

An Efficient Analytic Signal Generator

"DSP Tips and Tricks" introduces practical design and implementation signal processing algorithms that you may wish to incorporate into your designs. We welcome readers to submit their contributions to Associate Editors Rick Lyons (r.lyons@ieee.org) or Britt Rorabaugh (dspboss@aol.com).

Sometimes in digital signal processing (DSP) applications, one has a real-valued signal and needs to transform it into an analytic signal. Interest in analytic signals stems from the signals having one-sided Fourier transforms and the ease with which one may calculate their instantaneous amplitudes and frequencies. Traditionally, generating analytic signals is done by augmenting the real signal with its Hilbert transform, as shown in Figure 1(a). Specifically if our real signal is $i(t)$ and its Hilbert transform is $q(t)$, then our analytic signal is simply $i(t) + jq(t)$, where $j^2 = -1$ [1].

One shortcoming of the analytic signal generation scheme in Figure 1(a) is that the Hilbert transformer cannot be designed to have exactly unity gain, as we desire, over its full passband. Therefore the two paths exhibit unmatched frequency magnitude responses. (For completeness, we mention that [2] proposed an analytic signal generation scheme where both its $i(t)$ and $q(t)$ channels have identical frequency magnitude responses; however that scheme does not exhibit a linear-phase frequency response.) Since the absolute phase associated with a true analytic signal is rarely needed, we

can generate a pseudoanalytic signal by using a pair of unity gain filters having a phase response difference of 90° . This generality suggests that there are many such possible pairs of filters and there are! Reference [3] presents an analytic signal generation method where a half-band filter is created via a Remez algorithm, and the result is modulated by a complex sinusoid. A limitation of their method applies to the length of filters because the design algorithm encounters numerical problems for large filter lengths.

This article presents a simple way to construct a pair of quasi-linear-phase bandpass filters that have identical magnitude responses and differ in phase by 90° that can be used for analytic signal generation as shown in Figure 1(b). Furthermore, these filters have useful symmetry properties that significantly reduce their computational complexity and coefficient storage requirements, and there is no inherent length limitation to the size of the filters.

ANALYTIC SIGNAL GENERATION APPROACH

Since the filter pair's phase difference is key to the design, we will elect to use finite impulse response (FIR) filters with linear phase. And as is well known, to have linear phase, FIR filters must have impulse responses that are either odd or even

symmetric about their midpoint [4]. This fact will prove very useful momentarily.

DEFINING THE FREQUENCY RESPONSE

We start by defining our two filters' identical frequency magnitude response for positive frequencies. Then we will exploit the aforementioned symmetry rules of linear phase filters to find their impulse responses. We construct our bandpass response using two pieces of a sinusoid joined together with a horizontal line as shown in Figure 2.

In this filter design we will specify three parameters: The two half amplitude points ω_1 and ω_2 and the transition half bandwidth a . For this example they are, respectively, 0.05, 0.45, and 0.05 rad/s. The transition region width is $2a = 0.1$ rad/s.

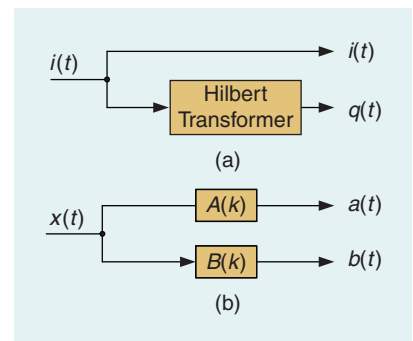


FIG1 Analytic signal generation methods: (a) traditional and (b) proposed.

$$H(\omega) = \begin{cases} 0, & \omega < \omega_1 - a \\ \sin^2 \left\{ \frac{\pi}{4a} [\omega - (\omega_1 - a)] \right\}, & \omega_1 - a \leq \omega \leq \omega_1 + a \\ 1, & \omega_1 + a \leq \omega \leq \omega_2 - a \\ \cos^2 \left\{ \frac{\pi}{4a} [\omega - (\omega_2 - a)] \right\}, & \omega_2 - a \leq \omega \leq \omega_2 + a \\ 0, & \omega > \omega_2 + a. \end{cases} \quad (1)$$

Algebraically, the positive frequency, $\omega \geq 0$, response of $H(\omega)$ is given by (1), shown at the bottom of the previous page.

