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An analytic center cutting plane method for pseudomonotone variational inequalities

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Abstract

We consider an analytic center algorithm for solving generalized monotone variational inequalities in \mathbb{R}^n , which adapts a recent result due to Goffin et al. (1993) to the numerical resolution of continuous pseudomonotone variational inequalities.

Keywords: Variational inequalities; Interior point methods; Cutting planes; Generalized monotonicity

1. Introduction

Although there exists a fairly vast literature on generalized monotonicity concepts (see [13]), little is known about algorithms for solving variational inequalities involving generalized monotone functions (see however [8], who analyzes double projection algorithms for solving pseudomonotone variational inequalities).

In this paper, we describe a cutting plane algorithm for solving pseudomonotone variational inequalities. This algorithm works on a reformulation of the variational inequality as a convex feasibility problem; it is an interior point method based on the computation of analytic centers; such

methods have been shown to be effective in practice. Our convergence proof does not rely on volume reduction arguments used for the ellipsoid and related methods (see [9, 10]). The algorithm generates sequences of points that are feasible with respect to the primal constraints; these sequences contain at least one subsequence that converges to an equilibrium solution. Complexity results are given, under the additional assumption that the variational inequality is pseudo-co-coercive.

Section 2 contains a short description of the notation; Section 3 presents an interior cutting plane method for solving pseudomonotone⁺ variational inequalities.

2. Basic concepts and definitions

Let C be a convex compact subset of \mathbb{R}^n and F a continuous function from C into \mathbb{R}^n . A vector

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x^* in \mathbb{R}^n is a solution of the variational inequality $\text{VI}(F, C)$ if and only if it satisfies the system of nonlinear inequalities

$$F(x^*)^T(x^* - x) \leq 0 \quad \forall x \in C. \quad (1)$$

It is well known that, under the above assumptions, the solution set C^* of $\text{VI}(F, C)$ is nonempty and compact (see [7]).

We say that the function F is *pseudomonotone* on C if, for all x, y in C ,

$$F(x)^T(y - x) \geq 0 \Rightarrow F(y)^T(y - x) \geq 0, \quad (2)$$

pseudomonotone⁺ on C if it is pseudomonotone on C and, for all x, y in C ,

$$\left. \begin{array}{l} F(x)^T(y - x) \geq 0 \\ F(y)^T(y - x) = 0 \end{array} \right\} \Rightarrow F(x) = F(y), \quad (3)$$

and *pseudo-co-coercive* with modulus α on C , for all x, y in C ,

$$\begin{aligned} F(x)^T(y - x) \geq 0 &\Rightarrow \\ F(y)^T(y - x) &\geq \alpha \|F(x) - F(y)\|^2. \end{aligned} \quad (4)$$

It follows from the previous definitions that pseudo-co-coercive functions are pseudomonotone⁺, pseudomonotone⁺ functions are pseudomonotone, and that monotone functions are pseudomonotone. The reader interested in these and other concepts of generalized monotonicity is referred to the papers of Schaible [13] and Zhu and Marcotte [14].

If F is pseudomonotone on C , x^* is in C^* if and only if it is in C and satisfies the system of linear inequalities

$$F(x)^T(x^* - x) \leq 0 \quad \forall x \in C. \quad (5)$$

The solution set C^* is characterized as the intersection of all valid cutting planes, and is consequently convex.

The inequality systems (1) and (5) can be recast into minimization problems by introducing the functions

$$g_P(x) = \max_{y \in C} F(x)^T(x - y) \quad (6)$$

and

$$g_D(x) = \max_{y \in C} F(y)^T(x - y) \quad (7)$$

Indeed, we have

$$\begin{aligned} C^* &= \arg \min_{x \in C} g_P(x) \\ &= \arg \min_{x \in C} g_D(x) \\ &= \{x \in C \mid g_P(x) = 0\} \\ &= \{x \in C \mid g_D(x) = 0\}. \end{aligned}$$

3. A cutting plane method based on analytic centers

In this section, we propose a cutting plane method to solve (5), based on the computation of (approximate) analytic centers of closed, convex polyhedra.

We will assume that F is Lipschitz continuous on C , with Lipschitz constant L , that the set C is a full-dimensional polyhedron $\{Ax \leq b\}$ and that these inequalities include the inequalities $0 \leq x \leq e$, where e is a vector of all ones.

We will set $M = \max_{x \in C} \|F(x)\|$ and denote by d the diameter of C , i.e., $d = \max_{x, y \in C} \|x - y\| \leq \sqrt{n}$.

