ISSN 0709-986X

OPTIMAL AND SUBOPTIMAL STRATEGIES FOR GROUP PREVENTIVE REPLACEMENT

par

Alain HAURIE
Pierre L'ECUYER

G-80-01

novembre 1980

SUMMARY

A group preventive replacement problem is formulated in continuous time for a multicomponent system having identical elements. The Dynamic Programming equation is obtained in the framework of the theory of optimal control of jump processes. A discrete time version of the model is used for the numerical computation of optimal and suboptimal strategies of group preventive replacement. A monotonicity property of the Bellman functional (or cost-to-go function) is proved and serves to reduce the size of the computational problem.

RESUME

On considère un problème de remplacements préventifs de groupe dans le cas d'un système à composantes identiques. On obtient l'équation de la programmation dynamique en temps continu dans le cadre de la théorie de la commande optimale des processus de saut. Une version en temps discret du modèle est utilisée pour obtenir numériquement des politiques optimales et sous-optimales de remplacement. On démontre la monotonicité de la fonction de Bellman (fonction de coût espéré actualisé) et cette propriété nous permet de réduire la taille du problème.

1. INTRODUCTION

The aim of this paper is to formulate and solve a group preventive replacement (GPR) problem, considered in the framework of optimal stochastic control theory.

A presentation of group preventive maintenance can be found in [1]. The GPR problem that we are considering occurs when a system consists of several elements of the same type which work under the same general conditions. All the elements have the same lifetime distribution. Only two actions are available for a single element: either replace it when it fails (emergency replacement) or replace the element before it fails (preventive replacement). The possibility to combine preventive replacements with emergency replacements in a multicomponent system can generate a substantial maintenance cost reduction if there is a large fixed cost associated with any intervention on the system, whatever be the number of elements replaced.

In Ref. [1] the GPR strategy is defined by a heuristic service rule which is clearly non optimal. In Ref. [2] a formulation of the GPR problem as a stochastic control problem has been proposed and a Dynamic Programming equation has been heuristically derived.

In section 2 of the present paper, the Dynamic Programming equation of Ref. [2] is more rigorously obtained by using some general results of the control theory of jump processes.

In section 3 a discrete time formulation of the problem is proposed and the approach for the numerical solution of this large scale problem is described.

In section 4 several numerical illustrations are fully developped to show the effect of the relative magnitudes of the fixed and the variable parts of the maintenance cost on the optimal GPR strategy. The optimal strategies are then compared with suboptimal strategies which are simpler to implement.

Finally the appendix contains the proof of the main theorem assuring that the Bellman function in the Dynamic Programming equation is non decreasing with the age of any element. This theorem and his corollaries permit the reduction of the number of candidates for the minimum in the DP algorithm.

2. THE GROUP PREVENTIVE REPLACEMENT PROBLEM

We consider a system comprised of m identical elements working independently under the same conditions. At time $t_0 = 0$ every element is new, at time $T > 0$, the whole system is replaced by a new one. During the time interval $[0,T]$, if one element fails it has to be replaced immidiately by a new one. This is called an emergency replacement (ER). Simultaneously with an ER the repairman can replace any number of working elements he wants. This is called the preventive replacement (PR) of a working element.

The cost of an intervention contains two parts: a fixed cost B is incurred whatever be the number of elements replaced; a variable cost varies linearly with the number of elements which are replaced. Thus the cost of an intervention where ν elements are replaced will be given by:

$$B + \nu b, \quad \text{for} \quad \nu \geq 1 .$$

Knowing this cost structure and the failure rate function $\ell(r)$ for one element, where r is the age of the element, the problem is to find the optimal strategy for preventive replacement (optimal PR strategy).

We assume that ERs or PRs are performed instantaneously.

In order to establish rigorously the optimality conditions characterizing the optimal strategy it will be convenient fo formulate the problem in the framework of the theory of optimal control for jump processes. Recently, several authors [3,4,5,6] have obtained very general optimality conditions for the control of jump processes. The results obtained by Rishel are appropriate for the solution of the present problem. In section 2.1 the Rishel's formalism is briefly recalled. In section 2.2 the controlled jump process associated with the GPR problem is identified and in section 2.3 the Dynamic programming equation leading to the optimal PR strategy is established.

2.1 Formulation of the general control problem for a jump process

Let us define a jump process $\underline{x} = (x(t);\ 0 \le t \le T)$ with value in a set X which with probability one has only a finite number of jumps. $\forall\ t \in [0,T]$, let n(t) be the number of jumps during $[0,t]$. Define the jump times $\tau_1, \tau_2, \ldots, \tau_n, \ldots$ by letting $\tau_o = 0$ and τ_n be the time of the n-th jump if $n(T) \ge n$ or $\tau_n = T$ if $n(T) < n$. Let

$$x_n \triangleq x(\tau_n). \tag{1}$$

Let X_n denote the random sequence:

$$X_n \triangleq (x_o, \tau_1, x_1, \tau_2, \ldots, \tau_n, x_n). \tag{2}$$

Rishel[6] has shown that the jump process is determined if one has given the conditional jump rate function:

$$q(t|X_n) \triangleq \lim_{dt \to 0} \frac{P[t \le \tau_{n+1} < t + dt \mid X_n]}{dt\ P[t \le \tau_{n+1} \mid X_n]}, \quad t \ge \tau_n \tag{3}$$

and the conditional state jump probabilities:

$$\pi(A|X_n, \tau_{n+1}) \triangleq P[x_{n+1} \in A|X_n, \tau_{n+1}], \quad A \subset X \tag{4}$$

Let U be a given set. Consider a family of controlled jump rates and state jump probabilities:

$$q\ (t|X_n, u), \quad u \in U \tag{5}$$

$$\pi\ (A|X_n, \tau_{n+1}, u), \quad A \subset X, \quad u \in U. \tag{6}$$

A control is a functional $u(t,X_n)$ on the past of the process, with value in U. A cost $\varphi(T,X_n)$ is paid at the terminal time T if $n(T) = n$. The control problem is to pick the control $u(t,X_n)$ in the admissible class so that

$$E\left[\varphi(T, X_{n(t)})\right] \tag{7}$$

is minimized.

Rishel has obtained [5,6] very general necessary and sufficient conditions for optimality of such a controlled jump process. In order to apply these results to the GPR problem we have to define the controlled jump process associated with the data of the GPR problem.

2.2 The controlled jump process associated with the GPR problem

The set X will be defined as:

$$X = M \times \mathbb{R}_+^m \tag{8}$$

and we shall write $x(t) \overset{\Delta}{=} \big(y(t), z(t)\big)$ where $y(t) \in M$ gives the identification number of the last element which has failed at or before t, while z(t) gives the age of each element just after the last intervention of the repairman.

Figure 1: Example of a sample path for the (y, z) process.