IMPULSE RESPONSE DERIVATIONS
To find our filters' impulse responses, we will now use the fact that they are related to the $H(\omega)$ frequency response

via the Fourier transform, and in finding the Fourier transform we will exploit the time symmetry demanded by the linear phase constraint. Our in-phase filter will have even time symmetry, thus its frequency response will also be even symmetric. So we perform an even extension $H(\omega)$ and then find its continuous inverse Fourier transform $I(t)$. By doing this, the in-phase impulse response is given by

$$I(t) = 2 \int_0^{\infty} H(\omega) \cos(\omega t) d\omega = 2\pi^2 \cos(at) \frac{\sin(\omega_1 t) - \sin(\omega_2 t)}{t(4a^2 t^2 - \pi^2)} \quad (2)$$

This method is similar to that used to find the impulse responses of raised cosine and root raised cosine filters [5]. Likewise, we find our quadrature filter's $Q(t)$ impulse response by forming an odd extension of $H(\omega)$ and finding its inverse Fourier transform. Thus,

$$Q(t) = 2 \int_0^{\infty} H(\omega) \sin(\omega t) d\omega = 2\pi^2 \cos(at) \frac{\cos(\omega_2 t) - \cos(\omega_1 t)}{t(4a^2 t^2 - \pi^2)} \quad (3)$$

Figure 3 shows the $I(t)$ and $Q(t)$ filters' impulse responses. Due to space limitations here, the derivations of (2) and (3) can be found at <http://signalprocessingsociety.org/publications/periodicals/spm/columns-resources>.

ROTATING FILTER IMPULSE RESPONSES

These filters essentially meet our design criteria, but when we sample the $I(t)$ and $Q(t)$ impulse responses, certain asymmetries in the resulting frequency response will occur potentially reducing the filters' magnitude and phase performance. So what we do is transform our 0° and 90° filters into a -45° and $+45^\circ$ pair via a 45° rotation. This is done by

$$\begin{bmatrix} A(t) \\ B(t) \end{bmatrix} = \begin{bmatrix} \cos(45^\circ) & -\sin(45^\circ) \\ \sin(45^\circ) & \cos(45^\circ) \end{bmatrix} \begin{bmatrix} I(t) \\ Q(t) \end{bmatrix} \quad (4)$$

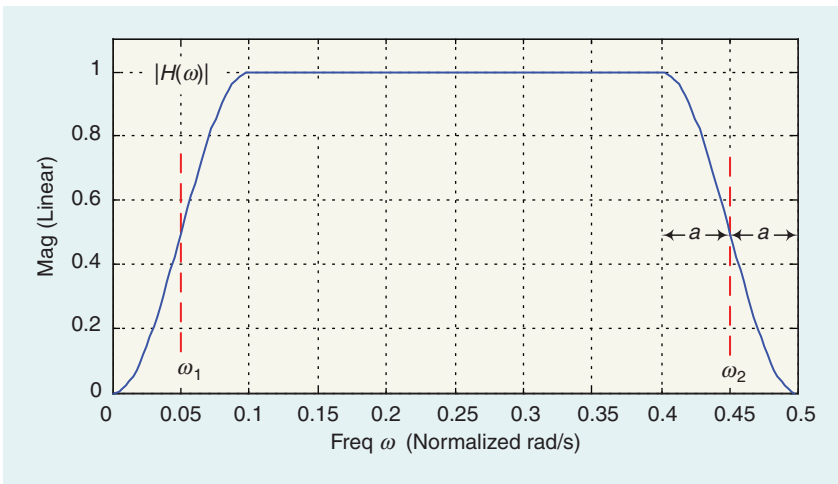


FIG2 Desired analytic signal generation filters' magnitude response.

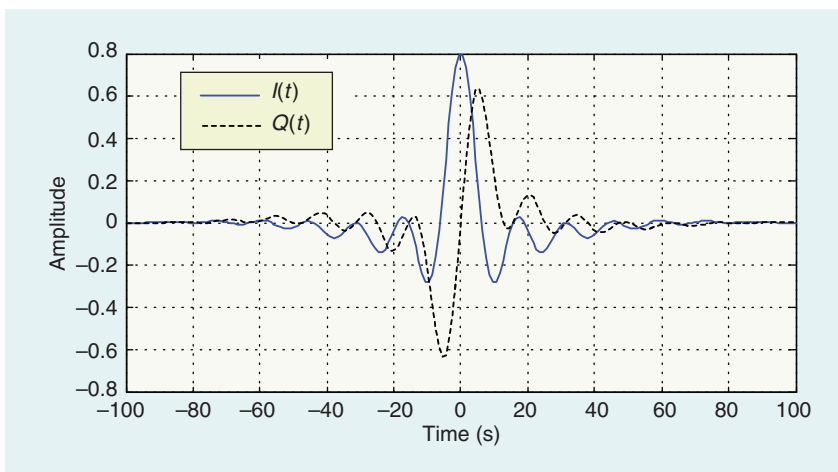


FIG3 In-phase and quadrature phase filter impulse responses.