We say that a vector x in C is an ε -solution of $\text{VI}(F, C)$ if $g_P(x) \leq \varepsilon$. It can be checked in polynomial time that x is an ε -solution by solving the linear program

$$\min_{y \in C} F(x)^T y.$$

We propose the following algorithm for identifying an ε -solution of $\text{VI}(F, C)$.

Algorithm 1

Step 0 (initialization):

$$k = 0, A^k = A, b^k = b,$$

$$C^k = \{x \mid A^k x \leq b^k\},$$

Step 1 (computation of an approximate analytic center): Find an approximate analytic center x^k of C^k .

Step 2 (stopping criterion):

$$\text{Compute } g_P(x^k) = \max_{x \in C} F(x^k)^T(x^k - x)$$

if $g_P(x^k)^T = 0$ **then STOP**

else GOTO Step 3.

Step 3 (generation of a cutting plane):

$$A^{k+1} = \begin{pmatrix} A^k \\ F(x^k)^T \end{pmatrix}, \quad b^{k+1} = \begin{pmatrix} b^k \\ F(x^k)^T x^k \end{pmatrix}$$

$H^k = \{x \mid F(x^k)^T(x - x^k) = 0\}$ is the new cutting plane.

Increase k by one.

RETURN to Step 1.

At Step 1 of Algorithm 1, the notion of approximate analytic center corresponds to that used in the paper by Goffin et al. [6].

To the current set of m^k linear inequalities

$$C^k = \{x \mid A^k x + s = b^k, s \geq 0\}$$

is associated the strictly concave dual potential

$$\varphi_D = \sum_{j=1}^{m^k} \ln s_j,$$

The exact analytic center ($\bar{s}^k > 0, \bar{x}^k$) of C^k is the maximizer of the dual potential and thus the exact analytic center is, together with a positive dual vector \bar{y} , the unique solution of the Karush–Kuhn–Tucker system

$$\begin{aligned} A^{k^T} \bar{y}^k &= 0, \\ A^k \bar{x}^k + \bar{s}^k &= b^k, \\ \bar{Y}^k \bar{s}^k &= e, \end{aligned} \tag{8}$$

where \bar{Y} is the diagonal matrix built upon the dual vector \bar{y} .

An approximate center ($s^k > 0, x^k$), together with a dual vector y^k , satisfies the system

$$\begin{aligned} A^{k^T} y^k &= 0, \\ A^k x^k + s^k &= b^k, \\ \|Y^k s^k - e\| &\leq \eta < 1. \end{aligned} \tag{9}$$

The analytic center also minimizes the primal (Karmarkar) potential

$$\varphi_P = m^k \ln b^{k^T} y - \sum_{j=1}^{m^k} \ln y_j.$$

This potential function has actually been used by Goffin et al. [4] to compute an approximate analytic center. An efficient updating procedure for computing an approximate analytic center after the

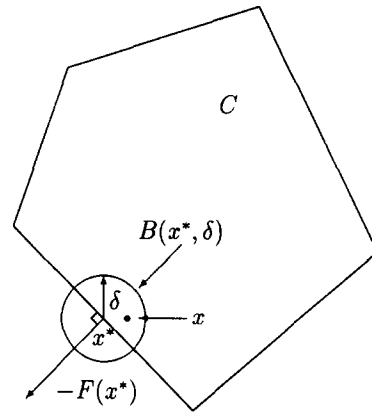


Fig. 1. The geometry of Lemma 1.

addition of a new cutting plane has been proposed by Mitchell and Todd [11] and studied in [4–6]. In fact, the computation of x^{k+1} given x^k can be achieved by performing (at most) four steps of Newton’s linear approximation method.

The convergence proof of Algorithm 1 relies on two preliminary results.

Lemma 1. Let $x^* \in C^*$, $\delta = \varepsilon / (M + Ld)$ and $B(x^*, \delta)$ be the closed ball of radius δ centered at x^* . We have that x is an ε -solution of VI(F, C) for all $x \in B(x^*, \delta) \cap C$, i.e., $g_P(x) \leq \varepsilon$ for all $x \in B(x^*, \delta) \cap C$

Proof. Let $x \in B(x^*, \delta) \cap C$ (see Fig. 1). For any $y \in C$ we have

$$\begin{aligned} F(x)^T(x - y) &= F(x)^T(x - x^*) + F(x)^T(x^* - y) \\ &\leq F(x)^T(x - x^*) \\ &\quad + (F(x) - F(x^*))^T(x^* - y) \\ &\leq M \|x - x^*\| + L \|x - x^*\| d \\ &\leq (M + Ld)\delta = \varepsilon, \end{aligned}$$

and x is an ε -solution, as required. \square

Assumption 1. The set C is the cube $0 \leq x \leq e$, where e is a vector of all ones.