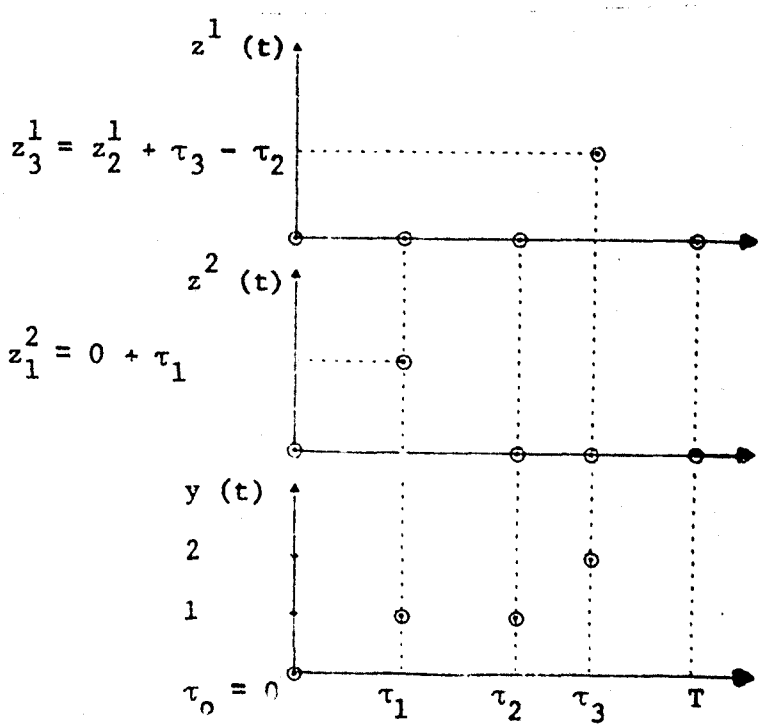


Figure 1 illustrates a possible sample path for the jump process (y,z) when $m = 2$.

At time τ_1 a jump occurs because the element 1 fails. The age of the element 1, which is replaced, jumps to the value $z_1^1 = 0$. The age of the element 2 which is not replaced jumps to the value $z_1^2 = \tau_1$.

At time τ_2 another jump occurs. Then a group replacement takes place and both elements jump to 0.

At time τ_3 the element 2 fails. Only this element is replaced thus one has:

$$z_3^1 = z_2^1 + \tau_3 - \tau_2 = \tau_3 - \tau_2$$

$$z_3^2 = 0.$$

At time T the whole system is replaced by a new one.

The set U of possible actions describes the choices of emergency and preventive replacements that the repairman can do during an intervention. We shall describe an element u of U as a vector $(u^j)_{j\in M}$ where each component u^j is a subset* of M which must contain j as an element.

$$j \in u^j \in P(M). \tag{9}$$

When element j fails, all elements contained in u^j are replaced. Thus u is already defined as a policy telling the repairman which working element he has to replace preventively knowing that an emergency replacement is necessary for the element j. This policy will be adapted to the time t and the information X_n through the control $u(t,X_n)$.

Now we shall derive the jump rate function. It is clear that, given the information X_n, the action $u(t,X_n)$ has no effect on the jump rate at t.

* $P(M)$ will denote the class of subsets of M.

If $x_n \triangleq x(\tau_n)$, let us write

$$y_n \triangleq y(\tau_n), \tag{10}$$

$$z_n = (z_n^j)_{j \in M} \triangleq z(\tau_n). \tag{11}$$

Thus, following the assumption of independence and given the failure rate function $\ell(r)$ for one element, the jump rate at time t is given by:

$$q\ (t|X_n) = \sum_{j \in M} \ell(z_n^j + t - \tau_n). \tag{12}$$

Once a jump has occured, the state jump probabilities depend on u.

Given z_n, τ_{n+1} and u, let us define the vector $\xi\ (j,u) \in \mathbb{R}^m$ by:

$$\left.\begin{aligned} \xi^i\ (j,u) &= 0 \quad \text{if} \quad i \in u_j \\ \xi^i\ (j,u) &= z_n^i + \tau_{n+1} - \tau_n \quad \text{if} \quad i \notin u_j \end{aligned}\right\} \tag{13}$$

Then we have:

$$P\left[x_{n+1} = (j,\ \xi(j,u)) \mid X_n,\ \tau_{n+1},\ u\right] =$$

$$\frac{\ell(z_n^j + \tau_{n+1} - \tau_n)}{\sum_{i=1}^{m} \ell(z_n^i + \tau_{n+1} - \tau_n)} \qquad \forall\ j \in M \tag{14}$$

For all subsets A of $M \times \mathbb{R}^m$ not containing an element $(j,\ \xi(j,u))$ the probability (6) is equal to zero. Thus at any time t, only m state jumps are possible, given an action u and an age vector z_n at $\tau_n < t$.

The control problem will be completely defined if the cost functional $\varphi(T,X_n)$ is characterized. Assume that $n(T) = n$, then:

$$\varphi(T,X_n) \triangleq \sum_{k=1}^{n} \left(B + b \sum_{j \in M} \prod (z_k^j = 0)\right) e^{\rho \tau_k} \tag{15}$$

At each jump k the fixed cost B is incurred and the variable cost b is multiplied by the number of elements replaced $\left(\sum_{j \in M} \mathbb{1}_{(z_k^j = 0)} \right)$. ρ is the discount rate.

Now the data (12), (14) and (15) define a problem of optimal control of a jump process.

Notice that, according to Eqs (12) and (14), we can write:

$$q(t|\tau_n, z_n) = q(t|X_n) \tag{16}$$

and

$$P\left[x_{n+1} = \left(j, \xi(j,u)\right) \mid \tau_n, z_n, \tau_{n+1}, u\right]$$

$$= P\left[x_{n+1} = \left(j, \xi(j,u)\right) \mid X_n, \tau_{n+1}, u\right] . \tag{17}$$

Thus the control can be restricted to be a functional $u(t, \tau_n, z_n)$.

2.3 Application of Rishel's necessary conditions

Using the dynamic programming optimality conditions of Ref. [6] we can say that an optimal control $u^*(t, \tau_n, z_n)$ is such that there exists a function $V(t, \tau_n, z_n)$ absolutely continuous in t such that:

$$V(T, \tau_n, z_n) \equiv 0 \tag{18}$$

$$-\frac{\partial}{\partial t} V(t,\tau_n,z_n) = \min_{u \in U} \left[\sum_{j=1}^{m} \ell(z_n^j + t - \tau_n) \Big\{ \right.$$

$$\left. e^{-\rho t} \left(B + b|u_j|\right) + V\left(t,t,\xi(j,u)\right) - V(t,\tau_n,z_n) \Big\} \right] \tag{19}$$

where the optimal PR strategy $u^*(t,\tau_n,z_n)$ solves the right-hand-side minimization. The dynamic programming equation (19) yields also a sufficient condition for optimality.

When looking for a solution of (18) - (19) we can take advantage of the fact that the age at time t of the element j is given by:

$$r_j = z_n^j + t - \tau_n, \quad j \in M \tag{20}$$

Thus, defining the vector $r = (r_j)_{j \in M}$, we shall restrict the function V to the class of functions satisfying:

$$V(t,\tau_n,z_n) \equiv W(t,r) \tag{21}$$

Whenever τ_n, z_n satisfy (20) and where W(t, r) is a given function.

Clearly the following relation will thus hold:

$$\frac{\partial V}{\partial t} = \frac{\partial W}{\partial t} + \sum_{j \in M} \frac{\partial W}{\partial r_j} \tag{22}$$

With this restricted class of function the sufficient conditions (18) - (19) become:

$$W(T,r) \equiv 0 \tag{23}$$

$$- \frac{\partial W}{\partial t}(t,r) = \sum_{j \in M} \frac{\partial W(t,r)}{\partial r_j}$$

$$+ \sum_{j=1}^{m} \ell(r_j) \min_{u_j \in U_j} \{(B+b|u_j|)\, e^{-\rho t}$$

$$+ W\big(t,\gamma(r,u_j)\big) - W(t,r)\} \tag{24}$$

Where $\gamma(r,u_j)$ is the vector defined by:

$$\gamma(r,u_j) \overset{\Delta}{=} \big(\gamma^i(r,u_j)\big)_{i \in M} \in \mathbb{R}_+^m \text{ with } \begin{cases} \gamma^i(r,u_j) = 0 & \text{if } i \in u_j \\ \gamma^i(r,u_j) = r_i & \text{if } i \notin u_j \,. \end{cases} \tag{25}$$

Going from (18), (19) to (23), (24) is going from optimality conditions which involve a single differential equation in t to a partial differential equation.

