k=2}^{\infty} P\{\nu(n) \geq k-1\} \\
&= \mathsf{E}[|u_2|]\mathsf{E}[\nu(n)] \ < \ \infty,
\end{aligned}$$

by the dominated convergence theorem we have

$$\mathsf{E}\left[\sum_{k=2}^{\nu(n)+1} u_k\right] = \mathsf{E}\left[\sum_{k=2}^{\infty} u_k I\{k \leq \nu(n)+1\}\right] = \sum_{k=2}^{\infty} \mathsf{E}[u_k I\{\nu(n) \geq k-1\}] = 0. \tag{C.6}$$

Analogous to 1), we can prove that

$$\frac{1}{n}\mathsf{E}[t_{R_{\nu(n)+1}} - t_{R_{\nu(n)-1}}] \xrightarrow[n\to\infty]{} 0. \tag{C.7}$$

Using (C.6) and (C.7), it follows from (C.5) that $\lim_{n\to\infty}(1/n)\mathsf{E}[W_{t_{R_{\alpha(n)}}}] = 0$.

□

Appendix D

In this appendix, we give the proof of Theorem 5.1 without condition (A6). We first introduce a new queueing system, identical to the original system, except that in the new system the service time of the nth customer at node i is defined as

$$\widetilde{S}_n(i) = \max\{H_0,\, S_n(i)\} \geq S_n(i), \quad 1 \leq i \leq c,\ n \geq 1,$$

where H_0 is a small constant such that the load condition in the new system satisfies

$$\widetilde{\rho}_i = \frac{\lambda_i}{\mathsf{E}[\widetilde{S}_1(i)]} < 1, \quad 1 \leq i \leq c.$$

Note that $\lambda_i,\ 1 \leq i \leq c$ are defined by (3.1).

Denote by $D_n(i)$ and $\widetilde{D}_n(i)$ the departure times of the nth customer from node i in the original and the new queueing systems, respectively. Then we have

Lemma D.1 $\quad D_n(i) \leq \widetilde{D}_n(i), \quad 1 \leq i \leq c,\ n \geq 1.$

Proof. See, e.g., Foss [1989] and Shanthikumar and Yao [1989].

□

For the new system, we define $\widetilde{X}_n = (\widetilde{Q}_n,\ \widetilde{Y}_n)$, where $\widetilde{Q}_n = (\widetilde{Q}_n(1),\ ...,\ \widetilde{Q}_n(c))$ is the queue lengths vector and $\widetilde{Y}_n = (\widetilde{Y}_n(1),\ ...,\ \widetilde{Y}_n(c))$ is the residual times vector at time t_n^-. Let $\widetilde{W}_t(i)$ be the total service time that node i will use to serve all of the customers that are in the network at time t, and define $\widetilde{W}_t = \sum_{i=1}^{c} \widetilde{W}_t(i)$.

Lemma D.2 $\quad W_t \leq \widetilde{W}_t$ *a.s. for all* $t \geq 0$.

Proof. By Lemma D.1, those customers that are in the original system at time t should be in the new system at time t, and stay less time in the original system before departure than in the new system.

□

We are now in a position to present the proof of Theorem 5.1 without condition (A6).

Proof of Theorem 5.1:

Since (A6) is satisfied by the new system described previously, Theorem 5.1 holds for the new system. Then there are regeneration points $\{\widetilde{R}_i,\ i \geq 1\}$ for the Markov chain $\{\widetilde{X}_n,\ n \geq 0\}$ such that $\{\widetilde{X}_{\widetilde{R}_i + k},\ k \geq 0\}$ are independent of $\{\widetilde{X}_k,\ 0 \leq k \leq \widetilde{R}_{i-2}\}$, $\forall\, i \geq 2$, and that

$$\sup_{x \in B(\widetilde{k}_0,\, \widetilde{b}_0)} \mathsf{E}_x[e^{s\widetilde{\sigma}_1}] \leq \widetilde{b}_3, \quad \forall\ s \in [0,\, \widetilde{a}_3], \tag{D.1}$$

where $\widetilde{a}_3,\ \widetilde{b}_3,\ \widetilde{k}_0$ and $\widetilde{b}_0$ are some constants, $\widetilde{R}_0 = 0$, and $\widetilde{\sigma}_i = \widetilde{R}_i - \widetilde{R}_{i-1}$ for $i \geq 1$.