This assumption is needed so that the complexity analysis of [5] can be used without change. It is not restrictive, in the sense that the nature of the results (i.e. the complexity as a function of ε) carries easily to the case of general linear constraints $C = \{x: \tilde{A}x \leq \tilde{b}, 0 \leq x \leq e\}$, but the exact algebraic expressions (i.e. the constants) change. This would need a complete and easy but tedious rewriting of [5].

The following lemma is simply a restatement of the result of Goffin et al. [5] in our context.

Lemma 2. *Let \bar{x} be a point in C and $B(\bar{x}, \rho)$ a closed ball of radius ρ , centered at \bar{x} and lying inside C . If, at iteration k of a cutting plane algorithm based on approximate analytic centers, we have that $B(\bar{x}, \rho)$ lies in C^k , then the following upper bound on ρ holds:*

$$\rho^2 \leq \frac{n}{2n+k} \left(\frac{1}{2} + 2n \ln \left(1 + \frac{k}{8n^2} \right) \right). \quad (10)$$

We will use Lemma 2 as follows: Given a ball of radius ρ lying in C , there exists an iteration index $k(\rho)$ such that C^k does not contain the given ball.

We are now in position to prove the main result of this section.

Theorem 1. *Let F be pseudomonotone⁺ on C . Then Algorithm 1 either stops with a solution of VI(F, C) after a finite number of iterations, or there exists a subsequence of the infinite sequence $\{x^k\}$ that converges to a point in C^* .*

Proof. We divide the proof into two mutually exclusive cases.

Case 1: For some iteration index k and some solution $x^* \in C^*$, x^* lies on the hyperplane H^k generated at Step 3 of Algorithm 1.

We have $F(x^k)^T(x^k - x^*) = 0$ and $F(x^*)^T(x^k - x^*) \geq 0$. Since F is pseudomonotone⁺, it follows that $F(x^k) = F(x^*)$. Now, for any x in C ,

$$\begin{aligned} F(x^k)^T(x - x^k) &= F(x^k)^T(x - x^*) + F(x^k)^T(x^* - x^k) \\ &= F(x^*)^T(x - x^*) + 0 \\ &\geq 0. \end{aligned}$$

Thus $g_P(x^k) = 0$, x^k is a solution to VI(F, C), and the algorithm has found a solution in finite time.

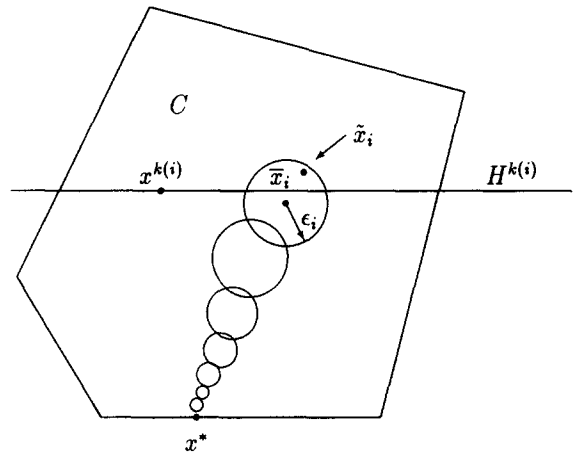


Fig. 2. Geometry of Theorem 1 (Case 2): first construction.

Case 2: There exists an optimal point $x^* \in C^*$ that never lies on H^k for any k .

First note that $x^* \in C^k$ for every index k , as $C^* \subset C^k$. Let $\{\tilde{x}_i\}_{i \in \mathbb{N}}$ be an arbitrary sequence of points in the interior of C converging to x^* , and ε_i a sequence of positive numbers such that $\lim_{i \rightarrow \infty} \varepsilon_i = 0$ and that the sequence of closed balls $\{B(\tilde{x}_i, \varepsilon_i)\}_{i \in \mathbb{N}}$ lies in the interior of C (see Fig. 2). Note that $\lim_{i \rightarrow \infty} \{B(\tilde{x}_i, \varepsilon_i)\} = \{x^*\}$.