It is not clear which system will be easier to solve numerically.

As the $(\underline{y},\underline{r})$ process is Markovian, the optimality conditions (23), (24) will have a discrete time counterpart related to the theory of the control of Markov chains. This is the approach followed in the next section.

A direct attempt for the numerical solution of (18), (19) or (23), (24) will be the object of a forthcoming paper.

3. A DISCRETE TIME FORMULATION

We consider now the case where the system is observed at discrete sampled times. Assume that the time interval $[0,T]$ is divided into n subintervals of length Δt. Thus, the sampled times of observation are:

$$t_o = 0,\ t_1 = \Delta t,\ \ldots,\ t_\sigma = \sigma\Delta t,\ \ldots,\ T = n\Delta t.$$

We assume that the ERs and PRs are possible only at sampled times. Denote by t_σ^+ the time t_σ immediately after the replacements (if any).

At a sampled time t_σ, the state of the system is a vector x, whose component x_j is d if element j is in failure state, and gives the age of element j expressed in number of time periods of length Δt otherwise. At time t_σ^+, the state of the system is a vector r whose component r_j is given by:

$$r_j = \begin{cases} 0 \text{ if element } j \text{ has been replaced at } t_\sigma \\ x_j \text{ otherwise.} \end{cases}$$

If an element j is in state r_j at time t_σ^+, then the conditional probability that it will fail during the next time period of length Δt given that it has not failed during $(t_\sigma - r_j\Delta t,\ t_\sigma]$ is given by:

$$\Delta Q(r_j) = 1 - \exp\left[-\int_0^{\Delta t} \ell(r_j\Delta t + \zeta)\,d\zeta\right]$$

$$= \ell(r_j\Delta t)\ \Delta t + o(\Delta t) \qquad (26)$$

where $o(\Delta t)$ is such that $\lim_{\Delta t \to 0} \frac{o(\Delta t)}{\Delta t} = 0$.

From now on, we assume that the failure rate is non decreasing, and so is $\Delta Q(r_j)$ as a function of r_j.

Given the state vector r at t_σ^+, the conditional probability that the subset G⊂M will contain all the elements which will fait during $(t_\sigma, t_{\sigma+1}]$ and only these ones is given by:

$$P(G|r) = \prod_{i\in G} \Delta Q(r_i) \prod_{j\in M-G} \left(1 - \Delta Q(r_j)\right). \tag{27}$$

We shall look for the solution of the GPR problem formulated on this discrete sampled time set and use this solution as an approximation to the exact solution of the continuous time GPR problem.

In a discrete time setting, there is no theoretical difficulty to assume that a preventive replacement can be made even if no failure have occured (in a continuous time setting this would have led to an impulse control problem). It will be shown later on that, with the chosen cost structure, preventive replacements are never made in the absence of failures.

3.1 The dynamic programming equation.

Let us assume a discount factor $\beta\in[0,1]$ per time period Δt and define the cost to go function $J_k(x)$ as the total minimal expected cost given that we are at time

$$t_\sigma = T - k\Delta t \tag{28}$$

and that the system is observed to be in state x.

By a standard dynamic programming approach (cf [7], [8]), we obtain the following equations defining the functions J_k, k = 0,...,m:

$$Z_o \equiv J_o \equiv 0 \tag{29}$$

and for k = 1,...,m:

$$J_k(x) = \min_{\{R|H\subset R\subset M\}} \{B\mathbb{1}_{R\neq\phi} + b|R| + Z_k(\sum_{i\in M-R} x_i e_i)\} \tag{30}$$

where*:

$$H \triangleq \{j | x_j = d\}$$

is the set of failed elements at time $t_\sigma = T - k\Delta t$ and:

$$Z_k(r) = \beta \sum_{G \subset M} P(G|r) J_{k-1}\Big(\sum_{j \in G} d\, e_j + \sum_{j \in M-G} (r_j + 1) e_j \Big). \qquad (31)$$

Thus, the control action at a sampled time t_σ is the subset R of elements replaced. This set must contain, as a subset, the set H of failed elements.

3.2 Reducing the size of the problem.

Part b) of the main theorem proved in Appendix shows that if no element have failed during the last time period $(t_{\sigma-1}, t_\sigma]$, that is if $H = \phi$, then it is optimal to do no preventive replacement at t_σ. Equation (30) can then be reduced to:

$$J_k(x) = \begin{cases} Z_k(x) & \text{if } H = \phi \\ \underset{\{R | H \subset R \subset M\}}{\text{Min}} \{B + b|R| + Z_k\Big(\sum_{i \in M-R} x_i\, e_i \Big) & \text{if } H \neq \phi. \end{cases} \qquad (32)$$

The equations (29), (31) and (32) are clearly the discrete time counterpart of (23)-(24).

At time T, the age of each element can take n+1 possible values (assuming that $x = 0$ at time t_o) and so there are $(n+1)^m$ possible states of the system.

We can define an order on $\mathbb{N} \cup \{d\}$ by setting $d > k \ \forall k \in \mathbb{N}$. As the elements are identical, it is possible to reduce the number of possible states of the system by considering that the components of x are always ranked by decreasing order of magnitude:

$$x_1 \geq x_2 \geq \ldots \geq x_m. \qquad (33)$$

---

* $|R|$ denotes the cardinality of the set R, e_i denotes the i-unit vector and $\mathbb{1}$ is the indicator function:

$$\mathbb{1}_{\langle \text{condition} \rangle} = \begin{cases} 1 & \text{if condition is true} \\ 0 & \text{otherwise.} \end{cases}$$

This is equivalent to say that the name of an element is no longer recorded. The number of possible states is now the number of combinations with repetitions of m elements among n+1, which is given by:

$$K^{m}_{n+1} = C^{m}_{n+m} = \frac{(n+m)!}{m!\ n!} . \tag{34}$$

In practice, there may be an age which can be attained by an element only with a very small probability. In that case, to reduce further the number of possible states, we may approximate $\Delta Q(x)$ by a function of the form:

$$p(x) = \begin{cases} \Delta Q(x) & \forall x \leq a \\ \Delta Q(a) & \forall x > a \end{cases} \tag{35}$$

where $a \leq n$. Hence, an element of age greater than a is equivalent to an element of age a.

This reduces the number of possible states of the system to:

$$K^{m}_{a+1} = C^{m}_{a+m} = \frac{(m+a)!}{m!\ a!} \tag{36}$$

Equation (27) then becomes:

$$P(G|r) = \prod_{i \in G} p(r_i) . \prod_{j \in M-G} (1-p(r_j)) . \tag{37}$$

In another way, it is possible to limit the life of any element to a+1 periods by taking

$$p(a) = 1. \tag{38}$$

It means that any element of age a+1 is replaced immediately.

In eq. (32), for a given H, we have $2^{m-|H|}$ possible candidates for R. This number can be very large if there are many elements. However, since we assume that $\Delta Q(x)$ is not decreasing in x, it seems intuitively clear that the elements preventively replaced, if any, will always be the oldest elements. This intuition is effectively correct, as stated in corollary 2 (see Appendix).

