1. (13 points) Ross (2014), page 346, Exercice 49.

Events occur according to a Poisson process with rate \( \lambda \). Each time an event occurs, we must decide whether or not to stop, with our objective being to stop at the last event to occur prior to some specified time \( T \), where \( T > 1/\lambda \). That is, if an event occurs at time \( t \), \( 0 \leq t \leq T \), and we decide to stop, then we win if there are no additional events by time \( T \) and we lose otherwise. If we do not stop when an event occurs and no additional events occur by time \( T \), then we lose. Also, if no events occur by time \( T \), then we lose. Consider the strategy that stops at the first event to occur after some fixed time \( s \), \( 0 \leq s \leq T \).

(a) Using this strategy, what is the probability of winning?

(b) What value of \( s \) maximizes the probability of winning?

(c) Show that one’s probability of winning when using the preceding strategy with the value of \( s \) specified in part (b) is \( 1/e \). (Encore Euler!)

2. (12 points) Ross (2014), page 346, Exercice 50.

Suppose the number of hours between successive bus arrivals at a given bus stop is uniformly distributed on \((0, 1)\). Passengers arrive according to a Poisson process with rate \( \lambda \) per hour. Suppose a bus has just left from the stop. Let \( X \) denote the number of people who will get on the next bus. Find \( \mathbb{E}[X] \) and \( \text{Var}(X) \). How can we compute the entire distribution of \( X \), i.e., \( \mathbb{P}[X = x] \) for each integer \( x \geq 0 \)?

Hint: In tables of integral, one can find that

\[
\int_0^1 y^n e^{-ay} \, dy = \frac{n!}{a^{n+1}} \left( 1 - e^{-a} \sum_{i=0}^{n} \frac{a^i}{i!} \right).
\]


There are two types of claims that are made to an insurance company. Let \( N_i(t) \) denote the number of type \( i \) claims made by time \( t \), and suppose that \( \{N_1(t), t \geq 0\} \) and \( \{N_2(t), t \geq 0\} \)
are independent Poisson processes with rates $\lambda_1 = 10$ and $\lambda_2 = 1$. The amounts of successive type 1 claims are independent exponential random variables with mean $1000$ whereas the amounts from type 2 claims are independent exponential random variables with mean $5000$. A claim for $4000$ has just been received; what is the probability that it is a type 1 claim?

4. (15 points)

Supposons que les clients arrivent à un magasin de détail selon un processus de Poisson de taux $B$ de 9 heures à midi, $2B$ de midi à 14 heures, puis de $1.2B$ de 14 heures à 17 heures. Ici, $B$ est une variable aléatoires de loi connue qui ne prend que des valeurs positives et prend la même valeur pour toute la journée. Une plus grande valeur de $B$ correspondra à une journée plus achalandée.

On dispose d’une fonction pour générer $B$ selon la bonne loi, et d’une autre fonction pour générner des (imitations de) variables aléatoires indépendantes uniformes sur $(0,1)$. Donnez un algorithme permettant de simuler les instants des arrivées successives pour ce processus, pour une journée, en utilisant ces deux ingrédients.