From Lemma 2, we know that there must exist a smallest index $k(i)$ and a point $\tilde{x}_i \in B(\tilde{x}_i, \varepsilon_i)$ such that \tilde{x}_i lies on the wrong side of the hyperplane $H^{k(i)}$, i.e.,

$$F(x^{k(i)})^T(x^{k(i)} - \tilde{x}_i) < 0.$$

As $F(x^{k(i)})^T(x^{k(i)} - x^*) \geq 0$, there exists a point \hat{x}^i on the segment $[\tilde{x}_i, x^*]$ such that $F(x^{k(i)})^T(\hat{x}^i - x^{k(i)}) = 0$ (see Fig. 3). Since C is compact, we can extract from the sequence $\{x^{k(i)}\}_{i \in \mathbb{N}}$ a convergent subsequence $\{x^{k(i)}\}_{i \in \mathbb{S}}$. Denote by \check{x} its limit point.

We have

$$F(x^{k(i)})^T(\hat{x}^i - x^{k(i)}) = 0$$

and, passing to the limit (recall that F is continuous, and that $\lim_{i \rightarrow \infty} \hat{x}^i = x^*$):

$$F(\check{x})^T(x^* - \check{x}) = 0.$$

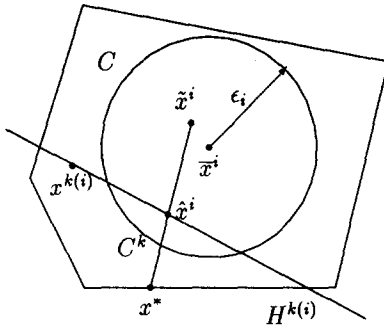


Fig. 3. Geometry of Theorem 1 (Case 2): second construction.

Invoking an argument similar to that used in Case 1, this equality implies that $g_P(\check{x}) = 0$ and that \check{x} is a solution of VI(F, C). \square

As described, Algorithm 1 is likely to generate an infinite sequence $\{x^k\}$. To make it finite, one must introduce an approximate stopping criterion, i.e., substitute Step 2a, described below, for Step 2:

Step 2a (stopping criterion):
If $g_P(x^k) \leq \varepsilon$ **then STOP**
else GOTO Step 3.

The resulting algorithm will be denoted Algorithm 1a.

We will now make the stronger assumption that F be pseudo-co-coercive with modulus α on C and consider the following construction. Let ε' be a pre-determined number, and set $\delta = \varepsilon' / (M + Ld)$. Define the ball $B(x^*, \delta)$ as in Lemma 1 and construct a close ball $B(\bar{x}, \rho)$ of radius ρ , centered at \bar{x} and lying in $C \cap B(x^*, \delta)$. Let also $\varepsilon = \varepsilon' + d\sqrt{\varepsilon' / \alpha}$.

From Lemma 2, we know that there exists a smallest index $k(\rho)$ such that $B(\bar{x}, \rho)$ does not lie in $C^{k(\rho)}$. Let $\tilde{x}^{k(\rho)}$ be such that

$$F(x^{k(\rho)})^T(x^{k(\rho)} - \tilde{x}^{k(\rho)}) < 0,$$

and $\tilde{x}^{k(\rho)} \in B(\bar{x}, \rho)$. As before, let $\hat{x}^{k(\rho)} \in [\tilde{x}^{k(\rho)}, x^*]$ be a point satisfying, $\hat{x}^{k(\rho)} \in B(x^*, \delta)$ and $F(\hat{x}^{k(\rho)})^T(\hat{x}^{k(\rho)} - x^{k(\rho)}) = 0$.

From Lemma 1, we know that $\hat{x}^{k(\rho)}$ is an ε' -solution of VI(F, C), i.e.,

$$F(\hat{x}^{k(\rho)})^T(\hat{x}^{k(\rho)} - x) \leq \varepsilon' \quad \forall x \in C.$$

Using the pseudo-co-coercivity of F on C there comes

$$\begin{aligned} \|F(x^{k(\rho)}) - F(\hat{x}^{k(\rho)})\|^2 &\leq \frac{1}{\alpha} F(\hat{x}^{k(\rho)})^T(\hat{x}^{k(\rho)} - x^{k(\rho)}) \\ &\leq \frac{\varepsilon'}{\alpha}, \end{aligned}$$

which implies that

$$\|F(x^{k(\rho)}) - F(\hat{x}^{k(\rho)})\| \leq \sqrt{\varepsilon' / \alpha}.$$

Now, for all x in C ,

$$\begin{aligned} F(x^{k(\rho)})^T(x^{k(\rho)} - x) &= F(x^{k(\rho)})^T(\hat{x}^{k(\rho)} - x) \\ &\quad + F(x^{k(\rho)})^T(x^{k(\rho)} - \hat{x}^{k(\rho)}) \\ &= F(\hat{x}^{k(\rho)})^T(\hat{x}^{k(\rho)} - x) \\ &\quad + (F(x^{k(\rho)}) - F(\hat{x}^{k(\rho)}))^T(\hat{x}^{k(\rho)} - x) + 0 \\ &\leq \varepsilon' + d\sqrt{\varepsilon' / \alpha} \\ &= \varepsilon. \end{aligned}$$