Hence, choosing R reduces to choosing a number η, $0 \leq \eta \leq m-|H|$ and replacing the η oldest elements. The number of candidates for R is thus reduced to $m+1-|H|$.

Notice that at a sampled time t_σ, $\sigma \geq 1$, no element can be of age 0. The set of all possible states of the system at that time is thus:

$$X = \{x \in \{1,2,\ldots,a,d\}^m \mid x_1 \geq x_2 \geq \ldots \geq x_m\}. \tag{39}$$

At t_σ^+, $\sigma \geq 0$, the set of all possible states of the system is:

$$X_o = \{x \in \{0,1,\ldots,a\}^m \mid x_1 \geq x_2 \geq \ldots \geq x_m\}. \tag{40}$$

3.3 Reformulation of the recurrence relations.

In the light of the preceeding remarks, the dynamic programming algorithm can be reformulated as follows:

1. Set, $\forall x \in X$:

$$J_o(x) = 0 \tag{41}$$

2. For $k = 1,\ldots,n$:

$\forall r \in X_o$, set:

$$Z_k(r) = \beta \sum_{G \subset M} P(G|r) J_{k-1}\left(\mathrm{ord}\left(\sum_{J \in G} de_j + \sum_{j \in M-G} (r_j+1)e_j\right)\right) \tag{42}$$

$\forall x \in X \cup X_o$ set:

$$J_k(x) = \begin{cases} Z_k(x) & \text{if } x \in X_o \\ B + \min\limits_{p \leq \theta \leq m} \left\{\theta b + Z_k\left(\mathrm{ord}\left(\sum\limits_{i=\theta+1-p}^{m-p} x_i e_i\right)\right)\right\} & \text{otherwise} \end{cases} \tag{43}$$

where:

$$p \triangleq \max \{i \mid x_i = d\} \tag{44}$$

represents the number of failed elements and ord(s) is the vector s whose components have been rearranged by decreasing order.

Let $\theta_k(x)$ be the value of θ for which the minimum is attained in (43). It represents the optimal number of elements to replace when we are in state x at time $T - k\Delta t$.

Another reduction of the computation time is possible by using the corollary 1 (see Appendix) which states that if x and $\tilde{x}$ are in X, $\tilde{x} \geq x$ and $\theta_k(x) = m$, then $\theta_k(\tilde{x}) = m$.

Notice that in the preceding statement, the hypothesis that $\theta_k(x) = m$ cannot be relaxed. In general, even if $\tilde{x} \geq x$, it is possible that $\theta_k(\tilde{x}) < \theta_k(x)$. We will see an example of this in section 4.1.

3.4 Infinite time horizon.

Consider finally the case where Δt is fixed and $T = \infty$. Let $J(x)$ be the expected discounted cost over an infinite horizon when $x \in X \cup X_o$ is the state of the system. Assume $\beta < 1$. From the proposition 1 on p. 227 of [8], we have, $\forall x \in X \cup X_o$:

$$J(x) = \lim_{k \to \infty} J_k(x) \leq \frac{B + bm}{1-\beta} < \infty. \tag{45}$$

According to Proposition 2 on p. 229 of [8], there exists a stationary optimal strategy $\{u^*(x)\}_{x \in X}$ telling which elements to replace when the system is observed to be in state x. Moreover, from Eq. (45) and Main Theorem (see Appendix), J(x) is a non decreasing fucntion of each component x_j. From this, it can be proved (see [13]) that u must comply with the three corollaries of the Appendix.

According to Proposition 4 on page 237 of [8], the function J_* obtained by the following algorithm is such that:

$$\max_{x \in X} |J_*(x) - J(x)| \leq \varepsilon.$$

Algorithm:

1. Set k=0 and define J_o as in Eq. (41).

2. Repeat n times:

   k: = k+1 and

   calculate $\{Z_k(r)\}_{r\in X_o}$, $\{J_k(x)\}_{x\in X\setminus X_o}$

   and $\{\theta_k(x)\}_{x\in X}$ as in section 3.3.

3. Set $\gamma_1 := \frac{\beta}{1-\beta} \max_{r\in X_o} (J_k(r) - J_{k-1}(r))$

   $$\gamma_2 = \frac{\beta}{1-\beta} \max_{r\in X_o} (J_k(r) - J_{k-1}(r)).$$

4. If $\gamma_2 - \gamma_1 > \varepsilon$ return to step 2.

5. Set $J_*(r) = J_k(r) + \frac{\gamma_1 + \gamma_2}{2} \quad \forall r\in X_o$

   $$\theta_*(x) = \theta_k(x) \quad \forall x\in X$$

   Stop.

4. NUMERICAL ILLUSTRATIONS.

In this section various examples are worked out. The analysis has been restricted to the infinite time horizon case.

4.1 Example 1.

Consider $m = 6$ elements, $\beta = 0.95$, a fixed cost $B = 8.0$ and a unit cost of $b = 6.0$ for each replaced element. The probability $p(t)$ that an element of age t will fail during the next period is given in table 1.

Table 1: Definition of the probability of failure.

t	p(t)
0	.05
1	.10
2	.20
3	.40
≥ 4	.90

We thus have $a = 4$. After 50 iterations (cpu time = 131 sec. on a cyber 173), we obtain the solution depicted in table 2, where the maximum error on the expected cost-to-go is less than $\varepsilon = .01$. For each possible observed state of the system, table 2 indicates the best action to do. It is already known that when there is no failure, we do no PR. So, these states without failed elements don't have to appear in the table. The expected discounted cost-to-go starting with a new system is $J_*(0) = 274.49$.

Notice that in state (33111*) we replace only the failed element but that in state (31111*) we replace 2 elements! Hence, $s \geq r$ does not implies that $\theta_*(s) \geq \theta_*(r)$.

This counterintuitive result can be explained as follows: when the system is in state (31111*), replacing 2 elements leads to a

Table 2: Solution for the numerical example.