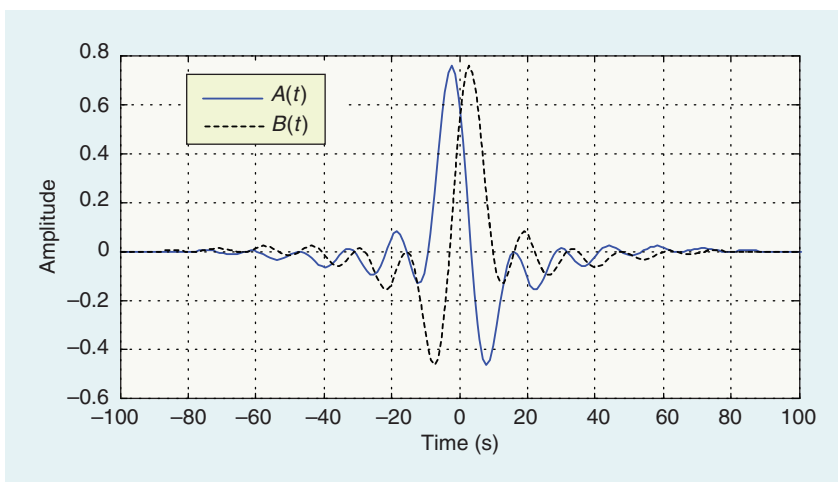


FIG4 Rotated ($\pm 45^\circ$) filters' impulse responses.

where $A(t)$ and $B(t)$ are the rotated in-phase and quadrature phase impulse responses. This simplifies to

$$A(t) = \frac{I(t) - Q(t)}{\sqrt{2}} \quad (5)$$

$$B(t) = \frac{I(t) + Q(t)}{\sqrt{2}}. \quad (6)$$

The resulting $A(t)$ and $B(t)$ impulse responses are shown in Figure 4.

If we apply the transformations, (5) and (6), to (2) and (3) respectively, we obtain

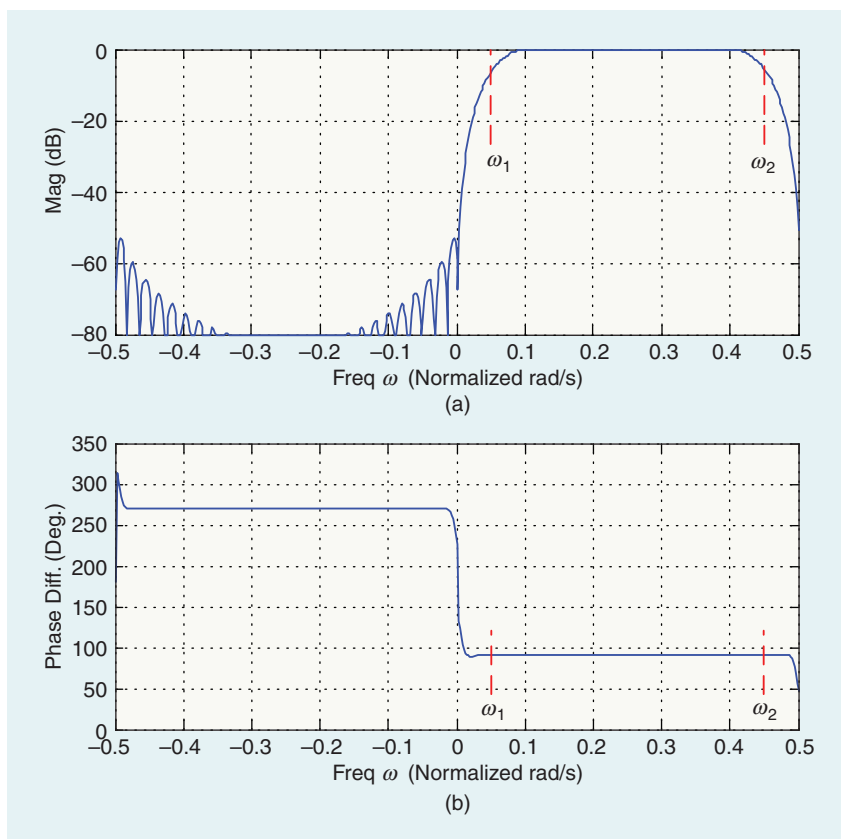
$$A(t) = \frac{2\pi^2 \cos(at)}{t(4t^2 a^2 - \pi^2)} \times [\sin(\omega_1 t + \pi/4) - \sin(\omega_2 t + \pi/4)] \quad (7)$$

$$B(t) = \frac{2\pi^2 \cos(at)}{t(4t^2 a^2 - \pi^2)} \times [\sin(\omega_1 t - \pi/4) - \sin(\omega_2 t - \pi/4)]. \quad (8)$$

This transformation allows us to eliminate half the coefficients. Both filters will now share the same set of coefficients because $B(t) = A(-t)$. Filter B 's coefficients are just a time-reversed version of filter A 's coefficients, and this "mirrored in time" relation will cut the coefficient memory requirement in half as well as making the two frequency magnitude responses identical.