Since the regeneration points $\{\widetilde{R}_i\}$ occur m^* step after $\{\widetilde{X}_i\}$ visit $B(\widetilde{k}_0,\, \widetilde{b}_0)$, there are at most $c\widetilde{k}_0 + m^*$ customers in the new system at time $t_{\widetilde{R}_i}$. By Lemma D.1, those customers that have departed from the new system before time $t_{\widetilde{R}_i}$ should also have departed from the original system. Thus we have $X_{\widetilde{R}_i} \in B(c\widetilde{k}_0 + m^*,\ \infty),\ \forall\, i \geq 1$. For the original system, define

$$\eta_i = \inf\{k :\quad X_{\widetilde{R}_{\zeta_{i-1}+k}} \in B(c\widetilde{k}_0 + m^*,\ b_0)\}, \quad \zeta_i = \sum_{j=1}^{i} \eta_j, \quad \zeta_0 = \eta_0 = 0, \tag{D.2}$$

where b_0 is some constant.

Since $\{\widetilde{X}_{\widetilde{R}_i},\ i \geq 1\}$ have the same distribution, $\{\widetilde{W}_{t_{\widetilde{R}_i}},\ i \geq 1\}$ are also identically distributed. By Lemma D.2, there is some constant α^* such that

$$\mathsf{E}[W_{t_{\widetilde{R}_i}}] \ \leq\ \mathsf{E}[\widetilde{W}_{t_{\widetilde{R}_i}}] \ =\ \mathsf{E}[\widetilde{W}_{t_{\widetilde{R}_1}}] \ \leq\ \alpha^*, \quad \forall\, i \geq 1,$$

which gives

$$\mathsf{E}\left[\max_{1 \leq j \leq c} Y_{\widetilde{R}_i}(j)\right] \leq \mathsf{E}\left[\sum_{j=1}^{c} Y_{\widetilde{R}_i}(j)\right] \leq \mathsf{E}[W_{t_{\widetilde{R}_i}}] \leq \alpha^*, \quad \forall\, i \geq 1. \tag{D.3}$$

Since $\{\widetilde{X}_{\widetilde{R}_i+k},\ k \geq 0\}$ are independent of $\{\widetilde{X}_k,\ 0 \leq k \leq \widetilde{R}_{i-2}\}$, the evolution of the new system after $t_{\widetilde{R}_i}$ is independent of that before $t_{\widetilde{R}_{i-2}}$. By the fact that the evolution for the original system after $t_{\widetilde{R}_i}$ and before $t_{\widetilde{R}_{i-2}}$ contains less information than that for the new system, respectively, it is seen that $\{X_{\widetilde{R}_i+k},\ k \geq 0\}$ are independent of $\{X_k,\ 0 \leq k \leq \widetilde{R}_{i-2}\}$. By (D.2) and (D.3) we have