Observed ages of the elements (* indicates failure) x	Number of elements to replace preventively $\theta_*(x)-\lvert H\rvert$	Total cost for the intervention $B+b\theta_*(x)$	Description of the system after the intervention r	Expected discounted cost to go $J_*(r)$
* * * * * *		44.00	0 0 0 0 0 0	274.49
1 * * * * *		38.00	1 0 0 0 0 0	276.85
2 * * * * *		38.00	2 0 0 0 0 0	279.47
3 * * * * *	1	44.00	0 0 0 0 0 0	274.49
4 * * * * *	1	44.00	0 0 0 0 0 0	274.49
1 1 * * * *		32.00	1 1 0 0 0 0	278.99
2 1 * * * *		32.00	2 1 0 0 0 0	281.56
3 1 * * * *	1	38.00	1 0 0 0 0 0	276.85
4 1 * * * *	1	38.00	1 0 0 0 0 0	276.85
2 2 * * * *		32.00	2 2 0 0 0 0	283.59
3 2 * * * *	1	38.00	2 0 0 0 0 0	279.47
4 2 * * * *	1	38.00	2 0 0 0 0 0	279.47
3 3 * * * *		32.00	3 3 0 0 0 0	286.35
4 3 * * * *	2	44.00	0 0 0 0 0 0	274.49
4 4 * * * *	2	44.00	0 0 0 0 0 0	274.49
1 1 1 * * *		26.00	1 1 1 0 0 0	280.92
2 1 1 * * *		26.00	2 1 1 0 0 0	283.42
3 1 1 * * *	1	32.00	1 1 0 0 0 0	278.99
4 1 1 * * *	1	32.00	1 1 0 0 0 0	278.99
2 2 1 * * *		26.00	2 2 1 0 0 0	285.50
3 2 1 * * *	1	32.00	2 1 0 0 0 0	281.56
4 2 1 * * *	1	32.00	2 1 0 0 0 0	281.56
3 3 1 * * *		26.00	3 3 1 0 0 0	288.52
4 3 1 * * *	2	38.00	1 0 0 0 0 0	276.85
4 4 1 * * *	2	38.00	1 0 0 0 0 0	276.85
2 2 2 * * *		26.00	2 2 2 0 0 0	287.08
3 2 2 * * *		26.00	3 2 2 0 0 0	288.96
4 2 2 * * *	1	32.00	2 2 0 0 0 0	283.59
3 3 2 * * *		26.00	3 3 2 0 0 0	290.03
4 3 2 * * *	2	38.00	2 0 0 0 0 0	279.47
4 4 2 * * *	2	38.00	2 0 0 0 0 0	279.47
3 3 3 * * *		26.00	3 3 3 0 0 0	290.58
4 3 3 * * *		26.00	4 3 3 0 0 0	291.89
4 4 3 * * *		26.00	4 4 3 0 0 0	292.11
4 4 4 * * *		26.00	4 4 4 0 0 0	292.14
1 1 1 1 * *		20.00	1 1 1 1 0 0	282.62
2 1 1 1 * *		20.00	2 1 1 1 0 0	285.03
3 1 1 1 * *	1	26.00	1 1 1 0 0 0	280.92
4 1 1 1 * *	1	26.00	1 1 1 0 0 0	280.92
2 2 1 1 * *		20.00	2 2 1 1 0 0	287.09
3 2 1 1 * *		20.00	3 2 1 1 0 0	289.26
4 2 1 1 * *	1	26.00	2 1 1 0 0 0	283.42
3 3 1 1 * *		20.00	3 3 1 1 0 0	290.43
4 3 1 1 * *	2	32.00	1 1 0 0 0 0	278.99
4 4 1 1 * *	2	32.00	1 1 0 0 0 0	278.99
2 2 2 1 * *		20.00	2 2 2 1 0 0	288.79
3 2 2 1 * *		20.00	3 2 2 1 0 0	290.85
4 2 2 1 * *	1	26.00	2 2 1 0 0 0	285.50
3 3 2 1 * *		20.00	3 3 2 1 0 0	292.18
4 3 2 1 * *	2	32.00	2 1 0 0 0 0	281.56
4 4 2 1 * *	2	32.00	2 1 0 0 0 0	281.56
3 3 3 1 * *		20.00	3 3 3 1 0 0	292.83
4 3 3 1 * *		20.00	4 3 3 1 0 0	294.21
4 4 3 1 * *		20.00	4 4 3 1 0 0	294.44
4 4 4 1 * *		20.00	4 4 4 1 0 0	294.48
2 2 2 2 * *		20.00	2 2 2 2 0 0	290.06
3 2 2 2 * *		20.00	3 2 2 2 0 0	291.93
4 2 2 2 * *	1	26.00	2 2 2 0 0 0	287.08
3 3 2 2 * *		20.00	3 3 2 2 0 0	293.20
4 3 2 2 * *	1	26.00	3 2 2 0 0 0	288.96
4 4 2 2 * *		20.00	4 4 2 2 0 0	295.39

Table 2 (continued): Solution for the numerical example.

Observed ages of the elements (* indicates failure) x	Number of elements to replace preventively $\theta_*(x)-\|H\|$	Total cost for the intervention $B+b\theta_*(x)$	Description of the system after the intervention r	Expected discounted cost to go $J_*(r)$
3 3 3 2 * *		20.00	3 3 3 2 0 0	293.95
4 3 3 2 * *		20.00	4 3 3 2 0 0	295.38
4 4 3 2 * *		20.00	4 4 3 2 0 0	295.72
4 4 4 2 * *		20.00	4 4 4 2 0 0	295.79
3 3 3 3 * *		20.00	3 3 3 3 0 0	294.41
4 3 3 3 * *		20.00	4 3 3 3 0 0	295.49
4 4 3 3 * *		20.00	4 4 3 3 0 0	295.74
4 4 4 3 * *		20.00	4 4 4 3 0 0	295.80
4 4 4 4 * *		20.00	4 4 4 4 0 0	295.81
1 1 1 1 1 *		14.00	1 1 1 1 1 0	284.11
2 1 1 1 1 *		14.00	2 1 1 1 1 0	286.39
3 1 1 1 1 *	1	20.00	1 1 1 1 0 0	282.62
4 1 1 1 1 *	1	20.00	1 1 1 1 0 0	282.62
2 2 1 1 1 *		14.00	2 2 1 1 1 0	288.36
3 2 1 1 1 *		14.00	3 2 1 1 1 0	290.63
4 2 1 1 1 *	1	20.00	2 1 1 1 0 0	285.03
3 3 1 1 1 *		14.00	3 3 1 1 1 0	292.04
4 3 1 1 1 *	2	26.00	1 1 1 0 0 0	280.92
4 4 1 1 1 *	2	26.00	1 1 1 0 0 0	280.92
2 2 2 1 1 *		14.00	2 2 2 1 1 0	290.03
3 2 2 1 1 *		14.00	3 2 2 1 1 0	292.21
4 2 2 1 1 *	1	20.00	2 2 1 1 0 0	287.09
3 3 2 1 1 *		14.00	3 3 2 1 1 0	293.76
4 3 2 1 1 *	1	20.00	3 2 1 1 0 0	289.26
4 4 2 1 1 *	2	26.00	2 1 1 0 0 0	283.42
3 3 3 1 1 *		14.00	3 3 3 1 1 0	294.65
4 3 3 1 1 *		14.00	4 3 3 1 1 0	296.14
4 4 3 1 1 *		14.00	4 4 3 1 1 0	296.43
4 4 4 1 1 *		14.00	4 4 4 1 1 0	296.49
2 2 2 2 1 *		14.00	2 2 2 2 1 0	291.38
3 2 2 2 1 *		14.00	3 2 2 2 1 0	293.38
4 2 2 2 1 *	1	20.00	2 2 2 1 0 0	288.79
3 3 2 2 1 *		14.00	3 3 2 2 1 0	294.88
4 3 2 2 1 *		14.00	4 3 2 2 1 0	296.79
4 4 2 2 1 *		14.00	4 4 2 2 1 0	297.27
3 3 3 2 1 *		14.00	3 3 3 2 1 0	295.88
4 3 3 2 1 *		14.00	4 3 3 2 1 0	297.48
4 4 3 2 1 *		14.00	4 4 3 2 1 0	298.01
4 4 4 2 1 *		14.00	4 4 4 2 1 0	298.15
3 3 3 3 1 *		14.00	3 3 3 3 1 0	296.44
4 3 3 3 1 *		14.00	4 3 3 3 1 0	297.67
4 4 3 3 1 *		14.00	4 4 3 3 1 0	298.06
4 4 4 3 1 *		14.00	4 4 4 3 1 0	298.16
4 4 4 4 1 *		14.00	4 4 4 4 1 0	298.18
2 2 2 2 2 *		14.00	2 2 2 2 2 0	292.42
3 2 2 2 2 *		14.00	3 2 2 2 2 0	294.20
4 2 2 2 2 *	1	20.00	2 2 2 2 0 0	290.06
3 3 2 2 2 *		14.00	3 3 2 2 2 0	295.54
4 3 2 2 2 *		14.00	4 3 2 2 2 0	297.41
4 4 2 2 2 *		14.00	4 4 2 2 2 0	298.00
3 3 3 2 2 *		14.00	3 3 3 2 2 0	296.49
4 3 3 2 2 *		14.00	4 3 3 2 2 0	298.09
4 4 3 2 2 *		14.00	4 4 3 2 2 0	298.75
4 4 4 2 2 *		14.00	4 4 4 2 2 0	298.99
3 3 3 3 2 *		14.00	3 3 3 3 2 0	297.12
4 3 3 3 2 *		14.00	4 3 3 3 2 0	298.43
4 4 3 3 2 *		14.00	4 4 3 3 2 0	298.98
4 4 4 3 2 *		14.00	4 4 4 3 2 0	299.19
4 4 4 4 2 *		14.00	4 4 4 4 2 0	299.26
3 3 3 3 3 *		14.00	3 3 3 3 3 0	297.56
4 3 3 3 3 *		14.00	4 3 3 3 3 0	298.62
4 4 3 3 3 *		14.00	4 4 3 3 3 0	299.06
4 4 4 3 3 *		14.00	4 4 4 3 3 0	299.22
4 4 4 4 3 *		14.00	4 4 4 4 3 0	299.27
4 4 4 4 4 *		14.00	4 4 4 4 4 0	299.28