COMPUTING FIR FILTER COEFFICIENTS

When using (7) to compute the filter coefficients, one may encounter a divide by zero error, which we resolve by applying L'Hospital's rule. Doing this, we



[FIG5] Analytic signal generator: (a) frequency magnitude response and (b) $A[k]$ and $B[k]$ filters' phase difference.

obtain the complete formulation for $A(t)$ in (9), shown at the bottom of the page.

Now we need to temporally sample $A(t)$ to generate our filters' coefficients. Using N to denote the length of the FIR filters' impulse responses, and $k = 0, 1, 2, \dots, N-1$ as the index of the impulse response samples (note that N can be either even or odd), we find our in-phase filter coefficients by evaluating the continuous expression in (9) at the time instants defined by

$$A[k] = A\left[2\pi\left(k - \frac{N-1}{2}\right)\right]. \quad (10)$$

Again, the quadrature phase coefficients, $B[k]$, are a reversed-order version of the in-phase $A[k]$ coefficients in (10).

As an example, when $N = 50$ and $\omega_1 = 0.05$, $\omega_2 = 0.45$, and $a = 0.05$, we obtain the frequency magnitude response shown in Figure 5(a) for our analytic signal generator.

As Figure 5(a) shows, our analytic signal generator has a flat frequency response over its positive-frequency passband, and the $A[k]$ and $B[k]$ filters have a phase response differential that differs by 90° as seen in Figure 5(b). The filters' peak-peak phase-difference error (deviation from 90°) is a mere 0.04° over the passband range of $0.1 \leq \omega \leq 0.4$.

Since we know that FIR filters with strictly linear phase must have either even or odd symmetry and our transformed filters do not possess this symmetry, then they can't be truly linear phase. However, their deviations from

$$A(t) = \begin{cases} \sqrt{2}(\omega_2 - \omega_1), & t = 0 \\ a \left\{ \sin\left[\frac{\pi}{4}\left(\frac{a + 2\omega_2}{a}\right)\right] - \sin\left[\frac{\pi}{4}\left(\frac{a + 2\omega_1}{a}\right)\right] \right\}, & t = \frac{\pi}{2a} \\ a \left\{ \sin\left[\frac{\pi}{4}\left(\frac{a - 2\omega_1}{a}\right)\right] - \sin\left[\frac{\pi}{4}\left(\frac{a - 2\omega_2}{a}\right)\right] \right\}, & t = \frac{-\pi}{2a} \\ \frac{2\pi^2 \cos(at)}{t(4a^2 t^2 - \pi^2)} \left[\sin\left(\omega_1 t + \frac{\pi}{4}\right) - \sin\left(\omega_2 t + \frac{\pi}{4}\right) \right], & \text{otherwise.} \end{cases} \quad (9)$$

linear phase are small and only occur for frequencies around dc and at half of the sampling-rate places where the magnitude response is nearly zero.

Our filter transformations also bring about a neat property. When the desired frequency response is chosen to be symmetric about one-fourth of the sampling rate, and our filter lengths are chosen to be even, then half of the filter's coefficients will be zero! So this, in combination with the mirror property, means our pair of filters will have one-half of computational effort of a pair of arbitrary filters of the same length.

CONCLUSION

We proposed a simple method for generating a pair of high-performance, phase-orthogonal FIR filters used to generate analytic signals. The filters have a 90°-phase differential, quasi-linear

phase responses and flat magnitude responses in their passbands. Due to their time reversal symmetries, only one set of filter coefficients need be stored in memory. If the frequency response of even-length filters is chosen to be centered at one-fourth the sampling rate, then half of the coefficients will be zero. And, finally, because the filters coefficients are defined by a closed-form formula, there is no filter length limitation imposed by a numerical process that fails to converge.

MathCAD and MATLAB code, modeling this dual quadrature filters technique, is available for downloading at <http://signalprocessingsociety.org/publications/periodicals/spm/columns-resources>.

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