Therefore, $g_P(x^{k(\rho)}) = \max_{x \in C} F(x)^T(x^{k(\rho)} - x) \leq \varepsilon$ and $x^{k(\rho)}$ is an ε -solution of VI(F, C). We can summarize this discussion in a theorem:

Theorem 2. Under the assumption that F is pseudo-co-coercive with modulus α on C , Algorithm 1a terminates with an ε -solution in at most $k(\rho)$ iterations, where $k(\rho)$ is the smallest index satisfying

$$\rho^2 \leq \frac{n}{2n+k} \left(\frac{1}{2} + 2n \ln \left(1 + \frac{k}{8n^2} \right) \right), \quad (11)$$

$\varepsilon = \varepsilon' + d\sqrt{\varepsilon' / \alpha}$, $\delta = \varepsilon' / (M + Ld)$ and $B(\bar{x}, \rho)$ is the largest ball inscribed in $C \cap B(x^*, \delta)$.

Remark 1. The convergence rate result crucially depends on the relationship between ρ and δ , which is essentially given by a condition number for the system of inequalities C . While this number is difficult to estimate for general convex polyhedra, it can be obtained in the case of simple structures (cubes, simplices, totally unimodular systems, etc.).

Clearly, for any $x^* \in C^*$, $C \subset B(x^*, \sqrt{n})$. For an arbitrary ball $B(x_c, \omega)$ included in C and for every $\lambda \in [0, 1]$, one has

$$B(\lambda x_c + (1 - \lambda)x^*, \lambda\omega) \subset B(x^*, \lambda\sqrt{n}),$$

and thus

$$\frac{\rho}{\delta} \geq \frac{\omega}{\sqrt{n}},$$

where the right-hand-side term ω/\sqrt{n} is related to the inverse of the asphericity of the set C .

For instance, if C is defined only by the box inequalities $0 \leq x \leq e$, where e is a vector of ones, one has

$$\frac{\rho}{\delta} \geq \frac{1}{2\sqrt{n}},$$

and we can derive, using the technique of Altman and Kiwiel [1],

$$\begin{aligned} \min \left\{ \frac{\min_{1 \leq j \leq k} \{g_P(x^j)\}}{L(1 + \sqrt{n})}, \frac{1}{4} \right\} \\ \leq \sqrt{\frac{\frac{1}{2}n + 2n^2 \ln(1 + k/8n^2)}{2n + k}}. \end{aligned} \quad (12)$$

Remark 2. Algorithm 2a, together with the proposed stopping rule, yields an ε -solution. It might be interesting to know how close an ε -solution is to an actual solution of VI(F, C). While it is difficult to estimate the proximity of $x^{k(\rho)}$ to C^* under our general hypothesis, this is easy to achieve if the function F is strongly monotone on C , with strong monotonicity constant β , say. In this case we can write

$$\begin{aligned} (F(x^{k(\rho)}) - F(x^*))^T (x^{k(\rho)} - x^*) \\ \leq F(x^{k(\rho)})^T (\hat{x}^{k(\rho)} - x^*) \leq \varepsilon, \end{aligned}$$

where x^* is the unique solution of VI(F, C). It follows that

$$\beta \|x^{k(\rho)} - x^*\|^2 \leq \varepsilon$$

and that

$$\|x^{k(\rho)} - x^*\| \leq \sqrt{\varepsilon/\beta}.$$

4. Extensions

As we alluded to in Assumption 1, the analysis given here extends easily to the case where C is a full-dimensional, bounded, polyhedron $\{Ax \leq b\}$.

The algorithm can also be extended to the case of a convex set $C = \{x \in \mathbb{R}^n: h_i(x) \leq 0\}$ defined by

pseudo-convex constraints; if all constraints are satisfied at the current point, then a functional cut is introduced, as before, but if the current point is not feasible (say $h_i(x^k) > 0$) then a standard feasibility cut would be added: $\{x: h_i'(x^k)^T(x - x^k) \leq 0\}$.

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