state where all elements are not old, whereas when the system is in state (33111*), the same replacement decision would leave one element at age 3 and thus a rather high probability of failure. To lower significantly the probability of failure in that latter state, we must replace all the elements. But this is too expensive, hence the best decision is to replace none of the elements of age 3 and wait for the next sampled time.

4.2 Comparison with suboptimal solutions.

One simple suboptimal strategy is the following:

a) Do no PRs when all elements are operative.

b) When there is at least one failure, replace only the failed elements.

This strategy will be called NOPR (No Preventive Replacements).

Another suboptimal strategy consists in the following:

a) Do no PRs When all elements are operative.

b) When there is at least one failure, replace all failed elements and all those elements whose ages are greater or equal to a certain threshold a_*, which is independent of the observed state of the system.

The number a_* is chosen in the set $\{1,2,\ldots,a+1\}$. Such a strategy will be called a FAT strategy (Fixed Age Threshold).

Adopting the NOPR strategy corresponds to taking $a_* = a+1$. Hence, there exists a number $a_* \in \{1,\ldots,a+1\}$ such that, starting with a new system and using the corresponding FAT strategy will yield an expected discounted cost-to-go not greater than the one obtained when using the NOPR strategy.

The following example is an illustration of what can be the percentage of increase of the expected cost when using NOPR or FAT instead of the optimal strategy.

4.3 Example 2.

Consider a system of $m = 6$ elements, each having a Gamma (4.1) lifetime distribution. Thus, $\forall t \in N$:

$$\Delta Q(t) = \frac{F(t+1) - F(t)}{1 - F(t)}$$

Where:

$$F(t) = \frac{1}{3!} \int_0^t z^3 e^{-z} dz$$

The failure rate $\Delta Q(t)$ can be approximated reasonably (taking $a = 7$) by:

$$p(t) = \begin{cases} \Delta Q(t) & t \leq 7 \\ \Delta Q(7) & t > 7. \end{cases}$$

We then obtain:

Table 3:

Definition of the probabilities of failure.

t	p(t)
0	.019
1	.126
2	.245
3	.330
4	.389
5	.429
6	.459
≥ 7	.482

For $\beta = 0.90$, $b = 1.0$ and for different values of B, the expected discounted cost-to-go starting with a new system ($J_*(0)$) has been computed using 1) the algorithm of section 3, which gives the optimal strategy, 2) NOPR, 3) FAT with different values of a_*. Table 4 gives the results of this computation.

Notice that:

1) The cost reduction in doing PRs increases as the ratio B/b of the fixed cost over the unit cost increases.

2) The optimal value of a_* for the strategy decreases as B/b increases.

3) The cost-to-go when using the optimal FAT strategy is in general not very much higher than when using the optimal strategy (1.3% increase in the worst case).

Table 4: Comparison of the expected total discounted cost to go for a new system for three kinds of strategies.

	Optimal strategy	N O P R		F A T		
B	$J_*(0)$	$J_*(0)$	% of increase	a_*	$J_*(0)$	% of increase
1.0	16.693	16.693	0.00%	8	16.693	0.00%
2.0	22.907	22.921	0.06%	8	22.921	0.06%
				7	23.025	
				8	29.149	1.31%
				7	29.17	
3.0	28.772	29.149	1.31%	6	29.18	
				5	29.20	
				4	29.27	
				8	35.38	
4.0	33.830	35.38	4.58%	4	34.7	
				3	34.21	1.12%
				2	34.90	
				8	41.61	
5.0	38.296	41.61	8.65%	3	38.84	
				2	38.627	0.86%
				1	39.41	
				8	72.75	
10.0	57.189	72.75	27.21%	2	57.253	0.11%
				1	57.322	

This numerical illustration suggests that, in a real life large scale problem, a FAT suboptimal PR strategy could be a good substitute to the optimal closed loop strategy. This could, however, be true only for systems observed over an infinite time horizon. When the time horizon is finite the age threshold should at least depend on time.

For $t > 7$, $p(t)$ has been approximated by $p(7) = .482$. Let us see what happens now, for the case $B = 3$, if $p(7)$ is raised to 0.6 or 0.8.

Table 5: Variations of $J_*(0)$ when $p(7)$ goes from .482 to .6 or .8.

	Optimal strategy		F A T		
$p(7)$	$J_*(0)$	% of increase	a_*	$J_*(0)$	% of increase
.482	28.772		8	29.149	
			7	29.17	
			6	29.18	
.6	28.779	< .03%	8	29.25	< 0.38%
			7	29.18	< 0.06%
			6	29.18	< 0.03%
.8	28.784	< .05%	8	29.36	< 0.75%
			7	29.20	< 0.14%
			6	29.18	< 0.03%

We can see, in table 5, that the optimal value of a_* changes with $p(7)$. However, the value of $J_*(0)$ does not change very much. Thus, the approximation of $\Delta Q(t)$ by $p(t)$ seems reasonable.

5. CONCLUSION

Multicomponent systems are commonplace in practice whereas the theory of optimal maintenance is mainly directed towards single component systems. (Cf. the recent survey of Ref. [9]). The results obtained in previous sections show that an optimal GPR strategy can be devised through the use of modern stochastic control theory in the case where all components have the same lifetime distribution.

The problem considered in this paper is related to the problem considered by Vergin in Ref. [10]. However, the methodology we have used permitted us to solve a much larger problem than the one considered in Ref. [10].

Two extentions of this work can be considered:

(i) the actions available for maintenance of one element could refer to a larger variety of repair and overhaul activities. This would necessitate a more detailed description of the state of wear of an element than the one given by the age of the element. A first attempt to generalize the model in this direction can be found in Ref. [2].

(ii) The elements could have distinct lifetime characteristics. This is the case for many systems like trucks (Ref. [11]), aircraft engines, etc.... There is no difficulty in obtaining the Dynamic Programming equation for the characterization of optimal GPR strategies in the case where the components are not identical (ref. [12]). However, the size of the state set could rapidly become much too large to allow for a direct adaptation of the numerical procedure presented in this paper to obtain an optimal solution. Numerical examples suggest that a "reasonable" or near optimal solution can be found in the class of FAT type policies defined in section 4. To restrict the search for a best policy to such a class of simpler policies can be the best way to obtain a practical solution to a very large problem.