$$\begin{aligned}
P\{\eta_1 > i\} &= P\{X_{\widetilde{R}_1} \notin B(c\widetilde{k}_0 + m^*,\ b_0),\ ...,\ X_{\widetilde{R}_i} \notin B(c\widetilde{k}_0 + m^*,\ b_0)\} \\
&\leq P\{X_{\widetilde{R}_i} \notin B(c\widetilde{k}_0 + m^*,\ b_0)\} P\{X_{\widetilde{R}_1} \notin B(c\widetilde{k}_0 + m^*,\ b_0),\ ...,\ X_{\widetilde{R}_{i-2}} \notin B(c\widetilde{k}_0 + m^*,\ b_0)\} \\
&= P\{\max_{1 \leq j \leq c} Y_{\widetilde{R}_i}(j) > b_0\} P\{X_{\widetilde{R}_1} \notin B(c\widetilde{k}_0 + m^*,\ b_0),\ ...,\ X_{\widetilde{R}_{i-2}} \notin B(c\widetilde{k}_0 + m^*,\ b_0)\} \\
&\leq \frac{\alpha^*}{b_0} P\{X_{\widetilde{R}_1} \notin B(c\widetilde{k}_0 + m^*,\ b_0),\ ...,\ X_{\widetilde{R}_{i-2}} \notin B(c\widetilde{k}_0 + m^*,\ b_0)\} \\
&\leq \delta_0^* P\{X_{\widetilde{R}_1} \notin B(c\widetilde{k}_0 + m^*,\ b_0),\ ...,\ X_{\widetilde{R}_{i-2}} \notin B(c\widetilde{k}_0 + m^*,\ b_0)\} \\
&\leq (\delta_0^*)^{i/2-1},
\end{aligned} \tag{D.4}$$

where b_0 is large enough such that $\alpha^*/b_0 \leq \delta_0^* \in (0,\, 1)$. From (D.4), it follows that η_1 is exponentially stable. By Proposition 2.5 (i) and (D.1), we can prove that $\sum_{i=1}^{\eta_1} \widetilde{\sigma}_i$ is exponentially stable. Let $A^* = B(k_0,\, b_0)$, where $k_0 = c\widetilde{k}_0 + m^*$. Then it is easy to see that $\tau^* \leq \sum_{i=1}^{\eta_1} \widetilde{\sigma}_i$ and this completes the proof.

□

Appendix E

We give an example which satisfies Condition (A2) but violates Conditions (B1) and (B2), and thus conclude that (A2) is weaker than (B1) or (B2).

We first define a function $g(x),\ x \geq 0$ as follows, where k is a positive integer.

$$g(x) = \begin{cases} 0, \quad \forall\, x \notin \bigcup_{k=1}^{\infty} [k^2-1,\ k^2+1]; \\ (x-(k^2-1))\frac{1}{k^2}\exp\left(-a_1(k^2+1)\right), \quad \text{if } x \in [k^2-1,\ k^2], \\ ((k^2+1)-x)\frac{1}{k^2}\exp\left(-a_1(k^2+1)\right), \quad \text{if } x \in [k^2,\ k^2+1], \quad \forall\, k \geq 1. \end{cases}$$

Then it is easy to see that

$$K \triangleq \int_0^{\infty} g(x)dx \leq \sum_{k=1}^{\infty} \frac{1}{2} \cdot 2 \cdot \frac{1}{k^2} < \infty.$$

Define a random variable X with density $p(x) = g(x)/K$. Then

$$\begin{aligned} \mathsf{E}[e^{a_1 X}] &= \int_0^{\infty} e^{a_1 x} p(x)dx = K^{-1}\int_0^{\infty} e^{a_1 x} g(x)dx \\ &\leq K^{-1}\sum_{k=1}^{\infty} e^{a_1(k^2+1)} \cdot \frac{1}{k^2} e^{a_1(k^2+1)} < \infty, \end{aligned}$$

which implies condition (A2).

We now show that X does not satisfy condition (B1). For any fixed $a > 0$, define

$$u(t) = P\{X > t+a | X > t\} = \frac{\int_{t+a}^{\infty} p(x)dx}{\int_t^{\infty} p(x)dx}, \quad \forall\, t > 0.$$

Let k be large enough such that $2k > a+1$. Since $k^2+1+a < (k+1)^2 - 1$, one has

$$u(k^2+1) = \frac{\int_{k^2+a+1}^{\infty} p(x)dx}{\int_{k^2+1}^{\infty} p(x)dx} = \frac{\int_{(k+1)^2-1}^{\infty} p(x)dx}{\int_{(k+1)^2-1}^{\infty} p(x)dx} = 1,$$

which gives $\limsup_{t\to\infty} u(t) = 1$, so (B1) is violated.

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