A realistic maintenance problem will in general involve a multi-component system with non identical elements and with a complex description of the state of wear of each element as well as of the set of available maintenance activities. Such problems offer stimulating challenges to stochastic control theorists.

APPENDIX

The main consequences of the following theorem are that $Z_k(r)$ as defined in eq. (31) is non decreasing with respect to each component r_i and that we do not have to consider the possibility of doing replacements when there is no failure.

MAIN THEOREM.

For any $k \in \mathbb{N}$:

a) If $x,r \in \mathbb{N}^m$ and $x \geq r$ then:

$$0 \leq Z_k(x) - Z_k(r) \leq \beta(B + b|Q|). \tag{A.1}$$

and

$$0 \leq J_k\big(\tau(x,G)\big) - J_k\big(\tau(r,(G\cap\bar{Q})\cup V)\big) \leq B + b|Q| \tag{A.2}$$

$\forall G \subseteq M$ and $\forall V \subseteq Q\cap G$, where

$$Q \triangleq \{i \in M \mid x_i > r_i\} \tag{A.3}$$

and

$$\tau(r,G)_i \triangleq \begin{cases} d & \text{if } i \in G \\ r_i + 1 & \text{otherwise.} \end{cases} \tag{A.4}$$

b) $\forall x \in \mathbb{N}^m$, the Min. in eq. (30) is attained for $R = \emptyset$.

PROOF:

The proof is by induction on k. For k = 0, (A.1), (A.2) and b) are clearly true.

We will show that a) and b) being true for k-1 implies that they are also true for k. For this, the following equation is needed:

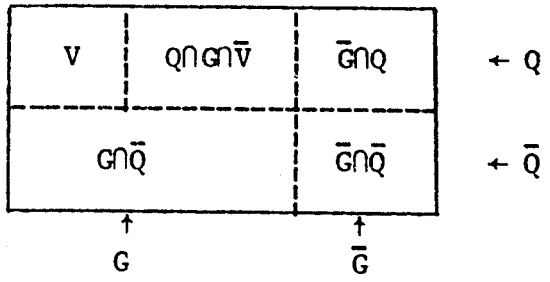
$$Z_k(x) - Z_k(r) =$$

$$\beta \sum_{G\subseteq M} \sum_{V\subseteq G\cap Q} \{ \left[J_{k-1}(\tau(x,G)) - J_{k-1}(\tau(r,(G\cap\bar{Q})\cup V))\right] \prod_{i\in(G\cap\bar{Q})\cup V} p(r_i) \prod_{i\in\bar{G}} (1-p(x_i))$$

$$\prod_{i\in Q\cap G\cap\bar{V}} (p(x_i) - p(r_i)) \}. \qquad \text{(A.5)}$$

The reader can refer to figure 2 as an illustration of the set relations involved in (A.5) which we now prove.

FIGURE 2: Partition of M.



From now on, we write $p(r_i)$ instead of $\Delta Q(r_i)$ to simplify the notation. Equation (31) then becomes:

$$Z_k(x) = \beta \sum_{G\subseteq M} J_{k-1}(\tau(x,G)) \prod_{i\in G} p(x_i) \prod_{i\in\bar{G}} (1-p(x_i)). \qquad \text{(A.6)}$$

But since $r_i = x_i \quad \forall i\in\bar{Q}$ and $p(x_i) \geq p(r_i) \quad \forall i\in Q$, we have:

$$\prod_{i\in G} p(x_i) = \prod_{i\in G\cap Q} (p(r_i)+(p(x_i)-p(r_i))) \prod_{i\in G\cap\bar{Q}} p(r_i)$$

$$= \left[\sum_{V\subseteq G\cap Q} \prod_{i\in V} p(r_i) \prod_{i\in Q\cap G\cap\bar{V}} (p(x_i)-p(r_i))\right] \prod_{i\in G\cap\bar{Q}} p(r_i) \qquad \text{(A.7)}$$

$$= \sum_{V\subseteq G\cap Q} \left[\prod_{i\in(G\cap\bar{Q})\cup V} p(r_i) \prod_{i\in Q\cap G\cap\bar{V}} (p(x_i)-p(r_i))\right]$$

so that:

$$Z_k(x) = \beta \sum_{G \subseteq M} \sum_{V \subseteq G \cap Q} \Big[J_{k-1}(\tau(x,G)) \cdot \prod_{i \in (G \cap \bar{Q}) \cup V} p(r_i) \prod_{i \in \bar{G}} (1-p(x_i)) \prod_{i \in Q \cap G \cap \bar{V}} (p(x_i)-p(r_i)) \Big]. \tag{A.8}$$

On the other end:

$$Z_k(r) = \beta \sum_{G' \subseteq M} J_{k-1}(\tau(r,G')) \prod_{i \in G'} p(r_i) \prod_{i \in \bar{G}'} (1-p(r_i)) \tag{A.9}$$

and:

$$\begin{aligned} \prod_{i \in \bar{G}'} (1-p(r_i)) &= \prod_{i \in \bar{G}'} \big((1-p(x_i)) + (p(x_i) - p(r_i)) \big) \\ &= \prod_{i \in \bar{G}' \cap \bar{Q}} (1-p(x_i)) \prod_{i \in \bar{G}' \cap Q} \big((1-p(x_i)) + (p(x_i)-p(r_i)) \big) \\ &= \sum_{W \subseteq \bar{G}' \cap Q} \prod_{i \in W} (1-p(x_i)) \prod_{i \in \bar{G}' \cap Q \cap \bar{W}} (p(x_i)-p(r_i)) \prod_{i \in \bar{G}' \cap \bar{Q}} (1-p(x_i)) \\ &= \sum_{W \subseteq \bar{G}' \cap J} \prod_{i \in (\bar{G}' \cap \bar{Q}) \cup W} (1-p(x_i)) \prod_{i \in \bar{G}' \cap Q \cap \bar{W}} (p(x_i)-p(r_i)) \end{aligned} \tag{A.10}$$

so that

$$Z_k(r) = \beta \sum_{G' \subseteq M} \sum_{W \subseteq \bar{G}' \cap Q} \Big[J_{k-1}(\tau(r,G')) \prod_{i \in G'} p(r_i) \prod_{i \in (\bar{G}' \cap \bar{Q}) \cup W} (1-p(x_i)) \prod_{i \in \bar{G}' \cap Q \cap \bar{W}} (p(x_i)-P(r_i)) \Big]. \tag{A.11}$$

Let us define a bijection between the sets $\{(G,V) \mid G \subseteq M,\ V \subseteq G \cap Q\}$ and $\{(G',W') \mid G' \subseteq M,\ W \subseteq \bar{G}' \cap Q\}$ by:

$$\begin{aligned} G' &= (G \cap \bar{Q}) \cup V \\ W &= \bar{G} \cap Q \ . \end{aligned} \tag{A.12}$$

It is lengthy but straightforward to verify that the relations (A.12) truly define a bijection and imply:

$$G = (G' \cup Q) - W$$

$$V = G' \cap Q$$

$$\bar{G} = (\bar{G}' \cap \bar{Q}) \cup W$$

$$Q \cap G \cap \bar{V} = \bar{G}' \cap Q \cap \bar{W}.$$

We can then rewrite equation (A.11):

$$Z_k(r) = \beta \sum_{G \subseteq M} \sum_{V \subseteq G \cap Q} \Big[J_{k-1}(\tau(r, (G \cap \bar{Q}) \cup V)) \prod_{i \in (G \cap \bar{Q}) \cup V} p(r_i) \prod_{i \in \bar{G}} (1-p(x_i)) \prod_{i \in Q \cap G \cap \bar{V}} (p(x_i)-p(r_i)) \Big]. \quad \text{(A.13)}$$

Equation (A.5) follows from (A.8) and (A.13).

Now, from (27):

$$\sum_{G \subseteq M} \sum_{V \subseteq G \cap Q} \prod_{i \in (G \cap \bar{Q}) \cup V} p(r_i) \prod_{i \in \bar{G}} (1-p(x_i)) \prod_{i \in Q \cap G \cap \bar{V}} (p(x_i)-p(r_i)) = \sum_{G \subseteq M} P(G|x) = 1 . \quad \text{(A.14)}$$

If (A.2) is true for k-1, the equations (A.5) and (A.14) imply that (A.1) holds for k.

To prove b), let $x \in \mathbb{N}^m$. From (A.1), we have:

$$Z_k(x) \leq B + b|R| + Z_k \left(\sum_{i \in M-R} x_i \, e_i \right) \quad \forall R \subseteq M \quad \text{(A.15)}$$

and so:

$$Z_k(x) \leq \min_{R \subseteq M} \left\{ B + b|R| + Z_k \left(\sum_{i \in M-R} x_i \, e_i \right) \right\} \leq J_k(x) . \quad \text{(A.16)}$$

But since $R = \emptyset$, $R = \emptyset$ is a candidate for the minimum in the definition of $J_k(x)$. Moreover, from (A.16), it is the best candidate.

Only (A.2) remains to be verified. Let R_r and R_x be sets for which the minimums are attained in the definitions of $J_k\left(\tau(r,(G\cap\bar{Q})UV)\right)$ and $J_k\left(\tau(x,G)\right)$ respectively. It is clear that:

$$(G\cap\bar{Q})UV \subseteq G \subseteq R_x \ . \tag{A.18}$$

and

$$G \subseteq R_r UQ$$

so that $R_r UQ$ is a candidate for the minimum in the definition of $J_k\left(\tau(x,G)\right)$. We thus have, using (A.1):

$$J_k\left(\tau(x,G)\right) \leq B + b\left|R_r UQ\right| + Z_k\Big(\sum_{i\in M-(R_r UQ)} (x_i+1)e_i\Big)$$

$$\leq B + b\left|R_r\right| + b\left|Q\right| + Z_k\Big(\sum_{i\in M-R_r} (r_i+1)e_i\Big)$$

$$\leq B + b\left|Q\right| + J_k\left(\tau(r,(G\cap\bar{Q})UV)\right)$$

which is the right-hand side inequality in (A.2).

If $(G\cap\bar{Q})UV = \emptyset$, then, from b) and (A.1) respectively, $R_r = \emptyset$ and

$$J_K\left(\tau(R,(G\cap\bar{Q})UV)\right) = Z_k(r+1) \leq Z_k(x+1) \ .$$

But

$$Z_k(x+1) \begin{cases} = J_k(x+1) = J_k\left(\tau(x,G)\right) & \text{if} \quad G = \emptyset \\ \leq \begin{cases} B + b\left|R_x\right| + Z_k\Big(\sum_{i\in M-R_x} (x_i+1)e_i\Big) \\ = J_k\left(\tau(x,G)\right) \end{cases} & \text{if} \quad G \neq \emptyset \end{cases}$$

so that the left inequality of (A.2) holds for that case.

If $(G\cap\bar{Q})\cup V \neq \emptyset$, then neither R_r nor R_x is empty. Hence, from (A.18) and the definition of R_r:

$$J_k\left(\tau(r,(G\cap\bar{Q})\cup V)\right) = B + b|R_r| + Z_k\Big(\sum_{i\in M-R_r}(r_i+1)e_i\Big)$$

$$\leq B + b|R_x| + Z_k\Big(\sum_{i\in M-R_x}(r_i+1)e_i\Big)$$

$$\leq B + b|R_x| + Z_k\Big(\sum_{i\in M-R_x}(x_i+1)e_i\Big)$$

$$= J_k\left(\tau(x,G)\right)$$

and the theorem is proved.

COROLLARY 1:

At a sampled time t_σ, if it is optimal to replace all the elements when we are in state x, then in any state $\tilde{x} \geq x$, it is also optimal to replace all the elements.

PROOF:

Let $H = \{i \,|\, \tilde{x}_i = d\}$. We assume $H \neq \emptyset$, since the case $H = \emptyset$ is trivial. We have:

$$J_k(\tilde{x}) = \min_{\{R\,|\,H\subseteq R\}} \{B + b|R| + Z_k\Big(\sum_{i\in M-R}\tilde{x}_i\, e_i\Big)\}$$

$$\geq \min_{\{R\,|\,H\subseteq R\}} \{B + b|R| + Z_k\Big(\sum_{i\in M-R}x_i\, e_i\Big)\}$$

$$\geq J_k(x) = B + bm + Z_k(0)$$

which proves the result.

COROLLARY 2:

At a sampled time t , if an optimal set of elements to replace preventively contains ℓ elements, then it is optimal to replace preventively the ℓ oldest elements among those being still operative.

PROOF:

This is a direct consequence of the definition of J_k and the non decreasingness of Z_k.

REFERENCES

1. GERTSBAKH, I.B., Models of Preventive Maintenance, North Holland 1977.

2. ALJ, A. and HAURIE, A., "Hierarchical Control of a Population Process with Application to Group Preventive Maintenance", Proceedings 2nd IFAC Workshop on Large Scale Systems, Toulouse, June 1980.

3. BOEL, R. and VARAYIA, P., "Optimal Control of Jump Process, SIAM Journal on Control and Optimisation, Vol. 15 (1977), pp. 92-119.

4. BREMAUD, P., "Bang-Bang Controls of Point Processes", Advances in Applied Probability, Vol. 8, pp. 385-394, 1976.

5. RISHEL, R., "Controls Optimal from time t onward and Dynamic Programming for Systems of Controlled Jump Processes" in R. Wets edit. Stochastic Systems Modeling Identification and Optimization, North Holland, 1977, pp. 125-153.

6. RISHEL, R., "Optimality for Completely Observed Controlled Jump Processes", IEEE Transactions on Automatic Control, Vol. AC-22, #6, pp. 906-908, 1977.

7. ROSS, S.M., "Applied Probability Models with Optimization Applications". Holden Day, 1970.

8. BERTSEKAS, D.P. "Dynamic Programming and Stochastic Control", Academic Press, 1976.

9. PIERSKALA, W.P. and VOELKER, J.A. "A Survey of Maintenance Models: The Control and Surveillance of Deteriorating Systems", Naval Research Logistics Quaterly, Vol. 18, #3, 1976.

10. VERGIN, R.C., "Optimal Renewal Policies for Complex Systems", Naval Research Logistics Quarterly, Vol. 15, #4, 1968.

11. BROSH, I., SHLIFER, E. and ZEIRA, Y., "Optimal Maintenance Policy for a Fleet of Vehicles", Management Science, Vol. 22, #4, 1975.

12. GOYETTE, H. et HAURIE, A., "Vers un modèle d'entretien préventif d'un système à plusieurs composantes adapté au cas d'un parc de camions", Proceedings ASAC Conference, 1980 Montréal.

13. L'ECUYER, P., "Politiques optimales et sous-optimales pour le remplacement des composantes d'un système", Rapport de stage de maîtrise, dépt. d'informatique, Université de Montréal, 1980.

* * *